BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 40, Number 3, Pages 421–427
S 0273-0979(03)00986-8
Article electronically published on April 17, 2003

An introduction to wavelet analysis, by David F. Walnut, Applied and Numerical Harmonic Analysis, Birkhäuser, Boston–Basel–Berlin, 2002, xx+449 pp., $69.95, ISBN 0-8176-3962-4

Michael Berry, summarizing for the *Bulletin* his recent AMS Gibbs Lecture [2], observes:

> Nowhere are the intimate connections between mathematics and physics more immediately apparent than in optics; with our own eyes, we can see through physical phenomena almost directly to the conceptual structures underlying them. Risking the wrath of philosophers, I use the term *mathematical phenomena* to describe these structures.

Once in a while a new trend in mathematics comes along. The skeptics would call it a new fad and ask what all the fuss is about. Those who are convinced will get on the wagon and drop the infinite series they are working on. Others will be looking for the lost remainder terms. Wavelet analysis is in a sense a new trend, but it started with Alfred Haar's paper [6] almost a hundred years ago. The significance of Haar's original construction was perhaps not fully understood until much later in the mid-1980's. Some of the reasons for the wavelet craze (a favorite term of the skeptics!) have to do with the need for fast algorithms, brought about by our better understanding of connections of wavelets to signal processing, optics, data compression, turning fingerprints into digital data files, subdivision algorithms in graphics, digital cameras, high-resolution television, and the JPEG 2000 encoding of images. As a mathematical subject, the theory of wavelets draws on tools from mathematics itself, such as harmonic analysis and numerical analysis. But in addition there are exciting links to areas outside mathematics. The connections to electrical and computer engineering, and to image compression and signal processing in particular, are especially fascinating. These interconnections of research disciplines may be illustrated with the two subjects (1) wavelets and (2) subband filtering (from signal processing). While they are quite different and have distinct and independent lives, and even have different aims and different histories, they have in recent years found common ground. It is a truly amazing success story. Advances in one area have helped the other: subband filters are absolutely essential in wavelet algorithms and in numerical recipes used in subdivision schemes, for example, and especially in JPEG 2000—an important and extraordinarily successful image-compression code. JPEG uses nonlinear approximations and harmonic analysis in spaces of signals of bounded variation. Similarly, new wavelet approximation techniques have given rise to the kind of data-compression which is now used by the FBI (via a patent held by two mathematicians) in digitizing fingerprints in the U.S. It is the happy marriage of the two disciplines, signal processing and wavelets, that enriches the union of the subjects, and the applications, to an extraordinary degree.

2000 *Mathematics Subject Classification.* Primary 41A15, 42A16, 42A65, 43A65, 46L60, 47D25.

We are pleased to thank Brian Treadway for excellent typesetting, and helpful suggestions.

Work supported in part by the U.S. National Science Foundation under grants DMS-9987777 and DMS-0139473(FRG).

While the use of high-pass and low-pass filters has a long history in signal processing, dating back more than fifty years, it is only relatively recently, say the mid-1980's, that the connections to wavelets have been made. Multiresolutions from optics are the bread and butter of wavelet algorithms, and they in turn thrive on methods from signal processing, in the quadrature mirror filter construction, for example. The effectiveness of multiresolutions in data compression is related to the fact that multiresolutions are modelled on the familiar positional number system: the digital, or dyadic, representation of numbers. Wavelets are created from scales of closed subspaces of the Hilbert space $L^2(\mathbb{R})$ with a scale of subspaces corresponding to the progression of bits in a number representation. While oversimplified here, this is the key to the use of wavelet algorithms in digital representation of signals and images. The digits in the classical number representation in fact are quite analogous to the frequency subbands that are used *both* in signal processing and in wavelets. There is a strong need for more interdisciplinary books along these lines. David Walnut's lovely book fills a need, but the questions, the applications and the interconnections are manifold: Most authors know one or more of the relevant subjects well, but probably not all. This is reflected in the well acknowledged but frustrating fact that the diverse subjects even have quite different languages and different terminology, resulting in a communications gap. Other recent textbooks on wavelets such as [10] and [3] have taken a direct and focused approach to the issue of communicating across traditional boundaries; [3] even has a dictionary of terms used in (1) mathematics, in (2) engineering, and in (3) quantum physics, in particular in quantum computing algorithms (QCA). Not surprisingly, the quantum algorithms have proved effective in wavelet-data compression, as they are based on the same kind of multiresolution analysis as described above. Also in quantum theory, it is the same kinds of scales of closed subspaces of a Hilbert space, now called resolutions, which form the basis for the algorithms. While these methods originate with engineering and physics, their use has now become widespread in mathematics.

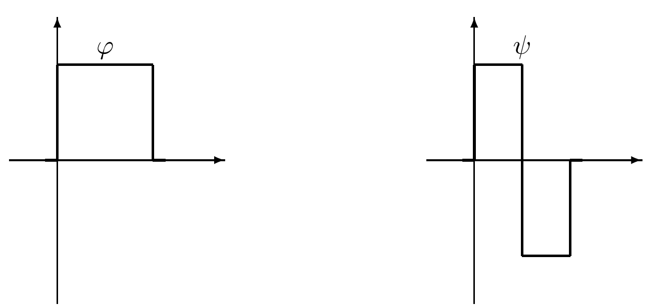
Some skeptics in mathematics departments might say that this is all very well, but applications like that are usually left to the engineers. True!—some of it, but what is unique here is that the new fashionable applications involve a number of core ideas from mathematics, such as classical concepts of transform theory, including discrete versions of Fourier transforms, matrix factorization (used both in the fast Fourier transform (FFT) and in some of the more effective wavelet algorithms), function spaces, geometry of Hilbert space, the Calderón atomic decomposition, operator theory, spectral theory, and the classical symbiotic interconnections between harmonic analysis and probability theory. And of more direct relevance to math departments is the fact that the engineering applications generate a need for new service courses for engineering students, or at least a rethinking of what we offer in our math courses for engineers, and for other needs which are dictated by applications.

This is where the book of David Walnut and other textbooks on wavelets come in. Are there more of them than we really need? On the contrary, there is good reason for the variety in outlook and presentation of new wavelet books, and in fact there is very little overlap from one to the next. The past ten years have indeed seen a number of such new book offerings: see, for example, [1] for books that appeared before Walnut's and [3] after. The ten books in the combined list of [1] include books that are primers (C. Blatter, Chui–Chan–Lin), case studies such as signal analysis (Chui), statistical estimation (Härdle), Fourier methods and filtering (Gasquet–Witomski), image processing and the multiscale approach (Starck–Murtagh–Bijaoui), comprehensive mathematical treatises (Resnikoff–Wells and Wojtaszczyk), and the signal processing viewpoint by one of the pioneers, Stéphane Mallat [7].

The first of these books which really caught on was Daubechies's [4], but the subject has branched out since then, and many new trends have materialized. This is perhaps one of the reasons why the fairly large number of recent wavelet books for the classroom in fact have not been overlapping as much as some might have thought. David Walnut's book is aimed at upper-level undergraduate students who have had advanced calculus, but not much more. As a result, he has included a number of preliminary topics such as Fourier series. These are topics that can also be found elsewhere, but they are given a twist in Walnut's presentation, which is directly aimed at the needs later in the book. Since they are not relegated to appendices, this means that readers will not get to the wavelets until around page 110. But the presentation of Shannon's sampling theorem and the fast Fourier transform is couched in terms that make very natural the fundamental ideas that go into the pyramid algorithms of wavelet analysis. At the outset, Walnut does not present these fundamental ideas axiomatically, but instead introduces them through the Haar systems, which are both clear and intuitive. At the same time, Haar's method has implicit in it the multiresolution idea. For example, the two functions

$$\varphi(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & \text{elsewhere} \end{cases} \quad \text{and} \quad \psi(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} \le x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(1)



Father function (a) — Mother function (b)

capture in a glance the refinement identities

$$\varphi(x) = \varphi(2x) + \varphi(2x-1) \qquad \text{and} \qquad \psi(x) = \varphi(2x) - \varphi(2x-1).$$

The two functions are clearly orthogonal in the inner product of $L^2(\mathbb{R})$, and the two closed subspaces $\mathcal{V}_0$ and $\mathcal{W}_0$ generated by the respective integral translates

$$\{\varphi(\,\cdot - k) : k \in \mathbb{Z}\} \qquad \text{and} \qquad \{\psi(\,\cdot - k) : k \in \mathbb{Z}\} \tag{2}$$

satisfy

$$U\mathcal{V}_0 \subset \mathcal{V}_0 \qquad \text{and} \qquad U\mathcal{W}_0 \subset \mathcal{V}_0 \tag{3}$$

where U is the dyadic scaling operator $Uf(x) = 2^{-1/2} f(x/2)$. The factor $2^{-1/2}$ is put in to make U a unitary operator in the Hilbert space $L^2(\mathbb{R})$. This version of Haar's system naturally invites the question of what other pairs of functions φ and ψ with corresponding orthogonal subspaces $\mathcal{V}_0$ and $\mathcal{W}_0$ there are such that the same invariance conditions (3) hold. The invariance conditions hold if there are coefficients a_k and b_k such that the scaling identity

$$\varphi(x) = \sum_{k \in \mathbb{Z}} a_k \varphi(2x - k) \tag{4}$$

is solved by the father function, called φ, and the mother function ψ is given by

$$\psi(x) = \sum_{k \in \mathbb{Z}} b_k \varphi(2x - k). \tag{5}$$

A fundamental question is the converse one: Give simple conditions on two sequences (a_k) and (b_k) which guarantee the existence of $L^2(\mathbb{R})$-solutions φ and ψ which satisfy the orthogonality relations for the translates (2). How do we then get an orthogonal basis from this? The identities for Haar's functions φ and ψ of (1)(a) and (1)(b) above make it clear that the answer lies in a similar tiling and matching game which is implicit in the more general identities (4) and (5). Clearly we might ask the same question for other scaling numbers, for example $x \to 3x$ or $x \to 4x$ in place of $x \to 2x$. Actually a direct analogue of the visual interpretation from (1) makes it clear that there are no nonzero locally integrable solutions to the simple variants of (4),

$$\varphi(x) = \frac{3}{2}\left(\varphi(3x) + \varphi(3x - 2)\right) \tag{6}$$

or

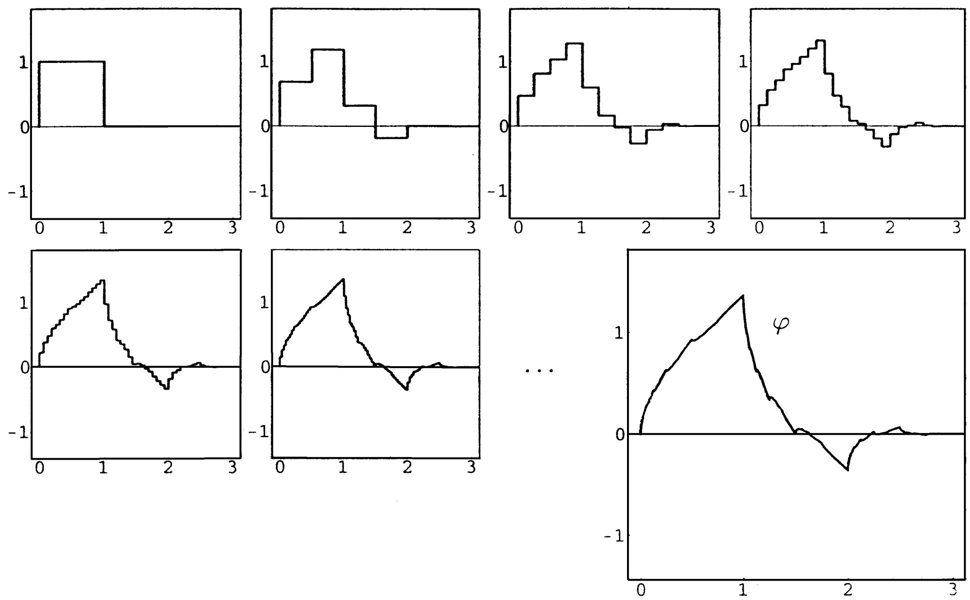
$$\varphi(x) = 2\left(\varphi(4x) + \varphi(4x - 2)\right). \tag{7}$$

There *are* nontrivial solutions to (6) and (7), to be sure, but they are versions of the Cantor Devil's Staircase functions, which are prototypes of functions which are not locally integrable.

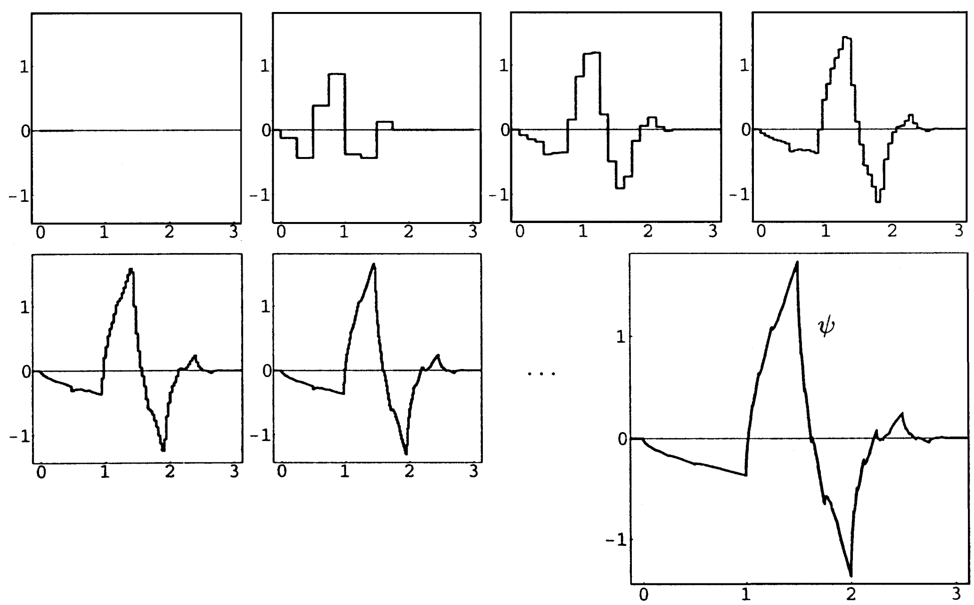
Since the Haar example is based on the fitting of copies of a fixed "box" inside an expanded one, it would almost seem unlikely that the system (4)–(5) admits finite sequences (a_k) and (b_k) such that the corresponding solutions φ and ψ are continuous or differentiable functions of compact support. The discovery in the mid-1980's of compactly supported differentiable solutions (see [4]) was paralleled by applications in seismology, acoustics [5], and optics [8], as discussed in [9], and once the solutions were found, other applications followed at a rapid pace: see, for example, the ten books in Benedetto's review [1]. It is the solution ψ in (5) that the fuss is about, the mother function; the other one, φ, the father function, is only there before the birth of the wavelet. The most famous of them are named after Daubechies and look like the graphs in Figure 1. With the multiresolution idea, we arrive at the closed subspaces

$$\mathcal{V}_j := U^{-j}\mathcal{V}_0, \qquad j \in \mathbb{Z}, \tag{8}$$

as noted in (2)–(3), where U is some scaling operator. There are extremely effective iterative algorithms for solving the scaling identity (4): see, for example,



(a): Father function



(b): Mother function

FIGURE 1. Daubechies wavelet functions and series of cascade approximants

[3, Example 2.5.3, pp. 124–125],[1] [4], and [10], and Figure 1. A key step in the algorithms involves a clever choice of the kind of resolution pictured in (13), but

[1]See an implementation of the "cascade" algorithm using Mathematica, and a "cartoon" of wavelets computed with it, at http://www.math.uiowa.edu/~ jorgen/wavelet_motions.pdf.

digitally encoded. The orthogonality relations can be encoded in the numbers (a_k) and (b_k) of (4)–(5), and we arrive at the doubly indexed functions

$$\psi_{j,k}(x) := 2^{j/2}\psi\left(2^j x - k\right), \qquad j,k \in \mathbb{Z}. \tag{9}$$

It is then not difficult to establish the combined orthogonality relations

$$\int_{\mathbb{R}} \overline{\psi_{j,k}(x)}\,\psi_{j',k'}(x)\;dx = \left\langle \psi_{j,k} \mid \psi_{j',k'} \right\rangle = \delta_{j,j'}\delta_{k,k'} \tag{10}$$

plus the fact that the functions in (9) form an orthogonal basis for $L^2(\mathbb{R})$. This provides a painless representation of $L^2(\mathbb{R})$-functions

$$f = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} c_{j,k}\psi_{j,k} \tag{11}$$

where the coefficients $c_{j,k}$ are

$$c_{j,k} = \int_{\mathbb{R}} \overline{\psi_{j,k}(x)}\,f(x)\;dx = \left\langle \psi_{j,k} \mid f \right\rangle. \tag{12}$$

What is more significant is that the resolution structure of closed subspaces of $L^2(\mathbb{R})$

$$\cdots \subset \mathcal{V}_{-2} \subset \mathcal{V}_{-1} \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \tag{13}$$

facilitates powerful algorithms for the representation of the numbers $c_{j,k}$ in (12). Amazingly, the two sets of numbers (a_k) and (b_k) which were used in (4)–(5) and which produced the magic basis (9), the wavelets, are the same magic numbers which encode the quadrature mirror filters of signal processing of communications engineering. On the face of it, those signals from communication engineering really seem to be quite unrelated to the issues from wavelets—the signals are just sequences, time is discrete, while wavelets concern $L^2(\mathbb{R})$ and problems in mathematical analysis that are highly non-discrete. Dual filters, or more generally, subband filters, were invented in engineering well before the wavelet craze in mathematics of recent decades. These dual filters in engineering have long been used in technology, even more generally than merely for the context of quadrature mirror filters (QMF's), and it turns out that other popular dual wavelet bases for $L^2(\mathbb{R})$ can be constructed from the more general filter systems; but the best of the wavelet bases are the ones that yield the strongest form of orthogonality, which is (10), and they are the ones that come from the QMF's. The QMF's in turn are the ones that yield perfect reconstruction of signals that are passed through filters of the analysis-synthesis algorithms of signal processing. They are also the algorithms whose iteration corresponds to the resolution sytems (13) from wavelet theory.

While Fourier invented his transform for the purpose of solving the heat equation, i.e., the partial differential equation for heat conduction, the wavelet transform (11)–(12) does not diagonalize the differential operators in the same way. Its effectiveness is more at the level of computation; it turns integral operators into sparse matrices, i.e., matrices which have "many" zeros in the off-diagonal entry slots. Again, the resolution (13) is key to how this matrix encoding is done in practice.

I take it as a healthy sign when there is a burst of new books in a sub-area of mathematics. In wavelet analysis and its applications, we have seen a number of recent books arrive in university bookstores. Surprisingly, there doesn't in fact seem to be much of an overlap of subject or scope from one book to the next. The possible variations of a presentation of the subject are infinite in many directions, for example the kind of student it is aimed at, the level, the specialized area within mathematics itself, and the kind of application stressed. D. Walnut's lovely book aims at the upper undergraduate level, and so it includes relatively more preliminary material, for example Fourier series, than is typically the case in a graduate text. It goes from Haar systems to multiresolutions and then the discrete wavelet transform, starting on page 215. The applications to image compression are wonderful and the best I have seen in books at this level. I also found the analysis of the best choice of basis and wavelet packet especially attractive. The later chapters include MATLAB codes. Highly recommended!

References

1. J.J. Benedetto, *Ten books on wavelets*, SIAM Rev. **42** (2000), 127–138.
2. M. Berry, *Making light of mathematics*, Bull. Amer. Math. Soc. (N.S.) **40** (2003), 229–237.
3. O. Bratteli and P.E.T. Jorgensen, *Wavelets through a Looking Glass: The World of the Spectrum*, Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, 2002.
4. I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF Regional Conf. Ser. in Appl. Math., vol. 61, SIAM, Philadelphia, 1992. MR **93e:**42045
5. D. Esteban and C. Galand, *Application of quadrature mirror filters to split band voice coding systems*, IEEE International Conference on Acoustics, Speech, and Signal Processing (Washington, DC, May 1977), Institute of Electrical and Electronics Engineers, Piscataway, NJ, 1977, pp. 191–195.
6. A. Haar, *Zur Theorie der orthogonalen Funktionensysteme*, Math. Ann. **69** (1910), 331–371.
7. S.G. Mallat, *A Wavelet Tour of Signal Processing*, Academic Press, Orlando–San Diego, 1998. MR **99m:**94012
8. D. Marr, *Vision: A Computational Investigation into the Human Representation and Processing of Visual Information*, W.H. Freeman, San Francisco, 1982.
9. Y. Meyer, *Wavelets: Algorithms & Applications*, SIAM, Philadelphia, 1993, translated from the French and with a foreword by Robert D. Ryan. MR **95f:**94005
10. G. Strang and T. Nguyen, *Wavelets and filter banks*, Wellesley-Cambridge Press, Wellesley, MA, 1996. MR **98b:**94003

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