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Two-dimensional wavelets and their relatives, by Jean-Pierre Antoine, Romain Murenzi, Pierre Vandergheynst, and Syed Twareque Ali, Cambridge University Press, Cambridge, UK, 2004, xviii+458 pp., £75.00, ISBN 0-521-62406-1

To some people, a *wavelet* $w = w(x)$ is a nice function which is localized in x, whose Fourier transform $\hat{w}(\xi)$ is localized in ξ, and which can be superposed, together with copies of itself produced by simple transformations like shifts, dilations, or modulations, to produce any desired function. For them, a *wavelet transform* is an expansion much like a Fourier series, with discrete coefficients. These expansion coefficients have a specific meaning in applications. When x is a single real variable it may be regarded as "time", and then w is said to be well-localized in time and frequency. These terms hint at the nonmathematical origins of the name and notion of wavelet. Algorithms arise naturally from truncation to finite series. The taxonomy of Fourier analysis accommodates wavelet series without trouble, and this kind of applied and computational "discrete wavelet analysis" has developed a huge literature over the past 20 years or so.

This is not how wavelet analysis began. The original notion of "wavelet of constant shape" first appeared in the fundamental work by Grossmann and Morlet [1], born of collaboration between a theoretical physicist (Grossmann) and a computational petroleum prospector (Morlet). The paper was in English, but the authors resided in France; and as "ondelettes" generated more and more interest, this class of special functions came to be known simply as wavelets.

In Grossmann and Morlet, the wavelet transform W is an isometry from a Hilbert space H to $L^2(G)$, where G is a locally compact topological group. To get the isometry, one specifies a strongly continuous irreducible unitary representation U of G on H. Existence of W is implied by the existence of a nonzero admissible element $w \in H$, namely one for which

$$c_w \stackrel{\text{def}}{=} \|w\|^{-2} \int_G |\langle U(g)w, w\rangle_H|^2 \, dg < \infty,$$

where dg is the left-invariant Haar measure on G. Clearly $c_w > 0$. Then W is defined for each $f \in H$ by

$$Wf : G \to \mathbf{C}; \qquad Wf(g) \stackrel{\text{def}}{=} c_w^{-1/2} \langle U(g)w, f\rangle_H .$$

This W is a linear isometry, so it is invertible on its range by its adjoint. Hence, $f \in H$ is recovered from Wf by an inner product in $L^2(G)$. Followers of this line of thought call it "continuous wavelet analysis". The admissible element is the "wavelet", but it clearly is not the star of the show. That role is played by the group.

An admissible wavelet always exists if G is compact, but Grossmann and Morlet were interested in the noncompact group of shifts and dilations (both positive and negative) of $\mathbf{R}$. This is also called the affine, or "$ax + b$" group (together with the reflection $x \mapsto -x$). Parametrized as $G = \{(a, b) : a \neq 0, b \in \mathbf{R}\}$, it has the

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following irreducible unitary representation and wavelet transform:

$$U(a,b)f(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{|a|}} f\left(\frac{x-b}{a}\right), \qquad Wf(a,b) \stackrel{\text{def}}{=} \frac{1}{\sqrt{c_w}} \int_{\mathbf{R}} \bar{w}\left(\frac{x-b}{a}\right) f(x)\, dx,$$

where $f \in L^2(\mathbf{R}) = H$, and $w \in H$ is admissible. The "constant shape" label applies to the set $\{U(g)w : g \in G\}$ for this group. What that shape is does not really matter, although Grossmann and Morlet provided a few nice examples. The admissibility condition for w may be stated in terms of its Fourier integral transform,

$$c_w = 2\pi \int_{\mathbf{R}} \frac{|\hat{w}(\xi)|^2}{|\xi|}\, d\xi,$$

from which it is seen that w must have some smoothness (so $|\hat{w}(\xi)| \to 0$ as $\xi \to \pm\infty$) and also some self-cancellation (so $|\hat{w}(\xi)| \to 0$ as $\xi \to 0$).

General layout

Two-Dimensional Wavelets and Their Relatives is a survey focused on applications of continuous wavelet analysis. Fundamentally, "a wavelet is a complex-valued function ψ ... satisfying an admissibility condition" (Definition 2.1.1, p. 34). The authors began their own large body of work on continuous wavelet analysis by letting $H = L^2(\mathbf{R}^2)$ and taking G to be positive isotropic dilations and all shifts of $\mathbf{R}^2$. Instead of just one reflection, they found that all plane rotations must be added to G. The Fourier-transformed admissibility condition remains essentially the same as for the affine group on $\mathbf{R}$, since rotations add only a compact factor to G. It again implies some smoothness and self-cancellation for the wavelet w. This first generalization formed the core of the Ph.D. thesis of one of the authors (Murenzi, [2]), who was a student of another (Antoine). It is already so rich in applications that the majority of the text need not stray beyond two dimensions.

The authors are clear about their preferences: the discrete wavelet transform "is not the main subject of the present book" (§5.6, p. 212), despite its being "used in the majority of applications." But there is in fact little direct competition between continuous and discrete wavelet applications. For some applications, like compact coding of communication signals such as music or television, discrete nonredundant expansions dominate. They combine well-engineered wavelets with highly refined source coding, all implemented with clever low-complexity arithmetic. On the other hand, automatic classification of complicated signals such as radar tracks gains nothing from compact coding, especially when the target is buried in noise.

Discrete wavelet transforms are presented as a special case of the continuous wavelet transform: its restriction to a discrete subset of the group elements. For sufficiently dense discrete subgroups (lattices) or semigroups $S \subset G$, the collection $\{\psi_g : g \in S\}$ is a frame in L^2, and the wavelet transform specializes to the expansion coefficients. Orthogonal wavelet bases are indexed by still smaller subsets which unfortunately are not subgroups or even semigroups. Missing algebraic tools are replaced by filters and multiresolution analysis (MRA), giving a contrasting perspective into discrete wavelets with its own rich theory. The authors briefly describe various 2-D MRA wavelet constructions, including the lifting scheme, the quincunx MRA, steerable filters, and wavelet orthonormal bases on the sphere.

The narrative occasionally stumbles, for example when the 2-D inversion formula Equation 2.133 loses the variable θ of the L^1-normalized continuous wavelet transform $\breve{S}(\vec{b}, a, \theta)$ defined in Equation 2.40 and redefined with inequivalent normalization in Equation 2.55. But given the scope of the material and the diversity of notations, one would expect far more inconsistency. The text is remarkably free of it.

The applications

After two chapters reviewing 1-D and 2-D wavelet transforms, both continuous and discrete, the authors devote Chapter 3 to constructions of wavelets w with specific properties. One can control "frequency content" by specifying the support of $\hat{w}$. For example, one can restrict it to a sector of the frequency plane; the admissibility condition is stated in terms of the Fourier transform and is easily checked in this case.

Of course, image processing commands top billing in Chapter 4. Redundant representations are useful for recognition algorithms like edge and texture detection, and the absence of a sampling grid eliminates certain artifacts in denoising. Chapter 5 continues with processing of image-like data from various sciences, including astronomy and fluid dynamics. Continuous wavelet analysis of turbulence modeled in 2-D has surprisingly much in common with texture analysis.

The group-theoretic point of view is thoroughly exploited in Chapters 6 and 7. Returning to the formalism of Grossmann and Morlet, the phase space of 2-D wavelet transforms is developed, setting the stage for Chapter 8 and the machinery of uncertainty inequalities and their minimizers, or "gaborettes".

Chapter 9 discusses 3-D wavelets and wavelets on spheres. The latter are not the "second generation wavelets" of Sweldens and Schröder [3], but instead derive from a shifts and dilations group on S^2. The sphere dilations are the image of dilations in the tangent plane under stereographic projection through the antipode. One can add local rotations by the same mechanism. There are technicalities involving computation with spherical harmonics, but the construction generalizes nicely to higher-dimensional spheres and further to general Riemannian symmetric spaces.

Chapter 10 describes some recent work on motion detection and motion estimation with the "spatio-temporal" wavelet transform. Here the group G is the affine Galilean group

$$(x, t) \mapsto (ax + b + vt, ct + d),$$

where the group element is parametrized by (a, b, v, c, d). The authors developed a missile-tracking algorithm based on finding the maxima of $Wf(a, b, v, c, d)$ from radar range profiles f. This was complicated by the complexity of the group, although it has a natural irreducible unitary representation just like the affine group.

Finally, Chapter 11 surveys a mix of "wavelet-like" algorithms that fit neither into the continuous nor discrete wavelet transform formalisms. In other words, there is neither a group nor an MRA. This covers curvelets, matching pursuit, and quasicrystals.

The book concludes with a 15-page appendix on group theory with examples relevant to the groups mentioned in the text. There are also two pages on spherical harmonic functions. These are a good starting point, though not complete enough for a novice who wants to understand all the proofs.

References

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