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**Manin, Yu. I.**

**New dimensions in geometry.**

*Workshop Bonn* 1984 (*Bonn,* 1984), 59–101, *Lecture Notes in Math.*, 1111, *Springer, Berlin*, 1985.

The present article is a speculative article, drawing on the author's enormous breadth and depth of mathematical culture, and it is difficult, indeed impossible, for the reviewer to do full justice to all its many facets. The article is a "lecture" by the author to the 25th Arbeitstagung in Bonn (Atiyah summarized orally the prepared text reproduced here). The author seeks to pull together several major strands of current developments in geometry, arithmetic and physics, specifically: (i) arithmetic geometry in the sense of Arakelov-Faltings, or "$A$-geometry"; (ii) Kähler-Einstein metrics on algebraic varieties and their generalizations; (iii) supersymmetry, or graded structures on manifolds.

The paper is naturally divided into three parts: an introduction, three sections on $A$-geometry, followed by three sections on supermanifolds. The topic (ii) above is laced through the $A$-geometry section. I will just mention a selection of what the author chooses to review from these various fields, and will rather emphasize reproducing here some of his almost "aphoristic" remarks as a way of trying to entice the reader to peruse his suggestions personally.

The first premise of the paper is that Diophantine problems and physics have forced the enlargement of our concepts of geometry. Put succinctly, one replaces the coordinate ring $\mathbf{R}[x_1, \cdots, x_n]$ of Cartesian $n$-space by $\mathbf{Z}[x_1, \cdots, x_n; \xi_1, \cdots, \xi_m]$. Here $\mathbf{Z}$ represents the arithmetic aspect of geometry, the $x_i$ are the usual (or "bosonic") geometric variables, and the $\xi_j$ are anticommuting (or "fermionic") variables. The author's first aphorism: "All three types of geometric dimensions are on an equal footing". That the arithmetic and geometric ($x_i$) variables are of an equal stature goes back about 100 years or more; the equivalence of the $x_i$ and $\xi_j$ is the relatively recent import of supersymmetry in physics. The author proposes simply completing the triangle, and the point of the paper is, in a way, to review current geometry with an eye towards evaluating whether we are evolving in that direction and to crystallize questions that he feels would help this trend.

$A$-geometry, the topic of Sections 1–3, seeks to compactify, in a natural way, a $\mathbf{Z}$-scheme by adding a variety at the infinite places. Of course, one knows what the variety is supposed to be, but one wants to add a metric structure at $\infty$, analogous to the norms at the $\infty$-places of a number field. In the case of an arithmetic surface, the key ingredient on the curve at infinity is a special metric and its Green's function. The author proposes using Kähler-Einstein metrics on varieties over infinite places, Hermite-Einstein metrics on stable bundles over these varieties, etc. He stops short of speculating on the role of odd variables in this $A$-geometry. Some of the questions he poses here are the following: (1) Do there exist "groups" mixing the arithmetic and geometric dimensions? (This means, obviously, more than group schemes/$\mathbf{Z}$ or adèle groups.) (2) If one considers Hermite-Einstein bundles as the "obvious" metrization of some coherent sheaves on $X_\infty$, is there a categorical way to generate all coherent sheaves from these semistable ones, and does such a construction still have differential geometric content? (3) Do "canonical examples", such as moduli spaces of vector bundles on a curve, have canonical $A$-structures? For example, could one describe an $A$-geometric $c_2$ for higher rank bundles over the moduli space? (4) What is intersection theory and the Riemann-Roch theorem in this higher-dimensional context?

Some progress in local index theory for families, due to Bismut-Freed, Bost, Gillet-Soulé and others, should hopefully be relevant to this last question.

In the sections on superspace, the author recounts some of the basic definitions (Section 4), and then reports on the results of Vaĭntrob, Skornyakov, Voronov, Penkov and the author himself. These latter concern "super"-analogues of Kodaira-Spencer theory and of the Bruhat decomposition and Schubert cells for complete flag superspaces of classical type. Several interesting differences from the classical (ungraded) case arise, some already highlighted in Kac's representation theory for such algebras. I would again rather record here some of the more aphoristic suggestions and conjectures of the author. (1) (only hinted at) Supergeometry change-of-coordinate formulae involve derivatives of coordinate changes in the transformations of the odd coordinates—what is the role, seemingly forced, of distributions in continuous (as opposed to $C^\infty$) supergeometry? (2) Is it possible to compactify superspaces along the odd directions? For example, Leĭtes asks what the purely odd projective space "$\operatorname{Proj}\mathbf{Z}[\xi_1,\cdots,\xi_n]$" should be. In much the same spirit, the author asks whether there is a cohomology on super-flag-manifolds with his super Bruhat cells (which are really "sub-super-schemes", and not just subvarieties of the flag supermanifolds as generators). In general, are there global geometric invariants of the odd dimensions? (Some steps in this direction might come from a paper by M. Rothstein [Trans. Amer. Math. Soc. **299** (1987), no. 1, 387–396].) (3) An even bolder question is, "Is the even geometry a collective effect in the infinite-dimensional odd geometry?" This has roots in both the equivalence of "wedge" and "spinor" pictures of representations of Kac-Moody algebras, and more philosophically from the perceived "primacy" of fermions in particle physics.

The final Section 6 treats the kinematics of supergravity from the point of view of creating a curved structure on superspace modelling the supergeometry of the Bruhat cells. Many interesting physical equations (especially Yang-Mills) have been treated "twistorially" in the last decade, and the equations of motion are usually related to such integrability conditions. (Cf. also a paper by the author [*Arithmetic and geometry, Vol. II*, 175–198, Birkhäuser, Boston, Mass., 1983; MR0717612 (85i:32043)].) His current point of view is an extension of the twistor correspondence, the interpretation being given in Section 6.1 of this paper. A final calculation equates these integrability conditions with the pre-potential formalism of V. I. Ogievetskiĭ and E. S. Sokachev [Yadernaya Fiz. **31** (1980), no. 3, 821–839; MR0607671 (82g:83030)].

A final topic, which is not really touched upon, but simply pointed out: A true super-symmetric Kähler geometry will probably be quite sophisticated. Even in algebraic supergeometry, the value of classical projective techniques seems unfortunately limited, so the development of such a Kähler alternative might prove important.

In summary, then, the author has tried to "seize the moment", to discern a pattern crystallizing out of what seems a tantalizing chaos in the rapidly exploding frontiers of geometry. He has, in the opinion of this reviewer, done an exciting job of updating the vision of the late nineteenth century in the discovery of the fecund "parallels" between the theory of numbers and the theory of algebraic functions.

The jewel of geometry is even more brilliant and fascinating with the dazzling interplay of flashes of light from any one of its new facets to another.

{For the entire collection see MR0797412 (86e:00018)}

*Daniel M. Burns, Jr.*

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**Atiyah, Michael**

**Commentary on the article of Yu. I. Manin: "New dimensions in geometry" [*Workshop Bonn 1984* (Bonn, 1984), 59–101, Springer, Berlin, 1985; MR0797416 (87j:14030)].**

*Workshop Bonn* 1984 (*Bonn,* 1984), 103–109, *Lecture Notes in Math.*, 1111, *Springer, Berlin*, 1985.

The author, who presented the oral summary of the paper reviewed above at the 1984 Arbeitstagung, here takes the opportunity to add his own brief reflections on Manin's theme. His main point is to emphasize the unity suggested on a more extensive scale when one considers the demands on geometry called for by quantum physics, due mainly to its inherent infinite-dimensionality and noncommutativity. He draws attention to several examples of arithmetic aspects of quantum-mechanical mathematics.

He points out, for example, that spectral invariants of finite-dimensional manifolds are "quantum" in nature ($\eta$-invariants, torsion invariants), in that they depend on the totality of the spectra of suitable Laplacians, while they are identifiable with arithmetic invariants in some cases (e.g., special values of $L$-series for totally real number fields). Similarly, D. G. Quillen [Funktsional. Anal. i Prilozhen. **19** (1985), no. 1, 37–41; MR0783704 (86g:32035)] found a Laplacian analogue of an important construction of Faltings in $A$-geometry which has subsequently proved enormously useful both in the theory of strings, and in local forms of the index theorem for families.

The author makes two more specific conjectural suggestions: (1) He proposes a definition of a structure like a supermanifold on a manifold $M$ using an embedding of $M$ in a larger manifold, $N$, and the complex of currents in $N$, supported on $M$ (I do not know whether anyone is currently pursuing this suggestion); (2) he points out a possible relation of Connes's "noncommutative differential geometry" and the arithmetic surfaces of Arakelov-Faltings (passing via suggestions of Roe and Vojta concerning Nevanlinna theory).

{For the entire collection see MR0797412 (86e:00018)}

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