MATHEMATICAL PERSPECTIVES

In the Perspectives section of this issue we have highlighted "dimension" from the title of the article by Yuri Manin. What follows concerns dimension in antiquity, illustrated on the cover and in the reprinting of two Mathematical Reviews discussing new dimensions in geometry.

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 43, Number 2, Pages 231–234
S 0273-0979(06)01111-6
Article electronically published on March 9, 2006

ABOUT THE COVER: THE MATHEMATICAL CONQUEST OF THE THIRD DIMENSION

BENNO ARTMANN

On the cover of this issue is an image of a regular icosahedron, accompanying Proposition XIII.16 of Euclid's *Elements*, taken from the manuscript d'Orville 301 now in the collection of the Bodleian Library of Oxford University. The image was made by Octavo Corporation with the collaboration of the Bodleian Library as part of a project to publish the entire manuscript. The manuscript was produced in 888 A.D., more than a thousand years after Euclid's death, and is one of the two earliest extant complete copies of the *Elements*.

At the beginning of his discussion of the concept of dimension, Y. Manin quotes the description of geometric dimensions by Euclid. Before commenting on Euclid's definitions and the figure on the cover, let us take a quick hypothetical look at the practical side of the problem as represented by architecture and sculpture.

The first conscious steps of humankind into the third dimension may well be connected to the first exactly executed stone buildings in Mesopotamia and Egypt. In Egypt we know about the sage Imhotep, the chief architect of pharaoh Djoser about 2750 B.C., who designed the first monumental stone temple and pyramid. He remained so famous throughout Egyptian history that in later centuries he even was deified and sometimes was identified with Thot, the god of wisdom.

We do not have, however, except for a few rough drafts, any good evidence in the sense of architectural drawings throughout antiquity. The only thing one can say for certain is that sculptors used orthogonal projections. This is testified by unfinished pieces recovered from excavated workshops. Apart from a few measurements of

Received by the editors November 22, 2005.

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volumes, the mathematical theory of threespace starts with the Pythagoreans about 500 B.C. in Greece.

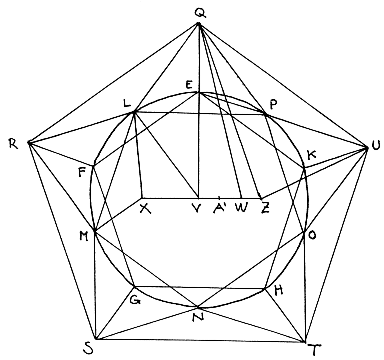

FIGURE 1. Theaetetus' figure redrawn. The letters are changed to the ones used in [Heath].

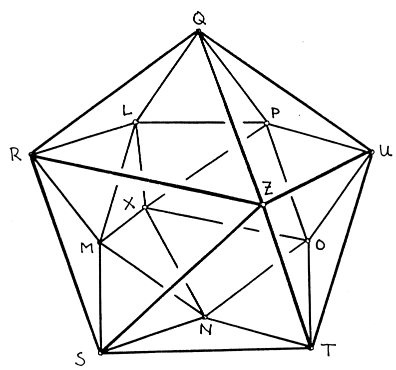

FIGURE 2. The icosahedron after Theaetetus completed, but without the lines of construction.

Greek mathematics from the time of Thales (about 580 B.C.) up to about 300 B.C., the time of Euclid, is preserved in Euclid's *Elements*. A careful reading of the text of the *Elements* reveals several historical layers. A typical example of one of these layers would be the definitions quoted by Manin, for example Def. 1 of Book XI (of the *Elements*): *A solid is that which has length, breadth, and depth.* In contrast to Euclid's axioms for plane geometry, his arithmetic and solid geometry are founded on definitions exclusively. This reflects the stage of mathematics in Plato's Academy (about 385 to 350 B.C.), where the explicit formulation of axioms seems to have been unknown. Plato says in his dialogue *Euthydemus* 290 bc [this is the customary way one quotes from Plato; 290 is the page of the first modern standard edition of the dialogues]: *Geometers…are not engaged in creating figures but in discovering those that really exist,* and in his sense a description given in a proper definition is a sufficient basis for further deductions. (For details see [Mueller], 1991.)

More specifically, in his *Republic* 528a-d, Plato discusses the state of solid geometry. On the one hand, he praises the *extraordinary attractiveness and charm* (528 d2) of the subject and observes that in spite of obstacles investigators made progress because of the inherent charm of their results. On the other hand, he deplores the state of affairs and speaks of an *absurd neglect* (528 d9/10) of solid geometry, the reasons for this being (a) the state gives no money for the research in this area and (b) there is no director coordinating the efforts of the investigators (528 b6-c4). In order to make sense of these remarks, we have to look at the historical situation around 380 B.C. and especially at the theory of the regular solids, which is preserved in the *Elements*, Book XIII. An ancient note written in the margins of the manuscript says: [*this book is about*] *the five so-called Platonic figures, which however do not belong to Plato, three of the aforesaid figures being due to the Pythagoreans, namely the cube, the Pyramid* (tetrahedron) *and the dodecahedron, while the octahedron and the icosahedron are due to Theaetetus* ([Heath, III, 438]). From another source we know that the Pythagorean Hippasus of Metapont (about 500 to 450 B.C.?) knew or constructed the "sphere of the twelve pentagons," that is, the dodecahedron.

The original knowledge of the dodecahedron may have come from crystals of pyrite [Art01, p. 305, photo], but in contrast the icosahedron is a pure mathematical creation. Here we have arrived at Theaetetus and the figure reproduced on the cover of this issue. Theaetetus was already famous for his mathematical genius when he was very young, as testified in Plato's dialogue *Theaetetus,* and from the same source we know about his early death in 369 B.C. We have no reason not to believe the ancient commentator to the *Elements* and may even go a bit further: It is very likely that Theaetetus investigated the regular polyhedra in the years about 380 to 370 B.C. The foundations of solid geometry in *Elements*, Book XI, may then be a slightly later answer to Plato's demands for a systematic treatment.

In Plato's dialogue *Timaios* we have another source about the polyhedra. He indicates in *Timaios* 53c1 that these things are relatively new and quotes (?) in 55a a definition of a rather modern type, namely that of a solid which *divides the circumsphere into congruent parts*. This or the definition given at the very end of Book XIII of the *Elements*, together with the complete list or classification of all possible examples, is the first prototype of a complete mathematical theory, certainly a result of *extraordinary attractiveness and charm.*

After the Greeks, nobody prior to the Renaissance painters appears to have been able to draw a mathematical figure as complicated as the one in the manuscripts of the *Elements*. Hence it is reasonable to believe that what we see on the cover of this issue is nothing but a copy of Theaetetus' own first drawing. And there is more to this: It is the first realization of an entity that existed before only in abstract thought. (Well, apart from the statues of the gods!)

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Mathematisches Institut, Bunsenstr. 3-5, D-37073 Goettingen, Germany
E-mail address: `artmann@uni-math.gwdg.de`