ABOUT THE COVER: EULER'S *INTRODUCTIO IN ANALYSIN INFINITORUM*

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The cover of this issue shows the frontispiece of Leonhard Euler's celebrated *Introductio in analysin infinitorum*, Chapter 16 of which is the source of the title and subject of Professor Andrews' article "Euler's 'De Partitio Numerorum'" [3, p. 81]. It is one of the great books in the mathematical literature, the only book of Euler's represented in that compendium of great printed books in the history of civilization, *Printing and the Mind of Man*, issued by the British Museum in 1963. Published in 1748, the *Introductio* is instructive and a pleasure to read even today. The eminent historian of mathematics, Carl Boyer, in his address to the International Congress of Mathematicians in 1950 [1, p. 748], called it the greatest modern textbook in mathematics. Boyer cited Euclid's *Geometry* as the greatest mathematical textbook of the classical period, perhaps of all time, appearing in over one thousand editions. For the medieval period, he chose the less well-known Al-Khwarizmi, largely devoted to algebra. But for "modern" times, Boyer made the case for Euler's *Introductio* as the greatest modern textbook—and, appropriately, this time a text in analysis. Gauss said, "The study of Euler's works will remain the best school for the various fields of mathematics, and nothing can replace it." In 1979 André Weil remarked that he was trying to convince "the mathematical community that our students of mathematics would profit much more from a study of Euler's *Introductio in analysis infinitorum*, rather than of the available modern textbooks" [5, p. xii].

During the 18th century there was competition. Other textbooks in analysis (or calculus) included L'Hôpital's *Analyse des infiniment petits* (1696), Gaetana Maria Agnesi's *Instituzioni analitiche* (1748), and Sylvestre Lacroix's *Traité de calcul différentiel et du calcul intégral* (1797–1798), which appeared in many editions. Euler's book, by contrast to some of these, was readable, highly original and full of ideas that stimulated research.

Euler wrote roughly 800 papers and 20 books and pamphlets, but of the latter only 8 were on topics that would be viewed today as strictly mathematical. Some of his great books—*Mechanica* and *Dioptricae*, for example—would now be considered to be physics, and others range from music theory to artillery, shipbuilding, and other fields, even religion. These often have some mathematical content but more properly lie in other disciplines.

The contents of the *Introductio* are for the most part topics that are considered now to be pre-calculus. It was only later that Euler published two calculus textbooks—*Institutiones calculi differentialis* (1751) (written in 1748, the year his *Introductio* was published) and *Institutionum calculi integralis* (1763–1770), in

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three volumes. Earlier he had written his famous *Methodus inveniendi lineas curvas* (1744) on the calculus of variations, but that is not so much a textbook as a monograph.

The topics in the *Introductio* include: the definition of a function and the distinction between single-valued and many-valued “functions”, or odd and even functions. There are questions of polynomial factorization related to the Fundamental Theorem of Algebra; substitutions, infinite series, exponentials and logarithms, and expansions into partial fractions; multiple angle formulas; evaluations of $\zeta(2k)$, for k a positive integer (the famous so-called Basel problem); infinite series-infinite product formulas; and partitions of integers. Much of the latter part of volume 1 is devoted to generating functions and continued fractions. Volume 2 is largely concerned with classification of curves and the study of the conics, and higher degree curves and surfaces. D. J. Struik referred to it as the first textbook in analytic geometry [8, p. 168].

What makes this remarkable as a textbook is that the later topics are closely related to Euler’s research, and in some cases their appearance here, and sometimes in earlier papers, opened up whole new fields of research. We cannot say that of most “elementary” modern textbooks. Whether an idea appeared first in the *Introductio* is not very important. To be sure, Leibniz and Johann Bernoulli had come close to our modern definition of function, but it was here that Euler spelled it out clearly. The question of the number of partitions of a positive integer was not original with Euler. In a letter dated August 29, 1740, Philipp Naudé the younger (1684–1745) had asked in how many ways a given positive integer can be written as a sum of different positive integers, a problem closely related to the general partition problem [2].

This is one of the first mathematics books that “looks modern”. Except for “ xx ” to denote “ x^2 ” and a few other conventions carried over from earlier times, it introduced modern notation: $f(x)$ for function, e for the base of the natural logarithm (appearing in print here for the first time but used earlier by Euler in a manuscript not published until 1862 in his *Opera posthuma mathematica et physica*), and π (though it had been used by Christian Goldbach two years earlier). Euler proved that e is irrational in the *Introductio*, though the fact could be deduced from work in a paper of 1737 [6, p. 246]. The proof in the *Introductio* using continued fractions was not, however, to everyone’s satisfaction. In 1748 Euler still used $\sqrt{-1}$ and did not introduce “ i ” for this number until 1777.

Other problems covered in the *Introductio* had occurred earlier in Euler’s reports to the St. Petersburg Academy and in published papers. The genius exhibited in the *Introductio* was in the way he formulated the problems and the methods he introduced. In the problem of partitions he developed powerful methods using generating functions. Again, using infinite series and products he came up with the relationships that inspired Riemann’s work on the ζ -function. It is often assumed that the formula usually called “Euler’s formula”, relating the exponential and the trigonometric functions, appeared first in the *Introductio*, but the formula was probably observed earlier by Roger Cotes [7, p. 4]. As was often the case, however, Euler did more with it. He was a master expositor, so when he took a subject he molded it into something that people could understand, thereby prompting subsequent development.

Beyond the beautiful mathematics contained in the book, however, there is the beauty of the original edition of this set of volumes as physical objects. Most of Euler's books were published by the press of the Academy in St. Petersburg. Those are rather sober, straightforward products of the printer's art. When Euler wrote the *Introductio*—probably around 1745—he was working in Berlin. Unfortunately the Berlin Academy published journals but not books, and the St. Petersburg Academy was having financial problems. So Euler had to look for a publisher elsewhere. In 1744 his *Methodus inveniendi lineas curvas* had been published by the firm of Marc-Michel Bousquet, located in Lausanne and Geneva. So he approached them about publishing the *Introductio*. This firm had also published in 1742 a beautifully decorated set of four volumes, the *Opera Omnia* of Jean Bernoulli. Handsome though these earlier volumes were, fortunately for us, the publisher seems to have given this set of two volumes an even more generous budget, so the book is decorated with an elegant engraving for a frontispiece, a title page in two colors, a fold-out, engraved dedication portrait of the physicist-astronomer-geologist Jean Jacques Dortous de Mairan, who was a patron of the publisher, secretary of the Académie Royale des Sciences and member of various other scientific societies. There are plenty of elaborately decorated pages with further engravings and florid capitals at the beginning of chapters and sections.

The engraved frontispiece is the work of Pierre Soubeyran, based on the design of a member of the de La Monce family of portrait painters (see Figure 1). It shows two classically attired women under an arch flanked by columns. Hovering overhead is a putto. One woman is gesticulating while the other is writing mathematical formulas in a book and the putto is offering inspiration. There are mathematical instruments on the floor and a book open to pages with “CALCUL DIFFERENTIEL” and “TABLE DES SINUS”. The inscription at the top of the frontispiece gives the title of the *Introductio* in French, *Analyse des infiniments petits*, which, curiously, happens to be the name of L'Hôpital's famous text of 1696. The title of the first French translation of the *Introductio* in 1796, on the other hand, was *Introduction de l'analyse infinitésimale*. There were German and Russian translations, but it wasn't until 1988 and 1990 that we saw English translations, by John D. Blanton, of the two volumes of the *Introductio* [5]. A Spanish translation appeared in 2000.

With the gradual disappearance of Latin from the schools, it is good to have this great classic once again available to a wide audience, even though the translation lacks the elaborate frontispiece. As L.-G. du Pasquier [4, p. 113] described it, the *Introductio* is “as marvelous in its clarity of exposition as for the richness of its contents.”

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FIGURE 1.