BOOK REVIEWS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 49, Number 3, July 2012, Pages 455–463
S 0273-0979(2012)01375-7
Article electronically published on April 18, 2012

Riemann surfaces, by Simon Donaldson, Oxford Graduate Texts in Mathematics, Vol. 22, Oxford University Press, Oxford, 2011, xiv+286 pp., ISBN 978-0-19-960674-0

Is there a need for another book on Riemann surfaces? "God, if She exists, created the natural numbers $\{0, 1, 2, \dots\}$, and compact Riemann surfaces, topologically

$$\{\textit{the sphere},\ \textit{the torus},\ \textit{the two holed torus}, \dots\}.$$

The rest of mathematics is man made."[1] Complex function theory, the content of many standard first year graduate courses, is calculus done right. Riemann surface theory is a natural second year follow up because it occupies a special, central place in mathematics. It can be regarded as the intersection of complex analysis, algebraic geometry, analytic number theory, and mathematical physics with applications to many fields including topology and differential geometry. The author views the subject in part as a prototype for more advanced results in global analysis. This alone provides a partial answer to the question. When the book is written by an outstanding mathematician who is expected to and does bring a new and novel point of view to the subject, it generates high expectations and will surely attract the attention of many. But Riemann surface theory is more than just a prototype. In its own right it is an elegant and beautiful field that continues to be studied; in part because it supplies beautiful examples that can be calculated and illustrated (see the appendix). It is my pleasure to thank the colleagues that reviewed and commented on earlier drafts of this review, and, in particular, to R. Rodríguez for help with the figures.

Riemann surfaces are globally 2–real dimensional orientable connected manifolds and locally pieces of the complex plane $\mathbb{C}$. They provide the right setting for the study of analyticity in one complex variable, equivalently for solutions of the Cauchy-Riemann equations,[2] $f_{\bar{z}} = 0$. The transition function $w \circ z^{-1}$ between two local coordinates z and w on a Riemann surface S with nonempty intersecting domains is analytic. There is a natural dividing point: closed (compact), the subject of the book under review as well as the remainder of this review, and open surfaces. The topology of a compact surface S is captured by a single integer $g \geq 0$, its *genus* (informally, the number of handles; formally, $\frac{1}{2}\text{rank } \mathrm{H}_1(S, \mathbb{Z})$).

2010 *Mathematics Subject Classification.* Primary 30F10, 30F30, 30F60.

[1] A version of this quote is attributed by many to Lipman Bers.

[2] We use complex coordinates z and $\bar{z}$ throughout.

Let q be an integer. The basic objects of study are *meromorphic q-forms* on Riemann surfaces: assignments of meromorphic functions f^z to local coordinates z such that $f^w(w(z))\left(\frac{dw}{dz}\right)^q = f^z(z)$, these are cross sections of the qth power of the canonical line bundle. The case $q = 0$ corresponds to meromorphic functions, while $q = 1$ to meromorphic differential 1-forms or *abelian differentials*. By the maximum modulus principle, there are no nonconstant holomorphic functions on a compact S and attention focuses on its field of meromorphic functions $K(S)$, spaces of q-forms, as well as holomorphic maps between surfaces.

Real parts of analytic functions are harmonic (satisfy Laplace's equation $u_{z\bar{z}} = 0$). A study of Riemann surfaces must at some point involve the construction of appropriate functions and/or differentials, usually labeled as a Main Theorem. Harmonic differentials with singularities can be constructed using Hilbert space methods or Perron families of subharmonic functions (real-valued smooth ones satisfy $u_{z\bar{z}} \geq 0$, but more generality is needed). From these, one constructs abelian differentials. Ratios of such differentials are meromorphic functions. Donaldson's approach starts by establishing in the compact case that $u_{z\bar{z}} = \rho$ is solvable if and only if the integral of the differential 2-form ρ over the Riemann surface is zero and thus also exposes the reader to connections with elliptic differential operators.

The classical period (the nineteenth and early twentieth centuries) for this subject starts with Bernhard Riemann [27], and the foundational work of that period emphasized meromorphic functions and abelian differentials on surfaces. Relations with 2-dimensional topology/geometry and connections to Fuchsian groups are established and exploited. The important milestones (the list below is certainly incomplete) are attached to the names of the mathematicians who laid the groundwork in the field (the topics marked with an asterisk (*) are discussed in the book):

- *The **Riemann Existence Theorem.** Every compact Riemann surface S carries a nonconstant meromorphic function, realizing S as a finite sheeted branched cover of the sphere. The theorem describes necessary and sufficient conditions for such covers to exist.
- *The **Riemann-Roch Theorem.** The points of the surface S are generators for the partially ordered abelian group of *divisors* on the surface. The zeros and poles of a meromorphic q-form on S define a *q-canonical divisor*, commonly called *principal* for $q = 0$. It has *degree* $q(2g - 2)$. The Riemann-Roch Theorem is an equality relating the difference between the dimension of the space the meromorphic functions defined by a divisor D and the dimension of the corresponding space of abelian differentials to the degree of the divisor and the genus of the surface. It—which is perhaps the most important result in the field—is particularly useful when one of the dimensions is zero.
- The **Weierstrass Gap Theorem** is an elegant description of the possible orders of the poles at a single point $x \in S$ of elements of $K(S)$ that are holomorphic except at x, showing, in particular, that not all points on a compact surface are alike and selects a finite number of distinguished points. Translations to the space of holomorphic abelian differentials and generalizations to spaces of meromorphic q-forms have wide applications. The results emphasize the unique nature of Riemann surfaces among their many generalizations.

- **Periods and bilinear relations.** We choose a basis
$$\{a_1, \ldots,\ a_g,\ b_1, \ldots,\ b_g\}$$
for $H_1(S, \mathbb{Z})$ whose intersection matrix is $\left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$, where I is the $g \times g$ identity matrix and a *normalized* basis $\omega_1, \ldots, \omega_g$ for the space of holomorphic abelian differentials whose a-periods are $I = [\int_{a_i} \omega_j]$. The resulting b-periods $\Pi = [\int_{b_i} \omega_j]$ yield a symmetric $g \times g$ matrix with positive definite imaginary part, *the period matrix.* These are the key (there are others) bilinear relations; all are obtained by expressing area integrals of the form $\int\int_S \omega \wedge \Omega$ and $\int\int_S \omega \wedge \bar{\Omega}$, where ω and Ω are abelian forms, as line integrals over curves in the homology basis. The period matrices are elements of the $\frac{1}{2}g(g+1)$-dimensional *Siegel upper half-space* $\mathcal{S}_g$ consisting of symmetric $g \times g$ matrices with positive definite imaginary parts (a generalization of the familiar upper half-plane, the case $g = 1$, in $\mathbb{C}$).
- ***Abel-Jacobi map and Jacobi inversion.** We let L be the lattice generated by the columns of the $g \times 2g$ matrix $[I, \Pi]$. The *Jacobi variety* is the g-dimensional torus $\mathbb{C}^g/L$. Fix a point $P_0 \in S$; the *Abel-Jacobi* map $\varphi : S \to \mathbb{C}^g/L$ is obtained by integrating the differentials in a normalized basis from P_0 to the variable point $P \in S$. The map φ is of maximal rank and an embedding for $g \geq 1$, a biholomorphic map for $g = 1$, and extends by linearity to the divisor group of the surface. Abel's theorem asserts that a divisor D is principal if and only if it is of degree 0 and $\varphi(D) = 0$. The Jacobi inversion theorem tells us that the map φ restricted to divisors of degree g is surjective, thus the group $\mathbb{C}^g/L$ can be described as the group of divisors of degree 0 modulo its subgroup of principal divisors.
- **Riemann Vanishing Theorem.** For $g \geq 1$, we study the *Riemann theta* function
$$\theta : \mathbb{C}^g \times \mathcal{S}_g \to \mathbb{C}.$$
For fixed values of $\Pi \in \mathcal{S}_g$, corresponding to a period matrix of the Riemann surface S, we can use ratios of translates of this function and the exponential function to produce elements of $K(S)$ via the Abel-Jacobi map φ. Although, $P \mapsto \theta(\varphi(P), \Pi)$ is not well defined on S, its zero set is; in fact, the zero set is a translate of the image under φ of the integral (positive) divisors of degree $g - 1$. The Riemann Vanishing Theorem relates dimensions of spaces associated to these divisors to the vanishing of the theta function on subspaces of the torus associated to the period matrix, connecting Riemann surface theory to analyticity in several complex variables and to algebraic geometry.
- ***Uniformization**. Up to biholomorphic equivalence, there are only three simply connected surfaces: the Riemann sphere $\mathbb{C} \cup \{\infty\}$, the plane $\mathbb{C}$, and the unit disc $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$. Although more or less a product of the nineteenth century, definitive proofs awaited the first two decades of the next century. Riemann-Roch makes this result unnecessary for the study of surfaces of genera 0 and 1.
- ***Algebraic curves**. The field of meromorphic functions $K(S)$ is an algebraic extension of a transcendental extension of $\mathbb{C}$. Hence there are functions z and $w \in K(S)$ that generate the function field and satisfy

an algebraic equation $\sum a_{ij} z^i w^j = 0$. These functions give a way of viewing the Riemann surface as a curve in projective space $\mathbb{CP}^2$ and show that the valuations of the field $K(S)$ are described by evaluating the orders of zeros and poles at the points of S. (The same theorem for an open surface is established using key results on the structure of the ring of holomorphic and the field of meromorphic functions on it, a theorem proven in 1966.)

- **Hurwitz's theorem** is a surprising, strong rigidity result. For $g \geq 2$, $|\mathrm{Aut}(S)| \leq 84(g-1)$. The result is not sharp for all g, and the automorphism group for a *generic* surface of genus $g > 2$ is trivial.

Many of the above topics are still areas of current activity and are essential for a transition to the modern period (twentieth century) that may be characterized as the study of moduli of surfaces using quadratic differentials and Teichmüller theory. Connections to 3-dimensional geometry and Kleinian groups play a key role beginning at about the midpoint of the century. A number of important developments of the last century (the list is, once again, incomplete because of the reviewer's taste and limited knowledge), most still active areas for research, are sketched below.[3]

- *__Moduli of Riemann surfaces__. B. Riemann observed that a compact nonsingular curve of genus $g \geq 2$ depends on $3g-3$ complex parameters. This leads to the construction and study of the *Riemann space* $\mathcal{R}$, a complex analytic orbifold of dimension $3g-3$ whose points are in one-to-one correspondence with the birational equivalence classes of such curves. Its compactification has additional interesting structure (strata). The space $\mathcal{R}$ has a manifold covering space, the *Teichmüller space*—the principal analytic tool to study moduli of Riemann surfaces.
- *__Teichmüller theory__. The second half of the twentieth century saw remarkable progress in Teichmüller theory, mostly as a result of the work of the complex analysts L.V. Ahlfors, L. Bers, the geometers W. Thurston and D. Sullivan, and their students. Whereas the Ahlfors-Bers school emphasized analytic aspects, Thurston (building on the work of Fenchel, Waldhausen, and Marden and using connections to 3-dimensional hyperbolic geometry) opened up the field in a way that has made the last 40 years very exciting. Thurston's approach to Riemann surfaces as boundaries of noncompact 3-manifolds has led to a wide ranging interesting mathematics. His work has helped develop the theory of moduli of surfaces and to clarify the relationships between geometric and analytic moduli for surfaces with contributions from many mathematicians, for example, Earle and Eells [9], Royden [28], Wolpert [36]. The distinctions between boundaries of different representations of Teichmüller space is the subject of much current research. The work of Sullivan connected Teichmüller theory to the study of dynamics and holomorphic motions. A transition is in process. During the last 20 years the emphasis has shifted away from analysis toward geometry and topology. It is unclear who first defined the Teichmüller space $\mathcal{T}$, a natural complex manifold covering of $\mathcal{R}$. Teichmüller spaces are already present in the work of Fricke [13] on families of Fuchsian groups and probably in some of the lectures of F. Klein. But Teichmüller was first to realize the importance of

[3] Again an asterisk (*) denotes a topic discussed in the book.

- **Quasiconformal maps** as a tool for the study of moduli of Riemann surfaces. Later, they found uses in complex dynamics. These maps are less rigid than the more obviously necessary (biholomorphic) conformal maps, but flexible enough to allow the use of important analytic tools. Teichmüller also introduced an important extremal problem [32] and a metric on Teichmüller space, essentially a distance between Riemann surfaces; both now bear his name. He did not settle the issue of the
- **Complex structure on $\mathcal{T}$**. Ahlfors [2] was the first to show that $\mathcal{T}$ is a complex manifold. His local coordinates, at most points, are periods of holomorphic abelian differentials; a point of view suggested by earlier work of H.E. Rauch [26] who computed the variation of the periods under quasiconformal deformations. The *Bers embedding* [5] maps Teichmüller space onto a bounded open set in a Banach space of holomorphic quadratic differentials using Schwarzian derivatives of univalent functions and thus obtaining global coordinates anchoring the modern theory of moduli to the classical theory of univalent functions.
- **Surface automorphisms**. Thurston's work [33] on boundaries of Teichmüller spaces and on diffeomorphisms of surfaces, classifying them into three types, revolutionized the field and led Bers [6] to reprove the classification by analytic methods and to formulate and solve an extremal problem that extends Teichmüller's problem: Fix a topological self-map f_o of a topological surface S_o. What is the the infimum of the dilatations of (how close to conformal are the) quasiconformal self-maps f of Riemann surfaces S, where f is homotopic to f_o and S is homeomorphic to S_o?
- **Kleinian groups**. *Fuchsian* groups are discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$. The study of their action on the upper half-plane in $\mathbb{C} \cong \mathbb{R}^2$ provides a deep connection between complex analysis and 2-dimensional topology and geometry. *Kleinian* groups are discrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$. Their action on the upper half-space $\mathbf{H}$ in $\mathbb{C} \times \mathbb{R}$ is the domain of 3-dimensional topology and geometry. Complex analysts require an additional hypotheses that theses groups be of the *second kind*—they act properly discontinuously on a dense open subset of the the boundary $(\mathbb{C} \cup \{\infty\}, 0)$ of $\mathbf{H}$. Ahlfors [3] revived the study of Kleinian groups in the twentieth century by describing the set of Riemann surfaces represented by a finitely generated Kleinian group of the second kind. Intrinsic, nonvariational coordinates for Teichmüller spaces are mostly based on the Klein-Maskit combination theorems [22]. Many different families of groups can be used to produce almost identical sets of intrinsic coordinates—much of the work on these coordinates is circulating in informal manuscripts.
- **Compactification of moduli space**. Deligne and Mumford [7] described an algebraic-geometric construction of compactified moduli space $\overline{\mathcal{R}}$. An analytic construction based on the augmented Teichmüller space, introduced in [1], factored by a naturally acting modular (also known as the mapping class) group, had to await the work of Hubbard and Koch [16] in the twenty-first century which proved that the complex structures of Deligne and Mumford agrees with the one resulting from the Ahlfors-Bers theory.

- **Iteration of rational maps**. Teichmüller spaces are used in the study of iterations of rational maps [30] and led to a dictionary by Sullivan between concepts in the fields of dynamics and Kleinian groups.
- **Holomorphic motions**. It is quite surprising that quasiconformal mappings appear naturally in the study of motions of points in the plane [31].
- The **Schottky-Jung problem** is to determine which elements of $\mathcal{S}_g$ are period matrices of Riemann surfaces. Although this natural problem arises quite early, most progress did not occur until the twentieth century. Classical approaches to a solution are found [11], followed by [35] and [8]. A more modern point of view can be seen in [29].

It is much too early to discuss the progress in this century. It is clear, however, that the field is alive and functioning. In addition to the recent work [16], we describe only one key recent result.

- The **Ehrenpreis conjecture.** Relying on Thurston's perspective, Kahn and Markovic [20] established a conjecture of Waldhausen: the fundamental group of an irreducible 3-manifold contains a subgroup isomorphic to the fundamental group of a closed hyperbolic Riemann surface. The 3-dimensional construction they exploited is adapted to 2-dimensional geometry and used to resolve a long standing conjecture of Leon Ehrenpries [19]: for any two compact surfaces X and Y, we can find finite sheeted unramified covers $\tilde{X}$ and $\tilde{Y}$ that are arbitrarily close in the Teichmüller metric.

The book under review originated during the author's graduate studies and was developed during several courses he taught as a faculty member in England and the United States. The content of the book, according to the author's aim, emphasizes connections to other areas of mathematics and presents much of the material from a novel point of view. The book is divided into four parts. The first consists of analytic and topological preliminaries addressed to a well-prepared reader. The second presents some of the basic material on surfaces. The third contains the main existence theorem and uniformization. The last part deals with algebraic and hyperbolic aspects (exposing the amazing centrality of the subject), moduli and ordinary differential equations. The exposition is lucid and the arguments in the proofs in the main part of the text are sufficiently detailed.

The choice of topics covered by the book has been dictated by the author's taste and wide-ranging expertise. It is quite reasonable for his emphasis to be different from mine. Even though no book can cover everything, I was particularly disappointed to find no mention of certain topics, for example, gaps and the finiteness of the automorphism group. On the other hand, it was a pleasure to find some topics not in most books, for example, the Riemann Existence Theorem and the Gauss-Manin connection.

There are many fine graduate texts on Riemann surfaces. Most are not listed in the book's reference section; particularly absent is [34], the first modern text on the subject. As an indication of the many books recently published and sources for topics not covered in this book, we list seven: [10], [12], [25], [23], [18], [4], [15]. For Teichmüller theory, [21], [14], [24], and [17] should be on most reference lists, in addition to the one entry listed in the author's bibliography.

Was there a need for another book on Riemann surfaces? I would say, yes, particularly one with as much good mathematics as in this one. The book's emphasis is on connections to topology, particularly through the mapping class group, and differential geometry, particularly Hodge theory and the $\bar{\partial}$-methods. It treats Riemann surface theory as a kernel of a more general field preparing the reader for higher dimensional, mostly geometric, generalization of the subject. This is the novelty of the author's approach to the subject and the resulting choice of topics. Were I teaching a year-long course on Riemann surfaces, I would probably choose another book as a basic reader for the students but would add this one as exposing a fascinating alternative. Most books on Riemann surfaces, this one included, are a mixture of topics of interest and importance primarily to the field itself and topics centered on the applicability to other important areas of current research. I would include more topics of the first group. Clearly, it is of interest to have a diversity of excellent books on this central topic in analysis.

Appendix

For those not familiar with Riemann surfaces, we provide two examples.

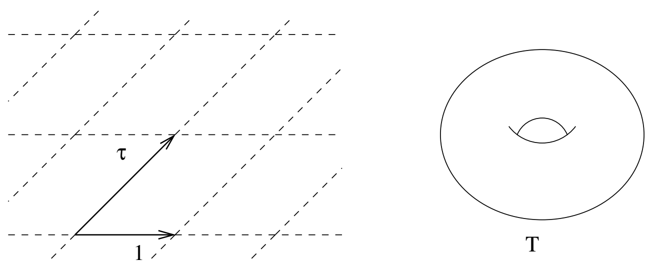

Figure 1. A torus.

All tori. The first figure (Figure 1) is a lattice in the complex plane $\mathbb{C}$ through the points $\mathbb{Z} \oplus \tau\mathbb{Z}$, with τ a complex number whose imaginary part is positive. For each τ we construct a Euclidean rank two free commutative group generated by the motions of the plane $z \mapsto z + 1$ and $z \mapsto z + \tau$. The torus is the result, identifying opposite sides of the rectangle with vertices $0,\ 1,\ 1 + \tau,\ \tau$. The corresponding algebraic curves can readily be computed using the Weierstrass $\mathcal{P}$-function and sums of infinite series. For $\tau = \imath$, the curve is $w^2 = z^4 - 1$. There are many ways to reach this conclusion because this case is well known. An elegant way, without use of series, is to observe that the torus with *period* $\tau = \imath$ and the last quartic curve each admit the action of a cyclic group of order four.

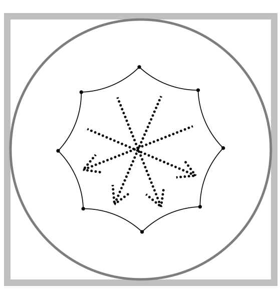

Figure 2. An octagon.

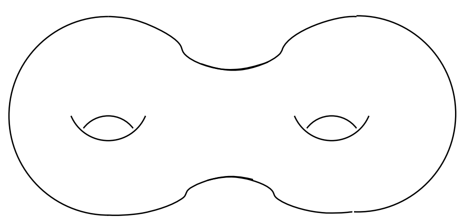

Figure 3. A genus 2 surface.

A surface of genus 2. The picture for genus ≥ 2 is more complicated. We discuss one special case where we can take advantage of symmetries. In Figure 2, we exhibit a non-Euclidean regular geodesic octagon. The side identifications produce a surface S of genus two pictured in Figure 3. All surfaces of genus two are so constructed if we drop the condition of regularity—the sides need not have the same lengths. For the regular octagon, we can compute explicitly the automorphisms of the unit disc that identify the sides and serve as generators for the Fuchsian group representing the surface S. From the construction we see that a cyclic group G of order 8 acts on the surface S and that the resulting quotient S/G is a sphere with three branch points; there is only one such sphere. The algebraic curve $w^2 = z^6 - 1$ also has many automorphisms allowing us to compare it to the surface S.

References

1. W. Abikoff, *Augmented Teichmüller spaces*, Bull. Amer. Math. Soc. **82** (1976), 333–334. MR0432919 (55:5898)
2. L.V. Ahlfors, *The complex analytic stucture of the space of closed Riemann surfaces*, Analytic Functions, Princeton University Press, 1960, pp. 45–66. MR0124486 (23:A1798)
3. ———, *Finitely generated Kleinian groups*, Amer. J. Math. **86** (1964), 413–429 and **87** (1965), 759. MR0167618 (29:4890)
4. A. Beardon, *A Primer on Riemann Surfaces*, London Mathematical Society Lecture Notes, 78, Cambridge University Press, Cambridge, 2011. MR0808581 (87h:30090)
5. L. Bers, *A nonstandard integral equation with applications to quasiconformal mappings*, Acta Math. **116** (1966), 113–134. MR0192046 (33:273)
6. ———, *An extremal problem for quasiconformal mappings and a theorem by Thurston*, Acta Math. **141** (1978), 73–98. MR0477161 (57:16704)
7. P. Deligne and D. Mumford, *The irreducibility of the space of curves of a given genus*, Publ. Math. I.H.E.S. **36** (1969), 75–110. MR0262240 (41:6850)
8. R. Donagi, *Big Schottky*, Invent. Math. **89** (1987), 569–599. MR903385 (88j:14052)
9. C.J. Earle and Jr. J. Eells, *On the differential geometry of Teichmüller spaces*, J. Analyse Math. **19** (1967), 35–52. MR0220923 (36:3975)
10. H.M. Farkas and I. Kra, *Riemann Surfaces (second edition)*, Graduate Texts in Mathematics, vol. 71, Springer-Verlag, 1992. MR1139765 (93a:30047)
11. H.M. Farkas and H.E. Rauch, *Period relations of Schottky type on Riemann surfaces*, Ann. Math. (1970), 434–461. MR0283193 (44:426)
12. O. Forster, *Lectures on Riemann Surfaces*, Graduate Texts in Mathematics, vol. 81, Springer-Verlag, 1981. MR648106 (83d:30046)
13. R. Fricke and F. Klein, *Vorlesungen über die Theorie der elliptischen Modulfunktionen*, B.G. Teubner, 1890-2.
14. F.P. Gardiner, *Teichmüller Theory and Quadratic Differentials*, Wiley-Interscience, 1987. MR903027 (88m:32044)
15. E. Girondo and G. Gonzáles-Diez, *Introduction to Compact Riemann Surfaces and Dessins d'Enfants*, vol. 79, London Mathematical Society Student Texts, 2012.
16. J.H. Hubbard and S.C. Koch, *An analytic construction of the Deligne-Mumford compactification of the moduli space of curves*, to appear.
17. Y. Imayoshi and M. Taniguchi, *An Introduction to Teichmüller Theory*, Springer-Verlag, 1992. MR1215481 (94b:32031)
18. J. Jost, *Compact Riemann Surfaces*, Springer-Verlag, 1992.
19. J. Kahn and V. Markovic, *The good pants homology and a proof of the Ehrenpreis conjecture*, Preprint arXiv:1101.1330 (2011).
20. ———, *Immersing almost geodesic surfaces in a closed hyperbolic three manofold*, to appear in Ann. of Math. arXiv:0910.5501.
21. O. Lehto, *Univalent Functions and Teichmüller Spaces*, Graduate Texts in Mathematics, vol. 109, Springer-Verlag, 1987. MR867407 (88f:30073)
22. B. Maskit, *Kleinian Groups*, Grundlehren der Mathematischen Wissenschaften, vol. 287, Springer-Verlag, 1988. MR959135 (90a:30132)

23. R. Miranda, *Algebraic Curves and Riemann Surfaces*, Graduate Studies in Mathematics, vol. 5, American Mathematical Society, 1995. MR1326604 (96f:14029)
24. S. Nag, *The Complex Analytic Theory of Teichmüller Space*, Wiley-Interscience, 1988. MR927291 (89f:32040)
25. R. Narasimhan, *Compact Riemann Surfaces*, Lectures in Mathematics, ETH Zurich, Birkhäuser Verlag, 1992. MR1172116 (93j:30049)
26. H.E. Rauch, *A transcendental view of the space of algebraic Riemann surfaces*, Bull. Amer. Math. Soc. **71** (1965), 1–39. MR0213543 (35:4403)
27. B. Riemann, *Collected Papers*, Kendrick Press, 2004 (translated by R. Basker and C. Christenson and H. Orde). MR2121437 (2005m:01028)
28. H.L. Royden, *Automorphisms and isometries of Teichmüller space*, Advances in the Theory of Riemann Surfaces, Ann. of Math. Stud. 66, Princeton University Press, 1971, pp. 369–383. MR0288254 (44:5452)
29. T. Shiota, *Characterization of Jacobian varieties in terms of soliton equations*, Invent. Math. **83** (1986), 333–382. MR818357 (87j:14047)
30. D. Sullivan, *Quasiconformal homeomorphisms and dynamics I, Solution of the Fatou-Julia problem on wandering domains*, Ann. of Math. **122** (1985), 401–418. MR819553 (87i:58103)
31. D. Sullivan and W.P. Thurston, *Extending holomorphic motions*, Acta Math. **157** (1986), 243–257. MR857674 (88i:30033)
32. O. Teichmüller, *Extremale quasikonforme Abbildungen und quadratische differentiale*, Abh. Preuss. Akad. Wiss. **22** (1940), 1–197. MR0003242 (2:187a)
33. W.P. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. **19** (1988), 417–431. MR956596 (89k:57023)
34. H. Weyl (translated by G.R. MacLane), *The Concept of a Riemann Surface*, Addison-Wesley, 1955. MR0166351 (29:3628)
35. B. van Geemen, *Siegel modular forms vanishing on the moduli space of curves*, Invent. Math. **78** (1984), 329–349. MR767196 (86d:14040)
36. S. Wolpert, *Thurston's metric for Teichmüller spaces*, J. Differential Geom. **23** (1986), 143–174. MR845703 (88c:32035)

Irwin Kra
State University of New York at Stony Brook
E-mail address: irwinkra@gmail.com