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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
DE LELLIS AND SZÉKELYHIDI

MR0864505 (90a:58201) 58G99; 35A99, 35B99, 53C42, 58-02

Gromov, Mikhael

Partial differential relations. (English)

Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 9.

Springer-Verlag, *Berlin*, 1986. x+363 *pp*. $60.00. *ISBN* 3-540-12177-3

Around 1970, the world of differential geometry was astounded by the news that a young Russian by the name of Mikhael Gromov had proved that any noncompact differential manifold admits a Riemannian metric of positive sectional curvature, and also one of negative sectional curvature. We were also told that this was achieved by a "soft" method of topological sheaves. Moreover, in one and the same setting, Gromov also proved generalizations of both the Hirsch-Smale immersion theorem and the A. Phillips submersion theorems. Many more results were promised. Slowly, Gromov's papers (some in collaboration with Ya. M. Èliashberg and V. A. Rokhlin) filtered to the West in the early seventies. Here are a sample of those particularly relevant to the present review: the author [in *Actes du Congrès International des Mathématiciens, Tome II* (Nice, 1970), 221–225, Gauthier-Villars, Paris, 1971;MR0420697 (54 #8709); Izv. Akad. Nauk SSSR Ser. Mat. **33** (1969), 707–734;MR0263103 (41 #7708)], the author and Èliashberg [Math. USSR-Izv. **5** (1971), 615–639;MR0301748 (46 #903)], the author [ibid. **7** (1973), no. 2, 329–343;MR0413206 (54 #1323)] and the author and Rokhlin [Russian Math. Surveys **25** (1970), no. 5, 1–57;MR0290390 (44 #7571)]. After a lapse of some fifteen years, the author has now presented what would appear to be his valedictory statement on the subject. Within the covers of the volume under review, he has deepened, generalized and synthesized the materials from the diverse earlier publications to arrive at a coherent account starting from first principles. The appearance of this book is a major event in geometry during the past decade.

The aim and scope of the book are succinctly set forth in the foreword: "The classical theory of partial differential equations is rooted in physics, where equations (are assumed to) describe the laws of nature. Law-abiding functions, which satisfy such an equation, are very rare in the space of all admissible functions. . . . Moreover, some additional conditions often insure the uniqueness of solutions. . . . We deal in this book with a completely different class of partial differential equations (and more general relations) which arise in differential geometry rather than in physics. Our equations are, for the most part, under-determined (or, at least, behave like those) and their solutions are rather dense in spaces of functions. We solve and classify solutions of these equations by means of direct (and not so direct) geometric constructions. Our exposition is elementary and the proofs of the basic results are self-contained." The partial differential relations alluded to above are usually either equations or inequalities. A typical example of the former is the system of partial differential equations arising from the isometric imbedding problem for Riemannian manifolds. Let M be an n-dimensional Riemannian manifold with metric $g = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j$. Let $f\colon M \to \mathbf{R}^q$ be a C^∞ map into a high-dimensional Euclidean space, and let $f = (f^1, \cdots, f^q)$. We want f to be injective and that its components $\{f^a\}$ satisfy: $(*)$ $\sum_{a=1}^{q}(\partial f^a/\partial x^i)(\partial f^a/\partial x^j) = g_{ij}$ for all $i, j = 1, \cdots, n$. This system $(*)$ expresses of course the fact that the induced metric on $f(M)$ equals g. Here we have $\frac{1}{2}n(n+1)$ equations in the q unknowns $\{f^1, \cdots, f^q\}$. Since q will be taken to be large, $(*)$ is grossly under-determined. A typical example of the kind of partial differential inequalities treated in this book is the following: Given differential manifolds V, W with dimensions n and q, respectively ($n \leq q$), we ask if there is an immersion $f\colon V \to W$. In terms of local coordinates, df can be represented as an $n \times q$ matrix. Let $\{D_i\}$ be the $\binom{q}{n}$ submatrices of dimension $n \times n$ in df. Then the property of f's being an immersion is expressed by the following inequality to be satisfied at each point: $(\#)$ $\sum_i (\det D_i)^2 > 0$.

Technically, the formulation of these problems takes a different form. In Part 1 of the book, which comprises four sections, one finds a general discussion of this formalism together with a survey of the basic problems and results. Thus let $p\colon X \to V$ be a smooth fibration and let $X^{(r)}$ be the space of germs of r-jets of smooth sections $V \to X$. Thus each $X^{(r)}$ is a bundle over X whose fibre at each $x \in X$ consists of all linear maps ψ from $T^r_{p(x)}V$ (the tangent space at $p(x)$ of V of order r) to $T^r_x X$, such that for all $1 \leq s \leq r$, $\psi(T^s_{p(x)}V) \subset T^s_x X$. By taking the rth order jet of a smooth section $f\colon V \to X$, we get a smooth section $J^r f\colon V \to X^{(r)}$. The set of all such sections $\{J^r f\}$ as f varies over all sections $f\colon V \to X$ is called the set of holonomic sections. Clearly $J^r f$ is locally nothing but the string of all partial derivatives of f up to order r. A differential relation on the sections of $p\colon X \to V$ is just a subset $\mathcal{R} \subset X^{(r)}$, and a solution of $\mathcal{R}$ is by definition a section $f\colon V \to X$ such that $J^r f(V) \subset \mathcal{R}$. Thus we may identify the solutions of a differential relation $\mathcal{R}$ with the holonomic sections $V \to X^{(r)}$ which map into $\mathcal{R}$. Usually it is easy to construct a continuous section $f\colon V \to \mathcal{R}$, or else one such is given. Then the obvious way to obtain a solution of $\mathcal{R}$ is to deform by homotopy the section f into a holonomic one (abbreviation: homotop f into a holonomic section), if this is possible. We say $\mathcal{R}$ satisfies the homotopy principle (h-principle for short) if every continuous section $V \to \mathcal{R}$ can be homotoped into a holonomic section. The main goal of this book is to show, often surprisingly, that in a wide variety of situations, the h-principle holds for the partial differential relation at hand. For the sake of simplicity, we take up three examples to give a flavor of this work. (I) Let $p\colon X \to V$ be a holomorphic fibre bundle with a Stein manifold V as base, with a complex Lie group G as structure group, and with G/H as fibre where H is a complex Lie subgroup of G. Define the relation $\mathcal{R} \subset X^{(1)}$ to consist of complex linear maps from the tangent space $T_{p(x)}V$ to $T_x X$ for each $x \in X$. The Cauchy-Riemann equations imply that every holonomic section of $\mathcal{R}$ must be holomorphic. The celebrated Grauert-Oka principle asserts that the h-principle holds for this $\mathcal{R}$ [H. Grauert, Math. Ann. **133** (1957), 450–472;MR0098198 (20 #4660)]. (II) Let V, W be differential manifolds of dimensions n and q, respectively, $n \leq q$. Let $X = V \times W$ and let $p\colon X \to V$ be the obvious projection. Now a section of $X^{(1)} \to V$ is a map $v \to ((v, w), \psi)$, where $v \in V$, $w \in W$ and ψ is a linear map $T_v V \to T_w W$. A holonomic section is then a map $v \to ((v, f(v)), df\colon T_v V \to T_{f(v)} W)$, where f is a map from V to W, and hence the holonomic section may be simply identified with the pair (f, df). Now define the immersion relation $\mathcal{R}$ on this $X^{(1)}$ to consist of only those $((v, w), \psi)$ where ψ is injective. It follows that the holonomic sections of

$\mathbf{R}$ in this case consist of immersions $V \to W$. The immersion theory of Hirsch and Smale [M. W. Hirsch, Trans. Amer. Math. Soc. **93** (1959), 242–276;MR0119214 (22 #9980)] guarantees that the h-principle is valid if either $n < q$ or if V is open. (III) Let V be a Riemannian manifold of dimension n, and let $X = V \times \mathbf{R}^q$, where $q \geq \frac{1}{2}(n+2)(n+3)$. As usual, $p\colon X \to V$ is given by the obvious projection. Now define $\mathcal{R} \subset X^{(2)}$ to be the set of all $((v,y),\psi)$ where $v \in V$, $y \in \mathbf{R}^q$ and ψ is a nonsingular linear map $T_v^2V \to T_y^2\mathbf{R}^q$ such that the restriction of ψ to T_vV is an isometric linear map into $T_y\mathbf{R}^q$. With the same reasoning as in (II), we see immediately that a holonomic section of this $\mathcal{R}$ is just an isometric immersion of V into $\mathbf{R}^q$ whose second order differential is everywhere nonsingular. Such immersions are called free isometric immersions, and they originated with the classical work of J. F. Nash, Jr. [Ann. of Math. (2) **63** (1956), 20–63;MR0075639 (17,782b)]. Now a theorem of the author states that this $\mathcal{R}$ also obeys the h-principle.

Of course the book considers a wide range of topics (submersions, C^∞ foliations, isometric imbeddings of Riemannian manifolds, contact structures, symplectic structures, etc.), but through these three examples one can already perceive many of the central features of this work. First, it does not give a proof of (I), i.e., the Grauert-Oka principle. This is perhaps due to the fact that this principle is a theorem in the theory of over-determined systems. But by the same token, the author does infuse every topic that gets discussed in the book with new results or a new viewpoint. Second, while the whole book is concerned with only one problem (when does the h-principle hold?), the consideration of the h-principle provides only the philosophical backbone to the book. Technically, each topic is treated from a perspective (sometimes up to three different perspectives, as in the case of the Hirsch-Smale theorem in (II)) uniquely its own. For example, (III) requires careful attention to the hard analysis inherent in the Nash imbedding theorem, while (II) is of course free from such considerations. Third, while some of the results treated in this book are partly or wholly known ((III) and (II) respectively, for example), the new ideas and improvements the author brings to these well-known topics are considerable. Thus he not only gives the Hirsch-Smale theorem three different proofs (recall Atiyah's dictum: if you only have one proof for a theorem then you cannot say you understand it very well), but the deeper understanding so achieved immediately allows him not only to treat submersion, k-mersions, etc. in the same setting, but also to draw new conclusions as well; for example, the theorem of the author and Èliashberg that holomorphic immersions of Stein manifolds into complex Euclidean space of strictly higher dimension also obey the h-principle. As another example, the result in (III) above on free isometric immersions leads immediately to the smallest known dimension of the receiving Euclidean space for isometric immersions of Riemannian manifolds. Last but not least, the reader will be happily surprised at every turn that, running through the many seemingly unrelated topics that show up in this book, there is the common thread of the h-principle (and sometimes more). The ability to draw together disparate topics under one roof is in fact a prominent feature of the author's work.

The heart of the book is Part 2, also comprising four sections. Its concern is the methods to prove the h-principle, to wit, removal of singularities, continuous sheaves, inversion of differential operators, and convex integration. These are, appropriately enough, highly technical matters so that a short discussion of these would make no sense and a sufficiently detailed discussion would be impossibly long. Perhaps a few peripheral remarks would suffice. The author's aim here is of course to prove theorems that would guarantee the validity of the h-principle in the most general and in the maximum number of situations. These theorems therefore tend to be abstract and do not make easy reading. The reviewer feels that reading the first four of the papers cited at the beginning could be helpful. In these papers, the intended applications of each method are always stated clearly, and the method comes through in a more transparent fashion because the machinery is less sophisticated. True, the exposition in these papers is sometimes sketchy in the technical details, but for the purpose of acquiring a general idea of the arguments this could even be an advantage. For example, to see how the Hirsch-Smale theorem could be proved by the method of continuous sheaves or convex integration, reading the 1969 paper [op. cit.;MR0263103 (41 #7708)] or the 1973 paper [op. cit.;MR0413206 (54 #1323)] would seem to yield this information much more readily. Another relevant remark is that Part 3 of the book contains the most substantive applications of the method of inversion of differential operators (and therefore to a certain extent, of the method of continuous sheaves), so perhaps they should be read simultaneously.

Finally, a few words about Part 3 on isometric C^∞-immersions. This part occupies 140 out of the 360 pages of the book, and this fact makes this topic unique among the many applications of the general theory developed here. Note that the explicit mention of "C^∞" is significant: this is to distinguish it from the C^1-immersion theory of Nash-Kuiper. The latter is of a totally different character; for example, every n-dimensional Riemannian manifold can be C^1 isometrically immersed into $\mathbf{R}^{2n}$. In this book, the C^1 theory is taken up in the context of convex integration in Part 2 (this part of the author's work seems to be appearing in print for the first time). What show up in Part 3 are various refinements of, and additions to the pioneering work of Nash concerning isometric imbedding of Riemannian manifolds into Euclidean space: improvements on the receiving dimension (alluded to above), what happens in low dimensions, the role of the second fundamental form, the case of indefinite metrics, and the case of a symplectic form (replacing the Riemannian metric). For this part, the paper of Gromov-Rokhlin referred to at the end of the first paragraph can serve as a good introduction.

As we mentioned above, the author intended this book to be an elementary exposition. This should not be taken literally. One should rather approach this as a research monograph where new ideas turn up almost in every page. Many of these ideas will undoubtedly inspire further developments.

From MathSciNet, May 2012

Hung-Hsi Wu

MR1231007 (94h:35215) 35Q35; 28A80, 76C99

Scheffer, Vladimir

An inviscid flow with compact support in space-time.

The Journal of Geometric Analysis **3** (1993), *no.* 4, 343–401.

The purpose of this paper is to construct a nontrivial weak solution of the incompressible Euler equations in two space dimensions with compact support in space-time. This solution satisfies the usual definition of weak solution, obtained by multiplying the equations by divergence-free test functions and integrating by parts, though the test functions do not need to have compact support. The solution has bounded kinetic energy almost everywhere in time.

The author introduces the notion of froth, a 4-tuple consisting of a bounded, open, nonempty set in space-time X, a countable set of indices P, a collection of disjoint open sets contained in X, indexed by P, such that the Lebesgue measure of the complement of their union relative to X is zero, and a collection of ordered pairs of complex numbers, also indexed by P. These numbers are such that the piecewise constant function defined on X, taking values in the set of ordered pairs of complex numbers $\mathbf{C}^2$, satisfies certain conditions. The author shows that if the collection of pairs of complex numbers in the definition of the froth all lie on a specific parabola E in $\mathbf{C}^2$ then they give rise to a weak solution of the 2D Euler equations. The proof amounts to the construction of a nontrivial E-froth. The author defines simple froths and develops a calculus of froths, to show how to construct a complicated froth from simple ones.

The resulting weak solution has the appearance of a fractal and is very irregular. The image of space-time is a countable set.

This paper shows nonuniqueness of weak solutions to the inviscid incompressible fluid flow equations if no further regularity is assumed other than bounded kinetic energy, a landmark result. The paper is very technical.

From MathSciNet, May 2012

Helena J. Nussenzveig Lopes

MR1476315 (98j:35149) 35Q30; 35D05, 76C99

Shnirelman, A.

On the nonuniqueness of weak solution of the Euler equation.

Communications on Pure and Applied Mathematics **50** (1997), *no.* 12, 1261–1286.

The aim of this paper is the analysis of weak solutions of the Euler equations for nonviscous incompressible flows. The author studies flows on the 2-dimensional torus and constructs a weak solution of the Euler equations that belongs to L^2 and has compact support in time. This construction is the main result of the paper. The existence of such a weak solution which is identically zero outside a finite time interval means, in particular, that the weak solution with zero initial velocity is not unique. The corresponding result for flows on the plane was obtained by V. Scheffer [J. Geom. Anal. **3** (1993), no. 4, 343–401;MR1231007 (94h:35215)], but the author in the paper under review presents a simpler construction of a nonzero weak solution so that the nature of these solutions becomes clearer. The main tool is the construction of a sequence of solutions of nonhomogeneous Euler equations with right-hand sides that become more and more oscillatory and a proof that the limit is a weak solution of the homogeneous equations.

From MathSciNet, May 2012

Alexander Yurjevich Chebotarev

MR1777341 (2002g:76009) 76B03; 35D05, 35Q30, 76F99

Shnirelman, A.

Weak solutions with decreasing energy of incompressible Euler equations.

Communications in Mathematical Physics **210** (2000), *no.* 3, 541–603.

In this paper A. Shnirelman proves the existence of a weak solution of 3D incompressible Euler equations which has a decreasing kinetic energy. More precisely there exists a weak solution $u(t,x)$ of the equation $\partial_t u + (u \cdot \nabla)u + \nabla p = 0$, $\nabla \cdot u = 0$ such that $\int |u|^2$ is decreasing with time. This is a major result and a great achievement in fluid mechanics; it provides a new insight into the complex structure of weak solutions.

Of course if u is sufficiently smooth (say C^1, sufficiently rapidly decreasing at infinity), the kinetic energy is conserved (just multiply the equation by u and integrate by parts). This is physically natural: if the flow is smooth, since the Euler equations model an inviscid fluid, there is no dissipative mechanism, and it is physically expected that the kinetic energy is preserved.

If u is only L^2, the former integration by parts cannot be justified. The author conjectures that this is linked with the behavior of turbulent flows for vanishing viscosities. Even if the viscosity is very small (or the Reynolds number very high), physically the kinetic energy decreases. It is therefore interesting to investigate the existence of weak solutions of Euler equations with a decreasing energy.

The construction of the weak solution is very intricate. The starting point is the notion of generalized flows, introduced by Y. Brenier, and that of sticky particles. The construction relies on a "fractal type" combination of multiphase flows, sticky behavior and deep understanding of mass exchange between phases of a multiphase flow.

This paper introduces many new ideas in the field and deserves careful study.

From MathSciNet, May 2012

Emmanuel Grenier

MR2600877 (2011e:35287) 35Q31; 34A60, 35D30, 76B03

De Lellis, Camillo; Székelyhidi, László, Jr.

The Euler equations as a differential inclusion.

Annals of Mathematics. Second Series **170** (20009), *no.* 3, 1417–1436.

The authors consider the Euler equations in $\mathbb{R}^n$:

$$\partial_t v + \operatorname{div}(v \otimes v) + \nabla p - f = 0, \quad \operatorname{div} v = 0. \tag{1}$$

They are interested in weak solutions of (1) and reformulate this system as a differential inclusion. In the introduction they survey many works; among them they cite results of V. Scheffer [J. Geom. Anal. **3** (1993), no. 4, 343–401;MR1231007 (94h:35215)] and of A. I. Shnirelman [Comm. Pure Appl. Math. **50** (1997), no. 12, 1261–1286;MR1476315 (98j:35149); Comm. Math. Phys. **210** (2000), no. 3, 541–603;MR1777341 (2002g:76009)]. These results follow from the main result of the present paper. In particular, the authors show that if $f = 0$, there exist a $v \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t; \mathbb{R}^n)$ and a $p \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t, \mathbb{R})$, solving system (1) in the sense of distributions, such that v is not identically zero, and v and p are compactly supported in spacetime $\mathbb{R}^n_x \times \mathbb{R}_t$.

The second section is devoted to giving a plane-wave analysis of the Euler equations in the spirit of L. C. Tartar [in *Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV*, 136–212, Res. Notes in Math., 39 Pitman, Boston, MA, 1979;MR0584398 (81m:35014)]. After a brief presentation of the work of Tartar, the authors give the adaptation to the Euler equations and the new variables to consider. As an exact plane-wave solution won't be compactly supported unless it is zero, they introduce plane-wave-like solutions of the system, called localized plane waves. In Section 3 they provide a proof of existence of such solutions. One of the ideas is to introduce oscillations in order to control the error.

The main theorem of the paper is stated and proved in Section 4. Another proof of this theorem using convex integration [see S. Müller and M. A. Sychev, J. Funct. Anal. **181** (2001), no. 2, 447–475;MR1821703 (2002c:35281)] is given in Section 5.

From MathSciNet, May 2012

Frédéric Charve

MR2564474 (2011d:35386) 35Q31; 35A02, 35L65, 76N15

De Lellis, Camillo; Székelyhidi, László, Jr.

On admissibility criteria for weak solutions of the Euler equations.

Archive for Rational Mechanics and Analysis **195** (2010), *no.* 1, 225–260.

This paper concerns the uniqueness problem for weak solutions to incompressible and compressible Euler equations.

In a previous paper [Ann. of Math. (2) **170** (2009), no. 3, 1417–1436; MR2011e:35287] the authors introduced a framework for constructing weak solutions to Euler equations by means of convex integration and analysis of oscillations in conservation laws. By means of the Baire category principle, they proved the existence of time-space compactly supported weak solutions.

A remaining question is whether some suitable dissipative estimates on the solution would force the uniqueness. The answer to this question is contained in the paper under review. The natural quantity to consider is the energy.

The authors consider two types of inequalities:

(1) the weak energy inequality is satisfied if the energy at any positive time is less than the energy at $t = 0$;
(2) the strong energy inequality is the requirement that the energy decreases as a function of $t > 0$.

The main result of the paper is that by suitable adaptation of the method introduced in [op. cit.], it is possible to construct infinitely many weak solutions such that

(1) both energy inequalities are satisfied;
(2) the strong is satisfied but not the weak;
(3) the weak is satisfied but not the strong.

The paper is particularly well written and clear, despite the difficulty of the subject and the importance of the results obtained.

From MathSciNet, May 2012

Stefano Bianchini