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Quasilinear hyperbolic systems, compressible flows, and waves, by Vishnu D. Sharma, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 142. CRC Press, Boca Raton, FL, 2010, xiv + 268 pp., ISBN 978-1-4398-3690-3

Hyperbolic partial differential equations (PDEs) arise in physical models of wave phenomena where information is transmitted at finite speeds. Paired with nonlinearities, the finite speed of propagation often leads to blowup phenomena and correspondingly low regularity of solutions. The other two general families of elliptic and parabolic PDEs model equilibrium and diffusive phenomena, and their solutions are as a rule more regular.

Nonlinear hyperbolic theory has witnessed major progress in recent years, both with respect to new methods and resolved problems, as well as in regard to its range of applications. At the same time it is embarrassingly simple to pose natural, fundamental questions that we do not know how to answer. This is particularly true for problems in several space dimensions, but even for one-dimensional (1-d) systems there are basic unresolved issues. The following two problems can illustrate the latter:

- Establish existence of global-in-time solutions to the Cauchy problem for the 1-d Euler system (which encodes conservation of mass, momentum and energy in compressible gas flow);
- For the same system but with spherically symmetric initial data in several space dimensions (radial distance is the only spatial coordinate): can/must a converging shock lead to blow up of the solution at the center of motion? How does the blow up take place?

These questions have been addressed in many works, starting from World War II when a better understanding of high speed fluid flow became imperative. The influential book by Courant and Friedrichs [4] describes the status of the field as of 1945, and it continues to be a source of important challenges. For fluid flow models, as well as for more general hyperbolic systems, many partial results are available. However, we are currently very far from a complete theory providing existence, uniqueness, and qualitative behavior of solutions. This is a frustrating state of affairs, and, according to Lax [12], a scandal of mathematical physics. For example, the 1-d Euler system is presumably a very good model for flow regimes far beyond the near-equilibrium situations addressed by most known results. Here is another simple-to-state question that illustrates how limited our knowledge is:

- Does the full, 1-d Euler system admit time-periodic solutions?

Recent work by Temple and Young [23] suggests that the answer is "Yes". If so, it will provide existence of *nonlinear* sound waves which propagate according to different rules than those of linear acoustics.

As is clear from above, the study of hyperbolic equations is intimately connected with physics. Let us quickly outline more generally how hyperbolic systems may be obtained. Classic works, starting with Euler's derivation of the field equations

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for a perfect fluid, via Cauchy's works on elasticity, combined with a (painfully) gradual clarification of thermodynamic concepts, provided a rich class of physical models. The resulting framework of continuum physics formulates a common setting for a wide range of physical situations. The various theories are identified by their constitutive relations and their symmetry properties, reflecting the physical properties of the material under consideration.

For example, a thermoelastic medium is one in which the constitutive functions for the internal energy, stress tensor, temperature, and heat flux are supposed to depend only on the pointwise values of the deformation gradient, entropy, and temperature gradient. These dependencies are subject to general thermodynamic constraints and material frame indifference. It turns out that these constraints severely limit the possible functional relationships. Further reductions follow by imposing material symmetry properties that encode whether the medium is a fluid or a solid, say.

More elaborate models are needed to incorporate viscous effects or internal friction in a body (visco-elastic medium); e.g., the Navier-Stokes equations for a compressible fluid are obtained by taking into consideration also the temporal rate of change of the deformation gradient. Finally, upon substituting the resulting constitutive relations into the general balance laws (field equations) for mass, momentum, and energy, one obtains a concrete system of partial differential equations. With this, the thermodynamic analysis is completed and the system is handed over to the mathematicians, so to speak. See [1, 6] for more details and references.

In many cases it is reasonable to disregard internal dissipation altogether; e.g., viscosity and heat conduction are typically negligible in gases. This results in hyperbolic systems of PDEs with finite transmission speeds. The prime example of a hyperbolic system is the Euler system for compressible gas flow. In the absence of external forces and sources of heat, the conservation of mass, momentum, and total (internal + kinetic) energy yield the following system of field equations:

$$\rho_t + \operatorname{div}_x(\rho u) = 0, \tag{0.1}$$

$$(\rho u)_t + \operatorname{div}_x(\rho u \otimes u) + \operatorname{grad}_x p = 0, \tag{0.2}$$

$$(\rho E)_t + \operatorname{div}_x\big((\rho E + p)u\big) = 0. \tag{0.3}$$

Here subscripts denote partial derivatives, ρ is the mass density, $u \in \mathbb{R}^{d\times 1}$ denotes the velocity of a fluid parcel ($u \otimes u \equiv uu^{tr}$), p is pressure, and $E = e + \frac{u^2}{2}$ is the total energy per unit mass. The internal energy e should obey basic thermodynamic constraints, which provide the constitutive relation for the pressure; see [15].

All in all this is a (very) complicated system of nonlinear PDEs with an exceedingly rich solution structure. It is worthwhile to consider simplified models as well as more general systems. Although these may be of limited physical relevance, they are of interest both for the general mathematical theory and as a test ground for probing properties of physical systems. For the Euler system, a common class of reduced models is obtained by assuming that the pressure p depends only on the density, $p = p(\rho)$. In this case the continuity equation (0.1) and the momentum equation (0.2) already form a closed system for ρ and u.

Next consider general systems of the form

$$U_t + \operatorname{div}_x f(U) = 0, \tag{0.4}$$

where $U = U(t,x) \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^{n\times d}$, n is the number of conserved quantities, and d is the spatial dimension (div_x is applied row-wise). Very roughly, we can summarize the current state of affairs concerning the initial value problem for (0.4) as follows: Little is known for general multidimensional systems (n, $d > 1$), more is known for 1-d systems ($n > 1$, $d = 1$), especially for systems of two equations, while a fairly complete theory is in place for single equations in any space dimension ($n = 1$, $d \geq 1$).

Let us stress that by "solving" the system (0.1)–(0.3) or (0.4) we mean to establish the global-in-time existence of a unique solution to the initial value problem. (The study of initial-boundary value (or "mixed") problems is a whole field in its own right; see [2].) The class of data should be reasonably general, in particular it should contain discontinuous functions (see below). Of course, one is also interested in understanding the range of phenomena and qualitative features covered by the model. Lest the reader thinks this should not be too difficult, let us quote from a recent review by Christodoulou [5]: "*...we are still, after the lapse of two and a half centuries, far from having achieved an adequate understanding of the observed phenomena which are supposed to lie within the domain of validity of Euler's fluid equations.*"

So, how come? Well, nonlinear PDEs are always challenging to analyze, and nonlinear systems all the more so. Already *local-in-time* existence requires hard work [2, 14]. What makes *hyperbolic* equations particularly hard is the *loss of regularity* in the solutions. As became clear to the pioneers in the field (Poisson, Stokes, Riemann), smooth initial data for the Euler system may lead to infinite gradients as time increases. The physical interpretation is that *shocks* appear: narrow regions where the solution suffers large gradients. In hyperbolic models, which do not include viscous (i.e., smoothing) effects, this manifests itself by the appearance of abrupt jumps (discontinuities) in the solution. Now, discontinuous functions are not differentiable in the classical sense, and one must therefore extend the solution concept and consider weak (distributional) solutions. Unfortunately weak solutions are highly nonunique, and one also needs to devise rules to select a unique "good" solution. There is a whole subfield dedicated to selection (entropy) criteria, and we will not go into this here. They are covered in some detail in the book under review; see also [6].

To investigate gradient blowup, one may consider the equations that monitor how the derivatives in the solution develop in time. While it seems reasonable from these that solutions typically develop singularities in finite time (as they contain super-linear source terms), it is nontrivial to establish precise theorems to this effect. A large number of works deal with the formation/nonformation of singularities in nonlinear wave equations.

Now, if we want to establish *global-in-time* results for (0.4), we need to include discontinuous solutions. However, even if we assume that we have at our disposal a reasonable method for generating approximate solutions (a nontrivial task!), the low regularity of the solutions means that compactness of the approximations is a challenging issue. As indicated above, in some cases there is a good theory in place: scalar equations, small variation data for 1-d systems, and systems of two equations. In the first case, one has a single equation of the form

$$u_t + \partial_{x_1} f_1(u) + \cdots + \partial_{x_d} f_d(u) = 0 \qquad \text{with } u = u(t,x) \in \mathbb{R},$$

and the solutions turn out to obey a maximum principle. This yields very favorable stability properties and leads to an existence and uniqueness theory [11] similar to that of parabolic equations: the data can be exceedingly rough, L^∞, say. Currently, the most precise results are available via kinetic methods [17]. (This method has also been applied to some systems, most notably the isentropic Euler system [13].)

For 1-d systems Glimm's fundamental work [10] shows that BV (the space of functions of bounded variation) is an appropriate space for near-equilibrium solutions. The relevant compactness result in this case is Helly's theorem. The space BV is natural in the sense that it contains discontinuous functions, such that solutions with shocks are covered. One-dimensional systems of two equations have also been analyzed using the framework of compensated compactness [16, 22]. Starting with DiPerna [9], this method has been applied to the case of isentropic gas-dynamics and provides existence even for large, rough (L^p) data; see also [8]. For technical reasons the method of compensated compactness applies mostly to systems of two equations.

The quest for a proper functional setting for multidimensional systems and 1-d systems with large data are two fundamental challenges in the field. It has been clear for some time [3, 7, 18] that BV is definitely *not* a relevant solution-space for general multidimensional systems. For example, DeLellis [7] considers a specific multidimensional hyperbolic system (0.4) and prescribes a sequence of initial data $\bar{u}_n(x)$ that converge in BV to a constant. However, the corresponding solutions $u_n(t, x)$ all attain an infinite amount of variation immediately at time $t = 0+$. Given the variety of phenomena displayed by multidimensional systems, it appears unlikely that a general result à la Glimm's theorem [10] in one dimension would be available for multidimensional systems—at least not without severe restrictions on the initial data.

It should be stressed that the picture is not as bleak as it might seem from the comments above. Given that nonlinear systems are inherently complicated, one should acknowledge the substantial progress that has taken place since WWII. There are certainly huge (and exciting!) challenges, but much has also been achieved since hyperbolic PDEs entered mainstream mathematics. Besides results on pure existence-uniqueness theory, let us mention two important aspects of this success: the numerical treatment of hyperbolic problems, and the ever-growing range of applications, from fluids to models for the motion crowds of individuals. The hyperbolic community currently includes an enormous range of directions and interests, from the purest mathematics to the most applied and computational. For an indication of this breadth, one may consult the latest proceedings from the biannual HYP meetings [21].

Let us return briefly to the Euler system and its relatives, which serve as a *leitmotif* in the book under review. For someone outside the field, it might appear strange that so much attention is paid to one specific system. However, there are important applications that require a good understanding of hyperbolic phenomena in fluid flow; high speed flight is an obvious case in point. To further justify the number of manhours invested in studying "the" Euler system, we note that it really covers a whole family of models. The constitutive relation for the pressure needs to be prescribed, and this makes any general insight applicable to different physical situations [15]. The Euler systems also appear as limiting cases of more elaborate models such as the Navier-Stokes model. Other derivations of the Euler system may be based on particle or kinetic models. At least at a formal level the Euler system can be derived from Boltzmann's equation. Some of the most challenging problems in the field concern the rigorous justification of the Euler system as the limit of more detailed models.

Until a few decades ago there were only a handful of books dealing with hyperbolic theory and compressible flow in a comprehensive manner. Courant and Friedrichs [4], Roždestvenskiĭ and Janenko [19], and Smoller [20] are classics. This has changed and there is now a wide collection of books available for beginners and specialists alike. Many of these are geared towards purely theoretical issues and/or numerical analysis. For a comprehensive bibliography, see [6].

As noted above, there are definite limits to our knowledge (even for 1-d systems), and some of the "big" issues remain elusive. However, the field of hyperbolic equations is so multifaceted, and so important, that there is both need and room for different approaches. Some of the most exciting recent results deal with concrete issues and particular phenomena such as vacuum formation, Mach reflection, vortex sheets, existence of periodic and "wild" solutions etc., rather than general existence results. One hopes that the former can lead the way toward the latter.

Whitham's monograph [24] is a seminal and highly influential book with focus on concrete phenomena. This is an application-driven account which treats similarity solutions, simplified models, approximations, and asymptotic methods. Professor Sharma's book belongs to this category and is a welcome addition to the literature on hyperbolic systems. The reader will find a careful discussion of 1-d, scalar problems, standard material about 1-d systems, discussion of entropy criteria, and a presentation of various fluid models. The core material concerns multidimensional problems and treats weak waves, asymptotics, and self-similar solutions. The first part of the book (Chapters 1–5) is suitable as course material, while the two last chapters summarize recent results of the author on self-similar solutions and shock kinematics.

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