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POINCARÉ AND THE EARLY HISTORY OF 3-MANIFOLDS

JOHN STILLWELL

ABSTRACT. Recent developments in the theory of 3-manifolds, centered around the Poincaré conjecture, use methods that were not envisioned by Poincaré and his contemporaries. Nevertheless, the main themes of 3-manifold topology originated in Poincaré’s time. The purpose of this article is to reveal the origins of the subject by revisiting the world of the early topologists.

1. INTRODUCTION

A century has now passed since the death of Poincaré, and it took most of that century to solve his most famous problem—the Poincaré conjecture. Since 1904, when Poincaré posed the conjecture, the theory of 3-manifolds has become vastly more sophisticated. The proof of the conjecture, by Grigory Perelman in 2003 (following a program outlined by Richard Hamilton in 1982), uses methods from differential geometry and PDEs that were foreign to topology until the late 20th century. For an account of the recent history of 3-manifolds, leading up to Perelman’s proof, see McMullen (2011) [39]. With the advent of these new methods, we may have reached a point where topologists are unaware of, and can barely imagine, what topology was like in Poincaré’s time. In this article I hope to recreate the almost-lost world of Poincaré and his immediate successors. Hopefully, this will give some insights into the themes and problems in 3-manifold topology today.

Poincaré’s work on algebraic topology is collected in the volume Poincaré (2010) [53], which shows Poincaré’s style of what Stephen Smale calls “research by successive approximation”. At first, Poincaré was not sure of the best way to define manifolds, or homology, or the fundamental group. He tried various approaches, often leaving it open whether a particular approach is completely general, and whether the objects he believes to be topologically invariant really are so. This left many opportunities for his successors to clarify his definitions, point out gaps, and

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sometimes to provide counterexamples. Indeed, the process of revision began before Poincaré was finished when Heegaard (1898) [34] discovered an error in Poincaré's homology theory, which led Poincaré to the discovery of torsion.

Poincaré's successors were naturally interested in the Poincaré conjecture, but they did not come close to proving it. Their attempts to understand the related concept of homology spheres, on the other hand, were remarkably fruitful. They led to the idea of *surgery* and to an early appreciation of the mysterious complexity of fundamental groups, which stimulated the development of what was later called combinatorial group theory and geometric group theory. As I hope to show, the idea that combinatorial problems could be algorithmically *unsolvable* grew out of difficulties in combinatorial group theory.

Much of the material in this article can be found, in scattered form, in my book Stillwell (1993) [64]. However, in writing this more cohesive account, I have taken the opportunity to include the results of more recent scholarship, particularly that of Eppele (1999b) [28], Gordon (1999) [31], and Volkert (2002) [68].

2. POINCARÉ AND THE FUNDAMENTAL GROUP

Before Poincaré, the only part of topology that was well understood was the theory of compact 2-manifolds ("closed surfaces" or "surfaces with boundary" as they were then known). It was known, in particular, that orientable closed surfaces were topologically classified by a single number, the *genus* p , or equivalently the *Euler characteristic* $2 - 2p$, or the *connectivity* $2p$. The concept of connectivity, originally due to Riemann (1851) [56], was generalized to higher dimensions by Betti (1871) [11], in what became known as the *Betti numbers*, later to become part of Poincaré's homology theory, as we will see.

As a natural outcome of this development, the initial aim of topology was to find topologically invariant *numbers*, hopefully enough of them to completely classify manifolds of all dimensions. In the case of 3-manifolds, this goal was articulated as follows by Dyck (1884) [26]:

The object is to determine certain characteristical numbers for closed threedimensional spaces, analogous to those introduced by Riemann in the theory of his surfaces, so that their identity shows the possibility of ... 'one-to-one geometrical correspondence'.

Poincaré himself was motivated by the search for invariant numbers, but in 1892 he made a discovery—the fundamental group—that was to lead to the algebraic topology of today, in which one searches for invariant algebraic structures rather than numbers. The early history of 3-manifolds is the story of the gradually dawning realization that the fundamental group is a new kind of invariant—one that cannot reasonably be encoded by a set of numbers.

Poincaré (1892) [47] introduces the fundamental group in terms of closed paths in a manifold, much as we do today, but in describing examples he assumes that the 3-manifold is defined by a polyhedral region with faces identified by certain geometric transformations. Clearly, he is building on his experience with Fuchsian groups in the early 1880s (see Poincaré (1985) [52]), where each group is associated with a fundamental polygon (usually in the hyperbolic plane), with edges identified by certain motions. These groups include the fundamental groups of all orientable surfaces of genus ≥ 1 . Indeed, the fundamental group of a surface of genus 2 is one of Poincaré's examples of a Fuchsian group given in Poincaré (1882) [46] (see

Poincaré (1985) [52, p. 81]). From now on we will denote the fundamental group by the modern symbol π_1 .

The 3-manifolds introduced by Poincaré (1892) [47] are what we would now call torus bundles over the circle. He defines them by taking the unit cube in \mathbb{R}^3 as fundamental region, and identifying opposite faces by the transformations

$$\begin{aligned}(x, y, z) &\mapsto (x + 1, y, z), \\ (x, y, z) &\mapsto (x, y + 1, z), \\ (x, y, z) &\mapsto (\alpha x + \beta y, \gamma x + \delta y, z + 1),\end{aligned}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ and $\alpha\delta - \beta\gamma = 1$. The cross-section of this manifold for fixed z is therefore a square with opposite edges identified by translation; that is, a torus. The bottom torus ($z = 0$) is identified with the top torus ($z = 1$) by an essentially arbitrary continuous bijection of the torus. The infinitely many quadruples $\langle \alpha, \beta, \gamma, \delta \rangle$ give infinitely many nonhomeomorphic manifolds, because they have infinitely many different groups π_1 . Poincaré is able to show this, thanks to his knowledge of the group of transformations $z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}$ (the *modular group*, one of the classical Fuchsian groups).

On the other hand, the Betti numbers of these torus bundles are each either 1, 2, or 3, so Poincaré's 3-manifolds necessarily include two with the same Betti numbers but different π_1 . Thus in 1892 Poincaré established that π_1 is a more discriminating invariant than the Betti numbers, giving group theory a foothold in topology that it has retained ever since.

Nevertheless, in his first long paper on topology Poincaré (1895) [48], Poincaré continued to explore the Betti numbers. He set up machinery for computing them by assuming that each manifold has a decomposition into cells homeomorphic to simplices, reading off linear equations he called homologies, and computing the Betti numbers by linear algebra. By considering the dual of the cell decomposition he discovered *Poincaré duality*, according to which the Betti numbers equidistant from the top and bottom dimension are equal. In particular, for a 3-manifold the 2-dimensional Betti number equals the 1-dimensional Betti number. In a footnote, Poincaré remarked that the 1-dimensional Betti number may be extracted from π_1 by “allowing its generators to commute”, so for 3-manifolds all Betti numbers are implicit in π_1 .

At the same time, Poincaré (1895) [48] continued to explore π_1 . He reintroduced his family of torus bundles, now giving a detailed proof that they have infinitely many different π_1 , but he also gave new examples that establish more simply that π_1 is a stronger invariant than the Betti numbers. In particular, by identifying opposite faces of an octahedron, he found a manifold with the same (trivial) Betti numbers as the 3-sphere \mathbb{S}^3 , but a different π_1 ; namely, the cyclic group of order 2. This manifold is in fact the real projective space \mathbb{RP}^3 , though Poincaré does not seem to have noticed.

\mathbb{RP}^3 is also a crucial example in the thesis of Heegaard (1898) [34], where it was used to pinpoint an error in Poincaré's account of the Betti numbers; namely, failure to account for the effects of *torsion*. The difference between \mathbb{S}^3 and \mathbb{RP}^3 can indeed be detected by homology, when torsion is included: \mathbb{S}^3 has no torsion, whereas \mathbb{RP}^3 has torsion number 2.

This led Poincaré to rework his homology theory in two *Compléments* to the 1895 paper, published in 1899 and 1900. The extended theory produced torsion

numbers as well as Betti numbers, and indeed Poincaré introduced the word “torsion”, since he saw it as a characteristic of manifolds that are somehow “twisted” upon themselves, like the Möbius band. His method for computing them, described in Poincaré (1900) [49], is to describe the cell structure of a manifold by an *incidence matrix*, from which the Betti and torsion numbers may be extracted by the elementary divisor theory of Smith (1861) [61]. He also attempted to put this theory on a sound foundation by proving that any smooth manifold admits a simplicial decomposition (a “triangulation”). His attempt was far from acceptable by modern standards.

It may seem strange to the modern mathematician that Poincaré never recognized the existence of homology *groups*, which is how we package Betti and torsion numbers today (following Noether (1925) [43], whose title is precisely “Derivation of elementary divisor theory from group theory”). But in the 19th century, when the goal of all mathematics was “arithmetization”, numbers were the most desirable kind of topological invariant. As late as 1934, the famous textbook of Seifert and Threlfall applauded Poincaré’s approach to homology in these terms:

By introducing the incidence matrices . . . Poincaré took the decisive step towards the arithmetization of topology.

(See the English translation, Seifert and Threlfall (1980) [58, p. 330].)

By complementing the Betti numbers with the torsion numbers, Poincaré made his homology theory strong enough to distinguish \mathbb{RP}^3 from \mathbb{S}^3 , and in Poincaré (1900) [49] this emboldened him to state his first “Poincaré conjecture”: *any 3-manifold with trivial homology is homeomorphic to \mathbb{S}^3* . This conjecture lay undisturbed for the next few years, while Poincaré published two more *Compléments* on the applications of topology to algebraic geometry. The main interest in these two papers, from the 3-manifold viewpoint, is the reappearance of Poincaré’s torus bundles in Poincaré (1902) [50], where they are shown to arise naturally in the study of families of algebraic curves. Volkert points out in Volkert (2002) [68] that this may be where Poincaré encountered these manifolds in the first place.

Finally, Poincaré (1904) [51] returned to π_1 with a new study of the difference between homology and homotopy, first in the case of curves on surfaces. Among other things, he found a remarkable algorithm for deciding whether a curve on a surface of genus ≥ 2 is homotopic to a simple curve. Using hyperbolic geometry, he formalized the idea of “pulling a curve tight” on the surface (finding the *geodesic representative* of its free homotopy class) and thereby showed that a curve is homotopic to a simple curve if and only if its geodesic representative is simple. This seems to be the first significant application of *geometrization* to topology; an idea that was taken further by Dehn and revived with great success by Thurston.

The results on simple curves, however, were only a warmup for the main event in Poincaré (1904) [51]: the construction of what we now call a *homology sphere*—a 3-manifold with the same homology as \mathbb{S}^3 , but with different π_1 . (Hence the homology sphere is not homeomorphic to \mathbb{S}^3 .) Poincaré’s construction is quite unmotivated and mysterious. The homology sphere is the result of pasting together two handlebodies of genus 2—that is, “filled” surfaces of genus 2—according to a scheme indicated in Figure 1. (Part of the mystery is how the homology sphere, which is an exceptionally symmetric object, arises from this utterly asymmetric diagram.) By some miracle, π_1 of the resulting manifold turns out to have generators

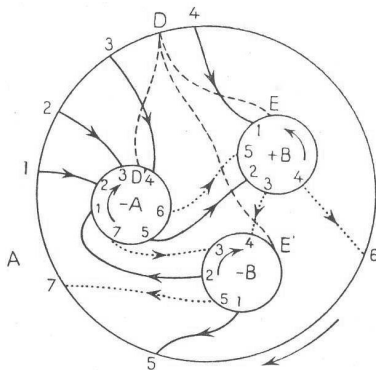


FIGURE 1. Poincaré's diagram of his homology sphere

a, b and defining relations

$$a^4ba^{-1}b = b^{-2}a^{-1}ba^{-1} = 1.$$

On the one hand, this group is nontrivial because setting $(a^{-1}b)^2 = 1$ maps it onto the 60-element *icosahedral group*

$$a^5 = b^3 = (a^{-1}b)^2 = 1.$$

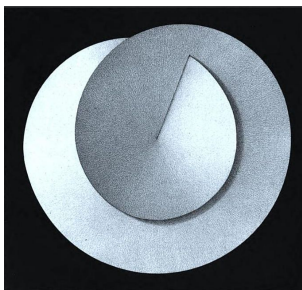
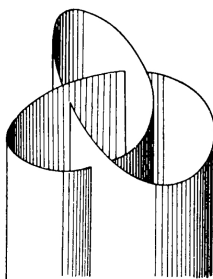
On the other hand, the group collapses to the single element 1 when generators are allowed to commute, which shows that the homology of the manifold is trivial. Thus the homology sphere shows, as strongly as possible, the superiority of π_1 over homology for distinguishing 3-manifolds.

With the construction of the homology sphere, Poincaré's earlier conjecture was demolished and the real Poincaré conjecture was born: *any closed 3-manifold with trivial π_1 is homeomorphic to S^3* . Poincaré does not offer an opinion on the truth of this conjecture, saying only that "this question would carry us too far away". As we now know, the Poincaré conjecture left topologists with enough work to keep them busy for the next 100 years. In the short term, just trying to understand the Poincaré sphere was hard enough. That and the broader question of the extent to which π_1 characterizes 3-manifolds were the questions that occupied Poincaré's immediate successors. We now look at these successors—the first generation of algebraic topologists.

3. HEEGAARD

Quite apart from his amendment to Poincaré's homology theory, Poul Heegaard (1871–1948) made other important contributions to the study of 3-manifolds. He found two new and interesting ways to construct them: as *branched coverings* of S^3 and by *Heegaard diagrams*.

Heegaard's branched coverings, or "Riemann spaces" as he called them, are the 3-dimensional counterpart of Riemann surfaces. Just as a Riemann surface consists of "sheets" covering S^2 which are fused together at branch *points*, a "Riemann space" consists of "sheets" (copies of S^3) which are fused together along branch *curves*. The fusion of sheets of a Riemann surface is comparatively easy to visualize, as Figure 2 shows.

FIGURE 2. Branch point of a covering of \mathbb{S}^2 FIGURE 3. Branch curve of a covering of \mathbb{S}^3

In Figure 2, which is from Neumann (1865) [42] by Neumann, the branch point lies at one end of a half line called a “branch cut”, which may be taken arbitrarily as the place where one sheet (the “upper” sheet) joins the other (the “lower” sheet). Figure 3 shows the analogous setup for a branched covering of \mathbb{S}^3 .

The branch curve is a trefoil knot, which lies at one end of a conical surface, which is where one passes from one sheet to the next. (The cone looks cylindrical in Figure 3, but let the parallel sides meet at infinity.) It is no longer possible to see separate sheets; one must simply imagine the surface as a portal to “another world” that is another copy of \mathbb{S}^3 . If the covering has finitely many sheets, then one will return to the initial copy of \mathbb{S}^3 after looping finitely many times around the branch curve.

In some cases, the branched covering is homeomorphic to the original manifold. For branched coverings of \mathbb{S}^2 , this happens when there are only two branch points. For branched coverings of \mathbb{S}^3 , it happens when the branch curve is a circle (or, more generally, any unknotted curve). Such coverings actually arise in potential theory, and were investigated by Appell (1887) [9] and Sommerfeld (1897) [62]. The simplest nontrivial covering of \mathbb{S}^3 , discovered by Heegaard (1898) [34], is the 2-sheeted covering branched over the trefoil knot. He showed that it is not homeomorphic to \mathbb{S}^3 because it has torsion number 3.

In doing so, he showed that knots have an important role to play in the construction of 3-manifolds. Conversely, he showed (perhaps unwittingly) that 3-manifolds are a tool for the investigation of knots. By finding a branched covering over the trefoil knot that is not homeomorphic to \mathbb{S}^3 , he proved that the trefoil knot is not

the same as the circle; that is, it really is knotted! At the time, it was not yet appreciated how hard it is to recognize knottedness, or to distinguish one knot from another, so Heegaard's proof went unnoticed. But 20 years later, as we will see, the construction of branched coverings turned out to be the first effective method for distinguishing a large number of knots.

Heegaard's other notable idea was the decomposition of 3-manifolds into handlebodies B_1 and B_2 of equal genus, n . Given a system of n canonical curves on B_1 , a 3-manifold M is determined up to homeomorphism by the images of these curves on B_2 —the so-called *Heegaard diagram of M* . The first important application of a Heegaard diagram was the construction of a *homology sphere* by Poincaré (1904) [51]. The handlebodies in this case are of genus 2. We will shortly see what kinds of manifold result from Heegaard diagrams of genus 1.

4. WIRTINGER

The name of Wilhelm Wirtinger (1865–1945) is known to all topologists from the so-called *Wirtinger presentation* for π_1 of a knot complement. He was a major influence on the development of knot theory, yet he published almost nothing on the subject, perhaps because his specialty was analysis. Word of his results trickled out, sometimes decades late, through the work of his students and colleagues at the University of Vienna. Wirtinger's role in the development of knot theory has recently become clearer, thanks to the work of Eppele (1999a) [27] and (1999b) [28].

Like Poincaré, Wirtinger found his way to topology from complex analysis—in his case by trying to generalize the theory of algebraic functions from one variable to two. This led him to the problem of describing singularities of algebraic curves, and he eventually found that knots form part of the description. His investigation began around 1896, was stimulated by Heegaard's idea of branched coverings, and produced the Wirtinger presentation around 1905. However, his derivation of the Wirtinger presentation surfaced only in Artin (1926) [10] (which is where Figure

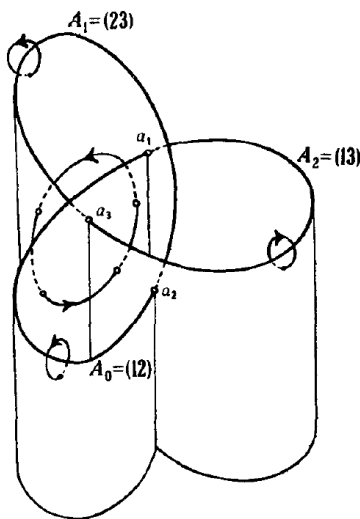


FIGURE 4. Brauner's trefoil knot diagram

3 comes from) and his ideas about singularities were fully expounded only in the work of his student Brauner (1928) [13].

Figure 4 is from Brauner (1928) [13], showing Wirtinger's setup for deriving relations for π_1 of the trefoil knot complement. In addition to the Heegaard-style picture of the branch curve, and the semicylinder through which one passes from one sheet to another, there are loops that generate π_1 of the trefoil knot complement and labels (12), (23), (13) describing how the sheets of a particular 3-sheeted covering are to be permuted. These three transpositions give a permutation representation of the trefoil knot group, showing that it is not abelian, which is another proof that the trefoil is really knotted.

The trefoil is the simplest knot, so it is not surprising that it was involved in the first theorems of knot theory. However, it is also the simplest example in Wirtinger's program of describing the singularities of algebraic curves: the curve $y^2 = x^3$, which has a *cusp* singularity at the origin. We know what the cusp looks like when x and y are real, but when x and y are complex, the curve is really a surface 4-dimensional space and we cannot visualize how it looks near the origin. The best we can do is intersect the curve with a decreasing series of 3-spheres centered on the origin, and see what kind of curve the intersection is. It turns out that each intersection is none other than a trefoil knot!

5. TIETZE

Wirtinger's work in topology seems not to have been influenced by Poincaré, except in his adoption of the fundamental group. Less surprisingly, Poincaré was not influenced by Wirtinger, since he could hardly have known about Wirtinger's work. The ideas of Poincaré and Wirtinger came together for the first time in the work of Heinrich Tietze (1880–1964). In the long paper Tietze (1908) [65], based on his *Habilitationsschrift* at the University of Vienna, Tietze took the fundamental group from Poincaré, handling it with great assurance and foresight, and from Wirtinger he took the knot concept, using it to throw new light on the basic problems of topology, for 3-manifolds in particular. His paper not only extended the work of Poincaré in many ways, but also exposed certain weak points and gaps.

In particular, Tietze used a *wild knot* to challenge the general approach of Poincaré, which was to assume that manifolds can be described by finite schema. For example, Poincaré assumed that curves can be replaced, for homological purposes, by polygons with finitely many sides. Tietze questioned whether this is the case for the curve in Figure 5, which he showed in both affine and projective form.

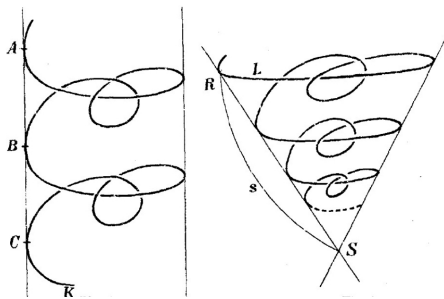


FIGURE 5. Tietze's wild knot

Even in the case of smooth compact manifolds, Tietze was aware that Poincaré's attempt to prove triangulation is unsatisfactory. Beyond this, he raised the problem of the *Hauptvermutung* (main conjecture): that any two triangulations of triangulable space have a common subdivision. The triangulation of smooth manifolds was eventually proved by Cairns (1934) [15], triangulation and *Hauptvermutung* for arbitrary 3-manifolds by Moise (1952) [41], and triangulation was shown to be *false* for 4-manifolds by Freedman (1982) [29].

Since Poincaré essentially used the *Hauptvermutung* to prove the topological invariance of Betti and torsion numbers, the invariance of these homological invariants was put into question by Tietze's critique. In fact, their invariance was proved by Alexander (1915) [2], by a different method, so the *Hauptvermutung* need not have been raised at this stage. Moreover, Tietze showed that, for 3-manifolds, the Betti and torsion numbers can be extracted from the fundamental group. So their invariance reduces to that of π_1 , which fortunately is not so hard to prove.

Thus Tietze was confident that the combinatorial approach to topology is sound, and this led him to a general study of groups given by generators and relations, and of π_1 in particular. Expanding on a footnote in Poincaré (1895) [48, §13], where Poincaré remarks that the first Betti number may be extracted from π_1 by allowing its generators to commute, Tietze defines what he calls the “Poincaré numbers of a discrete group” G . They are what we would call the rank and torsion numbers of the abelianization H of G , but Tietze finds them by the matrix computations—bypassing any consideration of H and its group structure. This was natural for Tietze and his contemporaries, whose goal was to compute invariant *numbers*, where possible. If manifolds have different numerical invariants we can tell immediately that they are not homeomorphic.

The invariance of the group π_1 was less helpful, because one computes only a *presentation* of π_1 by generators and relations. Different presentations can denote the same group, and it is *not* clear how to tell when this is the case. Although π_1 can contain more information than its “Poincaré numbers”, as Poincaré knew, there is no algorithm to extract all the extra information from a presentation. We cannot always extract enough information to characterize π_1 completely because, as Tietze presciently remarked in §14 of his paper,

While the equality of two series of numbers can always be decided, the question whether two groups are isomorphic ... is not solvable in general.

Tietze was right that this *isomorphism problem for groups* is unsolvable, though he wrote almost 30 years before Church (1936) [17] and Turing (1936) [66] gave a generally accepted definition of “solvability” for algorithmic problems, and almost 50 years before the isomorphism problem was proved unsolvable, by Adyan (1957) [1]. I believe that Tietze's remark was more than just a lucky guess, however, because Tietze really knew something about the isomorphism problem. He had solved it in one direction (the only direction in which it can be solved) by showing the following: if \mathcal{P}_1 and \mathcal{P}_2 are presentations of isomorphic groups, then \mathcal{P}_1 can be converted to \mathcal{P}_2 by a finite number of transformations. The requisite transformations are now called *Tietze transformations*, and they can be applied mechanically. So, if we are given presentations of isomorphic groups, this fact can be confirmed by a finite computation.

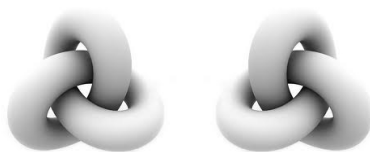


FIGURE 6. The two trefoil knots

Tietze’s unsolvability claim resurfaced in Reidemeister (1932) [55, p. 49], with the stronger claim that there is no algorithm for deciding, from a presentation, whether a group is free or trivial. Interestingly, Reidemeister organized the August 1930 conference in Königsberg at which Gödel announced his incompleteness theorem. The link between incompleteness and unsolvability was not yet clear—it became clearer in Church (1936) [17]—but it seems that a convergence of ideas in topology and foundations of mathematics was underway around 1930. Church was a student of the topologist Oswald Veblen, and the introduction to Church (1936) [17] gives the problem of finding

a complete set of effectively calculable invariants of closed three-dimensional simplicial manifolds under homeomorphisms

as an example whose solvability is open. We now know that the homeomorphism problem is solvable for 3-manifolds, but it is unsolvable for 4-manifolds. A solution for 3-manifolds depends on Perelman’s work (it was found by Sela (1995) [60], pending a proof of Thurston’s geometrization conjecture, which was proved by Perelman), while the unsolvability for 4-manifolds is due to Markov (1958) [38].

Tietze considered the homeomorphism problem for 3-manifolds, and realized that its difficulty is closely related to the difficulty of distinguishing between different knots. In §16 of his paper he gave the example of two submanifolds of S^3 : one obtained by removing two identical trefoil knots, and the other by removing right and left trefoil knots (Figure 6). Both manifolds have the same π_1 . The two manifolds are homeomorphic if the left trefoil knot can be deformed into the right, but Tietze said that this “is obviously out of the question”. It also seems obvious that the manifolds are different if the two trefoil knots are different, but neither of these “obvious” results were yet proved. Indeed, in the same section Tietze undermined his own confidence that the two trefoil knots are different by pointing out the lack of proof of two simple propositions about knot complements: that homeomorphic complements imply equivalent knots (this proposition remained open until confirmed by Gordon and Luecke (1989) [30]), and that a torus in S^3 necessarily bounds a knot complement (a proposition that follows from one already assumed by Dehn in 1907—see the next section).

In §18 of his paper, Tietze briefly discussed coverings of S^3 branched over a knot, reviewing the results of Heegaard and Wirtinger mentioned above. But he also foreshadowed a new construction of Heegaard’s double cover branched over the trefoil, as a *lens space*. The lens space construction is a return to Poincaré’s method of constructing 3-manifolds, by identifying faces of a polyhedron, and Tietze describes it in his §20.

The polyhedron is a lens-shaped solid with top and bottom divided into m equal sectors. These sectors are the faces of the polyhedron, and each top face is identified

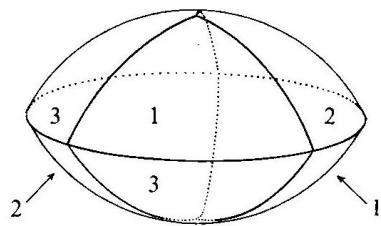


FIGURE 7. Face identifications for the (3,1) lens space

with the bottom face with a twist of $2n\pi/m$ —that is, the i th face on the top is identified with the $(i + n)$ th face on the bottom. The result is called the (m, n) lens space, and Figure 7 shows the faces to be identified in the case $m = 3, n = 1$.¹ Tietze asserted that the lens spaces are “the simplest possible type of two-sided closed three-dimensional manifold”. One way in which they are simplest is that they have *Heegaard genus* 1, that is, they can be obtained by pasting a pair of solid tori together. This can be seen by cutting the lens into two pieces: a cylindrical core surrounding the central axis of the lens, and the remainder. The core obviously becomes a solid torus when its top and bottom are identified with the required twist, and the same is true (less obviously) of the remainder. Figures 8 and 9 show these pieces in the (3,1) case.

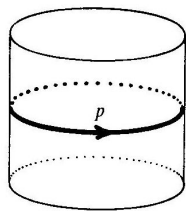


FIGURE 8. Core of the lens

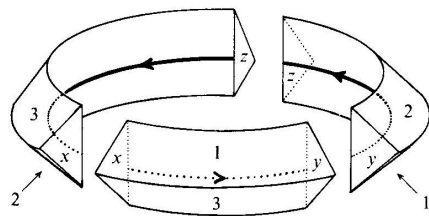


FIGURE 9. Remainder of the lens

¹One of Poincaré’s examples, the fifth example in §10 of Poincaré (1895) [48], is in fact the (2,1) lens space. Poincaré constructs it from an octahedron by identifying the four faces on the top with those on the bottom with a twist through angle π .

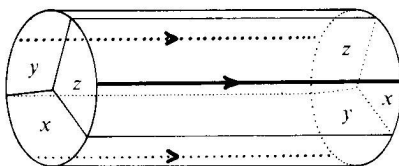


FIGURE 10. Assembling the remainder into a cylinder

Moreover, a curve around the “waist” of the core becomes identified with an (m, n) curve on the other solid torus, that is, the curve that results from m equally spaced lines on a cylinder when the ends are joined with a twist of $2n\pi/m$. Figure 10 shows how the cylinder arises in the $(3, 1)$ case. The ends of the cylinder must be joined so that the like-labelled triangles come together, forming a solid torus with a $(3, 1)$ curve.

Finally, since the $(3, 1)$ curve on one solid torus bounds a disk in the other, we get the relation $a^3 = 1$, where a is a loop that runs once around the solid torus in Figure 10. This relation defines the fundamental group of the $(3, 1)$ lens space. In general, π_1 of the (m, n) lens space has defining relation $a^m = 1$. So all the (m, n) lens spaces, for fixed m , have the same π_1 . This again raises the possibility that π_1 is not a complete invariant for 3-manifolds, because it seems doubtful that all the (m, n) lens spaces are homeomorphic, for fixed m . Indeed, in the final section of his paper, Tietze conjectured that the $(5, 1)$ and $(5, 2)$ lens spaces are *not* homeomorphic.

6. DEHN

Max Dehn (1878–1952) began his research career as a student of Hilbert at Göttingen in the late 1890s. During this period, Hilbert was working on the foundations of geometry, so this was the field in which Dehn made his first discoveries. He earned lasting fame for solving Hilbert’s third problem, by showing that a regular tetrahedron is not equidecomposable with a cube; see Dehn (1900) [18]. He also made a small contribution to topology, by proving the polygonal Jordan curve theorem from Hilbert’s axioms of incidence and order. This unpublished paper from 1899 is discussed by Guggenheimer (1977) [32]. Evidently, Dehn hoped to establish rigorous foundations for topology on the model of Hilbert’s foundations of geometry. And foundations were still uppermost in his mind when he co-authored an article on topology with Heegaard for Klein’s *Enzyklopädie der mathematischen Wissenschaften* in 1907.

The work by Dehn and Heegaard (1907) [25] is a survey of topology up to and including the work of Poincaré, with an attempt to construct foundations of a combinatorial/geometric nature. Observing how Poincaré relies on a simplicial structure to compute Betti numbers, torsion numbers, and the fundamental group, Dehn and Heegaard take a simplicial structure as part of the definition of a manifold, and they use it to define homology, homotopy, isotopy, and homeomorphism. As we now know, this is no restriction for dimensions ≤ 3 , and indeed the most important contribution of the Dehn–Heegaard paper was probably their proof of the classification theorem for 2-manifolds. This theorem—that compact surfaces

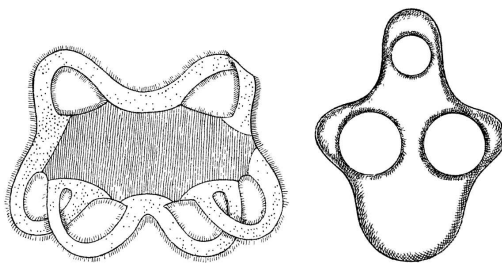
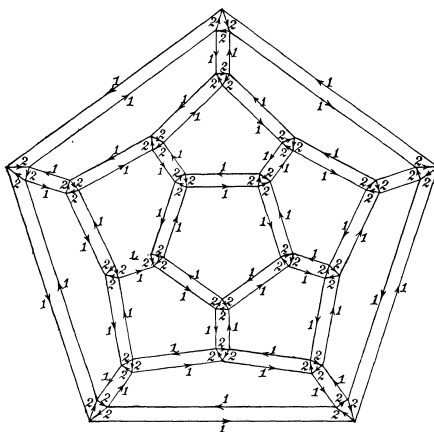


FIGURE 11. Picture of surfaces from the Dehn and Heegaard paper

are topologically classified by their genus and orientability—had been known in some sense since by Möbius (1863) [40], but the Dehn–Heegaard proof was the first to meet Hilbertian standards of rigor.

But the limitation to low dimensions is not the main problem with the Dehn–Heegaard article. Surprisingly, they fail to appreciate the power of group theory in topology, and they do not go beyond a brief definition of the fundamental group. They even attempt to revise the argument for Poincaré’s homology sphere so as to avoid using group theory. Shortly after the publication of his article Dehn (1907) [19], Dehn published a short note acknowledging an error in the Dehn–Heegaard account of the homology sphere and offering a new construction of his own.

The Dehn construction in Dehn (1907) [19] is very simple, and again it does not use group theory. He takes two copies of \mathbb{S}^3 minus the tubular neighborhood of a knot, and pastes them together in such a way as to kill the homology of the resulting manifold, M . Thus M is a homology sphere, but it is not homeomorphic to \mathbb{S}^3 , because a torus surface (the boundary of the knot neighborhood) cannot separate \mathbb{S}^3 into two parts, neither of which is a solid torus. It is not hard to justify the claim about the homology of M , but the claim that a torus surface cannot separate \mathbb{S}^3 into two knot complements is not so easy, and was in fact not proved until 1924.

FIGURE 12. Dehn’s *Gruppenbild* for the icosahedral group

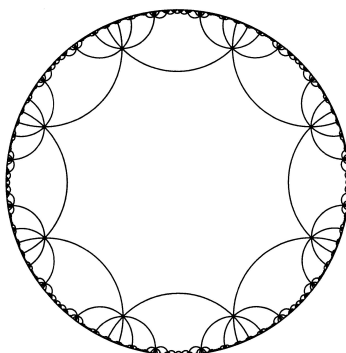


FIGURE 13. Universal covering of a genus 2 surface

Thus it seems that Dehn's early investigations in topology were hampered by his ignorance of group theory. Once he found his own way to work with groups, his creativity flourished. Dehn's approach to group theory developed in unpublished lectures on group theory around 1909–1910, the first two chapters of which may be seen in English translation by Dehn (1987) [24]. In the first chapter he introduces his *Gruppenbild* (group diagram), which allows groups to be studied by geometric methods. Strictly speaking, group diagrams had already been introduced by Cayley (1878) [16], but only for finite groups. Dehn includes some finite examples, such as the icosahedral group (Figure 12), but his diagrams are more enlightening for infinite groups, where they lead to the *geometric group theory* of today.

Dehn's most characteristic application of his group diagrams was the solution of the word problem for π_1 of surfaces with genus ≥ 2 , which is equivalent to the topological problem of deciding which closed curves on the surface contract to a point. As Poincaré (1904) [51] knew, curves on a surface of genus g may be studied by lifting them to the universal cover: a tessellation of the hyperbolic plane by $4g$ -gons. For a surface of genus 2, for example, the tessellation looks like that in Figure 13.

Dehn realized that the edges in the tessellation, which correspond to canonical closed curves on the surface, also correspond to generators for π_1 of the surface. So if we label the edges of the tessellation with names for the underlying generators, we have the diagram of the group. In fact, with the usual choice of canonical curves $a_1, b_1, \dots, a_g, b_g$, the boundary of each polygon spells out the “word”

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}.$$

Now a closed curve p on the surface is homotopic to a product of canonical curves, which lifts (from a given vertex in the tessellation) to a unique edge path \tilde{p} . And p contracts to a point if and only if \tilde{p} is a closed path. Thus the word problem is solved, in principle, by the construction of the group diagram. Dehn's great discovery was that the combinatorics of hyperbolic tessellations allows the word problem for surface groups to be solved by a simple and efficient algorithm, without actually constructing the group diagram.

His solution of the word problem evolved through several papers, reaching its purely combinatorial form in Dehn (1912) [22]. The idea is to view the polygons in the tessellation in successive layers: layer 1 is a single polygon, layer 2 consists

of the neighbors of polygon 1, layer 3 consists of the polygons (not in the previous layers) that touch polygons in layer 2, and so on. Any closed path \tilde{p} , starting in layer 1, must necessarily reach some outermost layer then turn back. When it does so, it must traverse *more than half* the edges of a polygon in succession, because at most three of the edges of each polygon in the outermost layer are not in the outer boundary of the layer. Such a sequence can be recognized, in the “word” spelled by the sequence of labels of edges in \tilde{p} , as more than half the boundary word of a polygon; and it can be replaced by complementary “word” representing the remainder of the boundary. By repeatedly shortening the word by such replacements, we find out, in a finite number of steps, whether it represents a closed path.

This method is now known as *Dehn’s algorithm*.

When Dehn returned to 3-manifolds in Dehn (1910) [20] from his promising but incomplete effort in Dehn (1907) [19], his newly acquired group theory skills made a great difference. He was now able to construct a whole infinite family of homology spheres, with complete proof that they are not 3-spheres, by computing their fundamental groups. The method he used, now called *Dehn surgery*, was to remove a solid torus from an \mathbb{S}^3 and “sew it back differently”. There are infinitely many ways to identify a pair of canonical curves on a solid torus with a pair of curves on the boundary of the knotted hole in \mathbb{S}^3 , producing infinitely many 3-manifolds. Infinitely many of these have trivial homology but nontrivial π_1 . One of them turned out to have the same π_1 as Poincaré’s homology sphere (and it was shown to be identical with it by Seifert and Weber (1933) [59]), while the others have infinite π_1 . Moreover, it was later shown that all 3-manifolds can be produced from \mathbb{S}^3 by Dehn surgery, by suitable choice of knotted, or linked, tori (see Lickorish (1962) [37]).

In the same paper Dehn derived a presentation for π_1 of the trefoil knot complement, similar to the Wirtinger presentation, and described its group diagram. The diagram has an interesting geometric structure, lying naturally in the 3-dimensional space equal to line \times hyperbolic plane. This happens to be one of the eight 3-dimensional geometries, discovered by Bianchi (1898) [12], which are the subject of Thurston’s geometrization conjecture.

Perhaps the most far-reaching result in Dehn (1910) [20] is a kind of “Poincaré conjecture for knots”, stating that a knot K with the most trivial possible group (namely, $\pi_1(\mathbb{S}^3 \setminus K) = \mathbb{Z}$) is in fact the trivial knot. Like the actual Poincaré conjecture, Dehn’s result was too hard for the techniques then available. It depended on *Dehn’s lemma*, the claim that a curve K in a 3-manifold, spanned by a singular disk but with no singularities on K itself, is spanned by a nonsingular disk. Dehn attempted to prove his lemma by a method of surface manipulation he called “switchover” (*Umschaltung*), but he overlooked certain cases where the method fails. It may be that Dehn began thinking about Dehn’s lemma in 1907, because the gap in his homology sphere construction can also be filled by an application of Dehn’s lemma (see Stillwell (1979) [63]).²

²Gordon (1999) [31] has pointed out an instance where Poincaré seems to assume Dehn’s lemma, without proof or comment. It is in §5 of Poincaré (1904) [51], where Poincaré assumes the existence of a disk “without double curves” (see Poincaré (2010) [53, p. 214]). This is just before his construction of the homology sphere, so it is also possible that Dehn picked up the idea there.

The mistake in Dehn's proof was discovered by Kneser (1929) [36], and a correct proof was first given by Papakyriakopoulos (1957) [45], using a combination of switchover with an elaborate covering space construction. Surface manipulation in 3-manifolds is indeed a viable idea, and Kneser himself used it successfully in his 1929 paper. The idea took off in the 1950s with its revival by Papakyriakopoulos, and it was used in the solution of two longstanding algorithmic problems for 3-manifolds: recognition of the trivial knot by Haken (1961) [33], and recognition of \mathbb{S}^3 by Rubenstein (1995) [57].

Dehn's work on the fundamental group and homology spheres was obviously inspired by Poincaré, but Tietze also had an important influence on him. In 1908, Dehn for a time believed that he had proved the Poincaré conjecture—until Tietze pointed out a mistake in his argument (see Volkert (1996) [67]). Thus Dehn had good reason to respect Tietze's work and it seems likely that Tietze (1908) [65] turned Dehn's thinking towards knots and combinatorial group theory. Like Tietze, Dehn (1911) [21] raised the isomorphism problem for groups, and with it the word and conjugacy problems. As mentioned above, the isomorphism problem was shown to be unsolvable by Aydan (1957) [1]. This was a consequence of the unsolvability of the word problem for certain groups, proved by Novikov (1955) [44].

Dehn (1911) [21] also answered two questions that had been raised in §14 of Poincaré (1895) [48]: whether each finitely presented group could be realized as π_1 of a manifold and, if so, how such a manifold might be constructed. In Chapter III of his paper, Dehn pointed out the (relatively trivial) fact that each finitely presented group G may be realized as π_1 of a 2-complex, and the less obvious fact that G is then π_1 of a 4-manifold; namely the boundary of a neighborhood of the 2-complex when it is embedded in \mathbb{R}^5 . This construction may be used to prove the result of Markov (1958) [38] that the homeomorphism problem for compact 4-manifolds is unsolvable.

Dehn's most spectacular contribution to knot theory, in Dehn (1914) [23], neatly capped off the era of knot theory initiated by Wirtinger and Tietze. In this paper Dehn gave the first rigorous proof that the two trefoil knots are distinct. Supposing there were a deformation of the left trefoil knot into the right, Dehn showed that it would induce a certain kind of automorphism of the trefoil knot group (see Figure 14). Then, by a laborious determination of *all* the automorphisms (made possible by knowledge of the trefoil knot group developed in his 1910 paper), Dehn was able to show that the hypothetical trefoil-reversing automorphism did not exist.

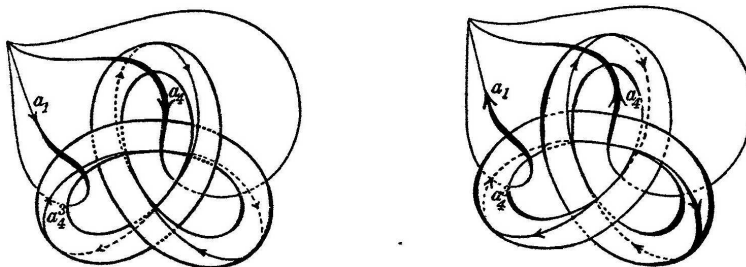


FIGURE 14. Dehn's picture of the two trefoil knots

7. ALEXANDER

James Alexander (1888–1971) was introduced to topology by Oswald Veblen while he was a student at Princeton. As Volkert says in Volkert (2002) [68, p. 182], Alexander can be described as the first topologist, in the sense of being the first person to work almost exclusively in the field of topology. In Alexander (1915) [2] gave the first proof that the Betti numbers and torsion numbers are topological invariants, thus putting Poincaré’s homology theory on a sound foundation for the first time. In 1916 he was involved in the production of the French translation of Heegaard’s thesis, Heegaard (1916) [35], which perhaps led him to study Tietze (1908) [65], because in the following years he solved some of Tietze’s open problems about lens spaces and knots. In particular, Alexander (1919b) [4] proved that the (5,1) and (5,2) lens spaces are not homeomorphic, thus giving the first example of nonhomeomorphic 3-manifolds with the same π_1 . And Alexander (1919a) [3] proved that any orientable 3-manifold is a branched cover of \mathbb{S}^3 . He also applied Heegaard’s branched coverings to obtain the first computable knot invariants, in 1920, and solved Dehn’s problem about the separation of \mathbb{S}^3 by a torus in 1924.

To illustrate Alexander’s way of thinking, we take a more detailed look at three of the results above: the invariance of the Betti and torsion numbers, the computation of knot invariants, and the embedding of surfaces in \mathbb{R}^3 .

Like Poincaré, Alexander took the subject matter of topology to be manifolds that admit simplicial decomposition, with a view towards computing Betti and torsion numbers. The *invariance* of these numbers is problematic, however, because it is not clear that the same numbers will arise from different simplicial decompositions of the same manifold. Following Riemann (1851) [56], one would like to *superimpose* two decompositions and obtain a “common subdivision”, obtainable from each of the original decompositions by elementary subdivisions (such as splitting an edge into two). It is easy to prove that the Betti and torsion numbers are invariant under elementary subdivisions, but *not* easy to prove that superposition always works, as Tietze (1908) [65] pointed out. The trouble is that the cells in a simplicial decomposition do not necessarily have straight edges—they are only homeomorphic images of true simplices—so two edges in different simplicial decompositions may intersect infinitely often. Alexander (1915) [2] got around this problem by *simplicial approximation*, a method introduced by Brouwer (1911) [14] to prove the invariance of dimension. 1

Thus, with a little help from Brouwer, Alexander put Poincaré’s homology theory on a sound foundation. It is therefore not surprising that Alexander was particularly attached to Betti and torsion numbers, and that he sought to apply them. He found a spectacular application with his discovery of the first computable knot invariants, as torsion numbers of branched coverings of \mathbb{S}^3 . To put his discovery in some perspective, it is useful to compare it with the first known knot invariant: the *knot group* discovered by Wirtinger and Dehn.

The group $\pi_1(\mathbb{S}^3 \setminus K)$ of a knot K is a good invariant only to the extent that we can distinguish different knot groups given by generators and relations. The first method available—Tietze’s process of abelianization and subsequent extraction of the “Poincaré numbers”—fails because all knot groups have the same abelianization (namely \mathbb{Z}).³ Before the 1920s, no general method of computing invariants that

³One wonders whether it was at this point that Tietze realized the difficulty of the isomorphism problem for groups. He does not explicitly mention the abelianization of knot groups, but it follows immediately from the Wirtinger presentation.

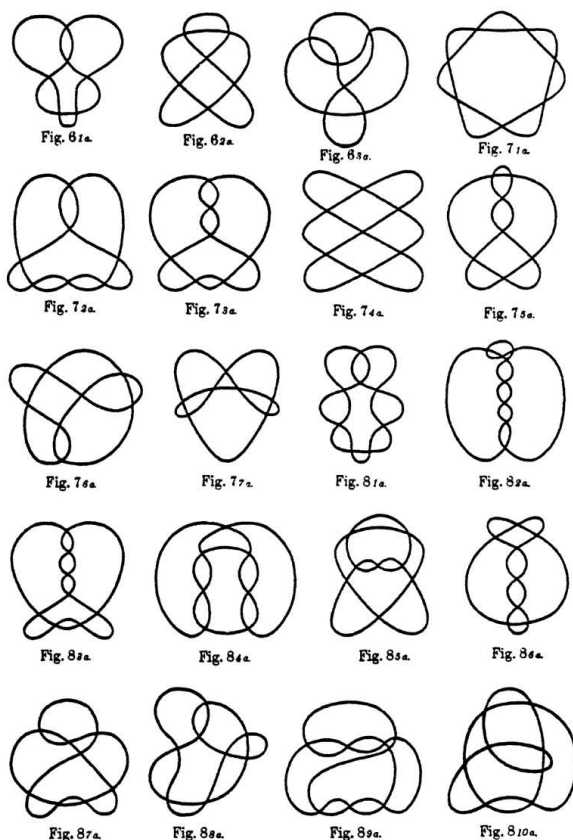


FIGURE 15. Some knots with 6, 7, 8 crossings

distinguish knot groups was known, so in this sense the knot group was not a *computable* knot invariant in 1920. In 1920 Alexander got around this problem by avoiding knot groups entirely and revisiting Heegaard's idea of studying coverings of \mathbb{S}^3 branched over knots.

Torsion is computable and, as we know, Heegaard observed torsion in a covering of \mathbb{S}^3 branched over the trefoil knot. Heegaard failed to exploit this discovery in the study of knots, but in 1920 Alexander exploited it with a vengeance, computing the torsion numbers for 2- and 3-sheeted covers of \mathbb{S}^3 branched over many different knots. Amazingly, he found that these invariant numbers are able to distinguish between all the knots that can be described by diagrams with up to eight crossings. Figure 15 shows some of these diagrams (in which the crossings are not shown, because they are understood to alternate between “under” and “over”). It is from the paper by Alexander and Briggs (1927) [8], which is a writeup of an announcement to the National Academy of Sciences by Alexander in 1920.

Alexander did not always write up full proofs of his results, but in this case he was forced to by the appearance of Reidemeister (1926) [54] by Reidemeister, a paper in which the same invariants were obtained from the knot group. Reidemeister replaced the covering of \mathbb{S}^3 , branched over a knot K , by an *unbranched* covering

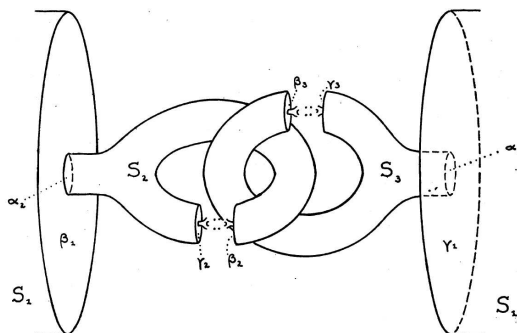


FIGURE 16. The Alexander horned sphere

of $\mathbb{S}^3 \setminus K$. He noticed that the π_1 of such coverings are subgroups of $\pi_1(\mathbb{S}^3 \setminus K)$ with nontrivial torsion numbers. The numbers were the same as Alexander's, but for the first time they were computable from the knot group. Reidemeister's method was a significant breakthrough, foreshadowing a new chapter in topology in which π_1 could play an effective role. Alexander's method, while more elementary, looked backward to the old world of Betti and torsion numbers. It was perhaps the swansong of the torsion numbers as they were conceived by Heegaard and Poincaré.

Finally, consider Alexander (1924b) [6], which contains a proof that any torus in \mathbb{S}^3 bounds a solid torus on at least one side, thus filling the gap in the Dehn construction Dehn (1907) [19] of a homology sphere, and answering Tietze's question whether a torus in \mathbb{S}^3 necessarily bounds a knot complement. In the same paper, Alexander showed that a *polyhedral* \mathbb{S}^2 in \mathbb{S}^3 bounds a ball on each side, a result that becomes significant in the light of the companion paper Alexander (1924a) [5]. In the latter paper, Alexander constructed the famous *Alexander horned sphere*—Figure 16—a non-polyhedral object, homeomorphic to a ball, whose complement is not simply connected. Rather in the spirit of Poincaré, Alexander takes it to be obvious that the complement of the horned sphere is not simply connected. He merely points to the example of the curve β_1 in Figure 16 as one that cannot be shrunk to a point in the complement space.

As these examples show, Alexander was able to fill most of the gaps in the work of Poincaré and his successors, and to make new advances, while staying quite close to their methods. In this sense, Alexander in the mid-1920s brought a natural close to the chapter of 3-manifold topology opened by Poincaré. By the late 1920s a new chapter had begun. The ideas of Poincaré continued to reverberate, but they were joined by many new ideas, such as the *Alexander polynomial* in Alexander (1928) [7]. If Alexander closed the Poincaré era, he also opened the post-Poincaré era.

ABOUT THE AUTHOR

John Stillwell is professor of mathematics at the University of San Francisco, and also is adjunct research associate at Monash University, Melbourne, Australia.

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UNIVERSITY OF SAN FRANCISCO, SAN FRANCISCO, CALIFORNIA; AND MONASH UNIVERSITY, MELBOURNE, AUSTRALIA