

Zariski geometries. Geometry from the logician's point of view, by Boris Zilber, London Mathematical Society Lecture Note Series, Vol. 360, Cambridge University Press, Cambridge, 2010, xii+212 pp., ISBN 978-0-521-73560-5

1. THE SUBJECT

Given an algebraic variety X over an algebraically closed field K , one can extract the following structure, in the sense of Bourbaki, loosely speaking. The underlying set is $X(K)$ the set of K -rational points of X , and for each $n \geq 1$ we have the collection $\mathcal{Z}_n(X(K))$ of *Zariski closed* subsets of the Cartesian product $X(K)^n$. Conversely, it is not unnatural to ask whether, given an arbitrary set Y and for each $n \geq 1$ a family $\mathcal{Z}_n(Y)$ of subsets of Y^n satisfying some suitable hypotheses, there is an algebraically closed field K and algebraic variety X over K such that $(Y, \mathcal{Z}_n(Y))_n$ is *isomorphic to* $(X(K), \mathcal{Z}_n(X(K)))_n$, in the sense that there is a bijection $f : Y \rightarrow X(K)$ inducing a bijection between $\mathcal{Z}_n(Y)$ and $\mathcal{Z}_n(X(K))$ for each n . Of course a key issue is what kind of suitable hypotheses are acceptable. The kind of thing we have in mind includes treating the elements of $\mathcal{Z}_n(Y)$ as the closed sets for an *abstract Zariski topology* on Y^n and requiring at least some continuity assumptions for the projection maps $Y^n \rightarrow Y^k$. An important result of Hrushovski and Zilber from the mid 1990s [3] says that, assuming Y is 1-dimensional, in an appropriate sense, this converse *does hold*: under a strong nondegeneracy assumption on $(Y, \mathcal{Z}_n(Y))_n$ (very ampleness), Y with its families $\mathcal{Z}_n(Y)$ is isomorphic to some $X(K)$ with families $\mathcal{Z}_n(X(K))$ where X is an *algebraic curve* over K . Under a weaker but natural nondegeneracy assumption (ampleness or nonmodularity) on $(Y, \mathcal{Z}_n(Y))_n$, but still assuming 1-dimensionality, they show that Y is a *finite cover* of an algebraic curve. Examples of such Y which are *not* themselves algebraic curves are given in [3], and this partly motivates Chapter 5 of the book under review, which tries to interpret such examples in terms of noncommutative geometry.

One of the aspects of the Hrushovski–Zilber theorem is the recovery of an ambient algebraically closed field from a set Y equipped with an abstract Zariski topology on each of its Cartesian powers. This is an analogue of the well-known result, sometimes called the Fundamental Theorem of Projective Geometry, which recovers a coordinatizing field (or division ring) from an abstract projective geometry of dimension at least 3.

The Hrushovski–Zilber theorem is very powerful, and its consequences are still being mined. One such application, discovered quickly by Hrushovski, was a proof of the functional Mordell–Lang conjecture, which required some more finessing in the positive characteristic case; see [2]. But as we describe below, what is really important is the *Zilber conjecture* for (strongly) minimal sets, and its validity in important cases, sometimes, but not exclusively, via the Hrushovski–Zilber theorem.

The background to and motivation for the Hrushovski–Zilber theorem is the problem of classifying *strongly minimal sets*, from model theory or more specifically the subarea geometric stability theory. There are several reasonable surveys of model theory for a general audience, including [7] and [5]. So rather than include

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a mini-course in model theory in the current review, I would rather refer the interested reader to these other sources and limit myself here to providing an informal description of the key notions.

Model theory studies *first order theories*. Such a first order theory T consists of a collection of first order sentences or axioms in a given *vocabulary* L , where L consists of some relation symbols and function symbols. For example if L consists of a binary function symbol \times , a unary function symbol inv , and a 0-ary function symbol e , we can write the axioms for groups as a (finite) set of first order sentences in the vocabulary L , and this collection of sentences is an example of a first order theory, the theory of groups. Likewise, if L consists of function symbols $+$, $-$, \times , 0 , e , and binary relation symbol $<$, we can write the axioms for ordered fields satisfying the intermediate value property for polynomials as a (infinite) collection of first order sentences in the language L , and this collection of sentences is called RCF (the theory of real closed fields). It is worth remarking that *completeness* of the ordered field cannot be expressed in a first order manner. A *structure* for a vocabulary L is simply a set M equipped with actual relations and functions corresponding to the symbols of L (and we often notationally identify this structure with its underlying set M in the same way that a group G is often notationally identified with its underlying set G). And given a theory T in the vocabulary L , M is said to be a *model of T* if the sentences in T are true in M (in the obvious or tautological sense). So a group is a model of the theory of groups, and a real closed field is a model of RCF.

A structure M for vocabulary L also comes equipped with its category $\text{Def}(M)$ of *definable sets*. These definable sets are the subsets of M^n , which are defined by first order *formulas* of the vocabulary L allowing parameters from M . Alternatively they are the subsets of M^n obtained from the basic relations on M and graphs of basic functions on M via finite Boolean combinations, projections, and fibres of projections. When M is the structure $(\mathbf{C}, +, \times)$, $\text{Def}(M)$ is the category of *constructible* subsets of the various Cartesian powers of \mathbf{C} (namely finite unions of locally Zariski closed sets).

A privileged class of first order theories consists of those which have a unique model in each uncountable cardinality κ , and these are often called *uncountably categorical* theories and are special cases of *stable* first order theories (which I will not define here). If M is a model of such an uncountably categorical theory, M is controlled in a suitable sense, by a *strongly minimal* definable set X in M . Strong minimality of X means that X is infinite but has no infinite co-infinite definable subsets. This definition of strong minimality of X only concerns definable subsets of X itself, but has implications for definable subsets Z of Cartesian powers X^n : for example to any such Z can be assigned in a canonical fashion a dimension. From this point of view the strongly minimal definable sets are the 1-dimensional ones.

The identification, in various senses, of such strongly minimal sets, becomes a key issue in the classification or description of uncountably categorical theories. Such a strongly minimal set X can be viewed as a structure in its own right, and there are three classical examples:

Example (i): X is a set with *no* structure, i.e., no basic relations other than equality (and no basic functions).

Example (ii): X is an (infinite) vector space over a field F where the basic functions are addition and scalar multiplication by r , for each $r \in F$.

Example (iii): X is an algebraically closed field where the field operations are the basic functions (and there are no additional basic relations).

There are various natural model-theoretic ways of distinguishing between Examples (i), (ii), and (iii) above. One such involves the notion of a pregeometry (or matroid) coming from model-theoretic *algebraic closure*. But I prefer to give another one: We consider definable families \mathcal{F} of strongly minimal (1-dimensional) subsets of $X \times X$. In Example (i) there are no positive-dimensional such families. In Example (ii) the only such families \mathcal{F} have dimension at most 1 (typically the family of graphs of translations $x \rightarrow x + a$ as a varies). In Example (iii) there are families \mathcal{F} of dimension ≥ 2 (such as the graphs of maps $x \rightarrow ax + b$ as a, b vary). For reasons coming from the matroid point of view, we call these features or properties of a strongly minimal set, (i) *triviality*, (ii) *nontriviality + modularity*, (iii) *nonmodularity*, respectively. Any strongly minimal set X will satisfy exactly one of these.

One of the important early contributions of Hrushovski (in his Ph.D. thesis) was that property (ii) above (modularity + nontriviality) corresponds essentially to Example (ii): there is a 1-dimensional definable abelian group which is generically a vector space over a division ring, with no additional induced structure. On the other hand what became known as Zilber's conjecture was that property (iii) (nonmodularity) corresponds essentially to Example (iii). The "essentially" means at the minimum that there is a 1-dimensional algebraically closed field K definable in X and at the maximum that the only induced structure on such K is the field structure.

Zilber's conjecture was shown to be false by Hrushovski in the late 1980s; see [1]. He found a clever variant on the well-known *Fraissé* method of constructing an infinite *homogeneous* (i.e., with many symmetries) structure from a suitable family of finite structures, and by these means he built a nonmodular strongly minimal set in which no infinite field (in fact no infinite group) is definable. The general technique that Hrushovski used now goes under the name Hrushovski construction and has developed into a subject in its own right, within model theory.

In model theory there is no a priori distinction between *closed* and *open* definable sets, although in natural *stable* examples, such as algebraically closed fields, differentially closed fields, and abelian groups, we do have such a distinction: the closed definable sets are the Zariski closed, Kolchin closed, and *positive-primitive*-definable sets, respectively; and moreover any definable set is a finite Boolean combination of closed definable sets. In the above examples the relevant topology is Noetherian (in various senses) and not Hausdorff. On the other hand there is another class of examples of structures which are equipped with a Hausdorff topology which has a *definable basis*, such as real closed fields and p -adically closed fields (in the language of rings). These are *unstable*. The kind of *topological* structures considered in the book under review belong to the first rather than second class of examples.

A major insight of Hrushovski and Zilber was that, under additional assumptions on a strongly minimal set X , which involve precisely the identification of certain definable sets being closed, the Zilber conjecture can be proved. The intuition is that one should then be able to define an abstract *tangent space* to $X \times X$ at a given point $(a, b) \in X \times X$, and that *composition* of suitable strongly minimal subsets of $X \times X$ passing through (a, b) should yield a (definable) group structure on the relevant tangent vectors. Repeating the procedure with this 1-dimensional definable group G in place of X (and under the nonmodularity assumption) should

yield an infinite definable field K . This intuition is realized in Chapters 3 and 4 of the book under review. The approach is, at least superficially, somewhat different from that in the paper [3], which used more formal methods that depended upon Hrushovski's group and field configuration theorems. These latter theorems give conditions under which one can recover definable groups (fields) from some model-theoretic configuration of points, a substantial generalization of Weil-type theorems on recovering an algebraic group from birational data. This review is not the right place to go into proper details about either proof. In any case, either route to the existence of a definable field poses formidable technical challenges, and the proofs are major accomplishments. In fact the maximum version of Zilber's conjecture is also proved: the induced structure on K is just the field structure. This is an abstract version of Chow's theorem and in fact yields Chow's theorem (that a closed analytic subvariety of $P^n(\mathbf{C})$ is algebraic).

The general idea of imposing some kind of *Zariski topology* on definable sets in a structure M originates with Gabriel Srouf and his notion of *equational theory* (see, for example, [6]). But Zariski structures (or geometries) involve a bit more, as I now describe, for the record:

A *Zariski geometry* or *1-dimensional Zariski structure* is a strongly minimal structure M , equipped for each n with a family \mathcal{Z}_n of distinguished definable subsets of M^n , which we will call the *closed* or *definable closed* sets in M , with the following properties:

- (i) *Quantifier elimination to closed sets.* Any definable subset of M^n is a finite Boolean combination of elements of \mathcal{Z}_n .
- (ii) *Noetherianity.* \mathcal{Z}_n has the descending chain condition (DCC), in the sense that the intersection of any collection of members of \mathcal{Z}_n is a finite subintersection.
- (iii) *Dimension theorem.* If X, Y are irreducible closed subsets of M^n with nonempty intersection, then the dimension of an irreducible component of $X \cap Y$ is $\geq \dim(X) + \dim(Y) - n$.

Condition (iii), the dimension theorem, is key, and realizing its importance was a great insight of Hrushovski and Zilber.

To summarize, the *fundamental theorem* of [3], and also of the book under review (except that (d) below is not paid much attention in the book), is:

Fundamental Theorem. Suppose M is a Zariski geometry and is nonmodular. Then

- (a) There is a 1-dimensional definable field K (necessarily algebraically closed) in M .
- (b) Any subset of K^n definable in M is definable in the structure $(K, +, \times)$.
- (c) Up to removing finitely many points, M is a *finite cover* of the affine line over K .
- (d) When M has the property (slightly stronger than nonmodularity) that there is a definable family of strongly minimal subsets of $M \times M$ which generically separates points, then M has the natural structure of an algebraic curve over K .

Two important and nontrivial examples to which the Hrushovski–Zilber theorem applies are strongly minimal differential algebraic varieties X (in the sense of Kolchin [4]) and strongly minimal compact complex manifolds X . In each of these

cases there are natural choices for the closed sets, namely the Kolchin closed subsets of X^n , and the closed analytic subvarieties of X^n , respectively. In both these examples, the consequences of Hrushovski–Zilber (truth of the Zilber conjecture) are substantial, with impact on the structure of solutions of differential equations, and bimeromorphic geometry, respectively. On the other hand in each of these cases the Zilber conjecture can be proved relatively directly (although these proofs were found later). The function field Mordell–Lang conjecture in characteristic zero follows from the differential algebraic case. Among the consequences (with some additional work) for bimeromorphic geometry is a trichotomy theorem for *simple* compact complex manifolds X : X is either an algebraic curve, a simple nonalgebraic complex torus (up to correspondence), or X has no positive dimensional family of *correspondences*. When X is not of *Kaehler-type* it is of the third kind.

2. THE BOOK

The book is based on notes that Zilber wrote in the early 1990s. The original notes have been massaged in various ways, a couple of additional chapters have been added around Zilber’s more recent interests, and there is an appendix on model theory. Chapters 1 to 4 exposit the basic theory of Zariski structures and geometries, culminating in a proof of the fundamental theorem, as described earlier, and using ideas with a nonstandard analysis flavour (infinitesimal neighbourhoods). Chapter 5 attempts to relate some *nonalgebraic* examples of nonmodular Zariski geometries (from [3]) to *noncommutative geometry*, and Chapter 6 discusses the notion of analytic Zariski structure.

Chapters 5 and 6 are more speculative than the earlier part of the book. Chapter 5 takes as its starting point an example in [3] of a nonmodular Zariski geometry M which is *not* an algebraic curve and is not even interpretable in an algebraically closed field (basically because of the structure of $\text{Aut}(M)$). Zilber attaches to such examples a noncommutative K -algebra. Conversely, to a suitable K -algebra, he associates a Zariski structure. The K -algebras considered here have large central subalgebras. I understand that for applications to noncommutative geometry it would be interesting to also consider simple algebras, in particular with no non-trivial centre. Concerning Chapter 6, I must admit that I understand neither the (mathematical) content of the notion of analytic Zariski structure nor its (mathematical) motivation. I guess that the author believes that, as with nonalgebraic Zariski geometries, various exotic structures (such as Hrushovski’s examples) should nevertheless have some classical origin, but coming maybe from analytic rather than algebraic geometry. And the analytic Zariski structure machinery is supposed to make this more precise. My feeling is that one has to buy into or accept this ideology in order to appreciate the mathematics. The actual definitions or axioms for analytic Zariski structures have developed over a few years through several approximations and Ph.D. theses. I leave the reader to explore these independently. Some examples are given, rather briefly, including the author’s *pseudo-exponential field*, coming out of his rather influential model-theoretic approach to the complex exponential function.

Boris Zilber has been a dominant force in model theory for around 40 years, and his ideas and results have shaped what we now call geometric model theory. These wide ranging ideas, sometimes rather speculative, are on show throughout the book. But the book is marred by a considerable amount of loose definitions,

ambiguities, and typographical problems, which might make it rather difficult for a graduate student or interested mathematician to actually learn something at a technical level. In fact one wonders about the nature of the refereeing process. Also what is missing from the book is a treatment of *type-definable* Zariski geometries, which are relevant to the really new part of Hrushovski's proof of the functional Mordell–Lang conjecture, namely the positive characteristic case. Nevertheless, this book is an important achievement. Any model-theorist should have a copy, and it can serve as a motivated introduction to model theory for a general mathematician.

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