BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 50, Number 3, July 2013, Pages 523–526
S 0273-0979(2012)01394-0
Article electronically published on December 7, 2012

Fusion systems in algebra and topology, by M. Aschbacher, R. Kessar, and B. Oliver, London Mathematical Society Lecture Notes, Vol. 391, Cambridge University Press, Cambridge, 2011, vi+320 pp., ISBN 978-1-107-60100-0

The theory of fusion systems: an algebraic approach, by David A. Craven, Cambridge Studies in Advanced Mathematics, Vol. 131, Cambridge University Press, Cambridge, 2011, xii+371 pp., ISBN 978-1-107-00596-9

A standard technique for understanding the structure theory of certain algebraic objects consists of filtering information relative to a chosen prime number p. In finite group theory, an early highlight of this philosophy are Sylow's theorems. The first of Sylow's theorems states that a finite group G has a subgroup P whose order is the highest power of p which divides the order of G; these subgroups are the *Sylow-p-subgroups of* G. Sylow's second theorem states that any two Sylow-p-subgroups P, P' of G are *conjugate*; that is, there is an element x in G such that $P' = P$. In particular, all Sylow-p-subgroups of G are isomorphic.

Wildly different finite groups can, of course, have isomorphic Sylow-p-subgroups at a fixed prime number p. In order to refine this approach, one needs to take into account information on how a Sylow-p-subgroup of a finite group G is embedded in G. This line of thought is formalised under the notion of *G-fusion in a Sylow-p-subgroup P*: two elements in P are *G-fused* if they are conjugate by an element in G. Since any finite p-group P is obviously its own Sylow-p-subgroup, fusion is in that case the same as conjugacy in P. One of the early results on fusion in finite groups is Burnside's theorem stating that if G has an abelian Sylow-p-subgroup P, then any two G-conjugate elements in P are in fact already conjugate by an element in the normaliser $N_G(P) = \{x \in G \mid xPx^{-1} = P\}$ of P in G. In contemporary terminology, *$N_G(P)$ controls G-fusion in P*, whenever P is abelian. The proof of Burnside's theorem is a consequence of Sylow's theorems applied to the centraliser $C_G(P)$ of P in G, noting that P is contained in $C_G(P)$ as P is abelian. The hypothesis on P being abelian is indeed crucial: the symmetric group S_4 on four letters has a dihedral Sylow-2-subgroup D_8 of order 8 which is equal to its own normaliser, and which has the property that certain involutions are conjugate in S_4 but not in D_8.

Burnside's theorem set the stage for a panoply of results for when a subgroup H of a finite group G containing a Sylow-p-subgroup controls G-fusion in P. Frobenius showed that P itself controls G-fusion in P if and only if G is a semidirect product of the form $G = K \rtimes P$ for some normal subgroup K of order prime to p. Glauberman's celebrated ZJ-theorem from 1968 states that if p is odd and if G has no subquotient isomorphic to a semidirect product ${\mathbb{F}_p}^2 \rtimes \mathrm{SL}_2(p)$, then the G-fusion of G in P is controlled by the normaliser $N_G(ZJP)$, where ZJP is the center of the *Thompson subgroup* JP of P, generated by all abelian subgroups of maximal order in P. Alperin's fusion theorem, another cornerstone of fusion in finite groups which appeared in 1967, shows that if two elements in P are conjugate in G, then

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2010 *Mathematics Subject Classification.* Primary 16W22, 18A25, 18A30, 18G30, 20C05, 20C20, 20E25, 20F99, 20J05, 55P10, 55P60, 55Q05, 55Q52.

they are conjugate by an element which is a product of elements in normalisers of certain nontrivial subgroups of P.

Investigating the relationship between the structure of a finite group G and its fusion on a Sylow-p-subgroup P works both ways: in the run-up to the classification of finite simple groups, a key strategy during the 1970s for proving or disproving the existence of a finite simple groups consisted of considering "plausible" fusion patterns on finite p-groups and asking whether there is actually a finite group whose fusion yields this pattern. Ron Solomon ruled out the existence of a finite simple group whose Sylow-2-subgroup P is that of the spinor group $\mathrm{Spin}(7,3)$ such that in addition to fusion from $\mathrm{Spin}(7,3)$, all involutions in P are conjugate.

An important step towards formalising fusion systems was taken in work of Puig from 1976, translating the concept of fusion into a *category*: G-fusion on a Sylow-p-subgroup P is described as the category $\mathcal{F}_P(G)$ having the subgroups of P as objects, with morphisms induced by conjugation in G. More precisely, for any two subgroups Q, R of P, the set of morphisms in $\mathcal{F}_P(G)$ from Q to R consists of all group homomorphisms $\varphi : Q \to R$ for which there exists an element $x \in G$ such that $xQx^{-1} \subseteq R$ and such that $\varphi(u) = xux^{-1}$ for all $u \in Q$. The composition of morphisms in $\mathcal{F}_P(G)$ is given by the usual composition of group homomorphisms. The notion of control of fusion takes an elegant form: a subgroup H of G containing a Sylow-p-subgroup P of G controls G-fusion if we have an equality of categories $\mathcal{F}_P(G) = \mathcal{F}_P(H)$. Burnside's theorem states that if P is abelian, then $\mathcal{F}_P(G) = \mathcal{F}_P(N_G(P))$. Frobenius' theorem says that $\mathcal{F}_P(G) = \mathcal{F}_P(P)$ if and only if $G = K \rtimes P$ for some normal p'-subgroup, and the conclusion of Glauberman's ZJ-theorem reads $\mathcal{F}_P(G) = \mathcal{F}_P(N_G(ZPJ))$.

The first hints for the relevance of fusion systems in modular representation theory emerged in work of Brauer on p-blocks of finite groups. Bringing some of Brauer's ideas to their logical conclusion, Alperin and Broué developed in the late 1970s the concept of fusion in p-blocks of finite groups by replacing p-subgroups by what are now called *subpairs* or *Brauer pairs*, effectively associating a category to any p-block, where the Sylow-p-subgroup is replaced by a defect group of the block. While the details of associating fusion systems to blocks seem rather tedious, the formal similarities between fusion systems of blocks and of finite groups are so striking that they led Puig to the abstract notion of a fusion system in 1992. A *saturated fusion system on a finite p-group P* is a category whose objects are the subgroups of P, and whose morphisms are group homomorphisms between subgroups of P, satisfying a list of certain axioms. Those axioms are easily seen to be satisfied by the category $\mathcal{F}_P(G)$ above, as well as the fusion systems of blocks defined in terms of Brauer pairs. Rather remarkably, Puig's axioms seem to leave limited room for "exotic" fusion systems—that is, saturated fusion systems which do not arise as fusion system of some finite group with P as a Sylow-p-subgroup. Benson predicted in 1996 that there should be a fusion system associated with the 2-fusion pattern on a Sylow-2-subgroup of $\mathrm{Spin}(7,3)$ considered by Solomon, and constructed a space which should play the role of the 2-completed classifying space of a finite group with the above 2-fusion. Levi and Oliver showed that there is indeed an exotic fusion system on the Sylow-2-subgroup of $\mathrm{Spin}(7,3)$ with a fusion pattern that had been ruled out by Ron Solomon as a candidate for a finite simple group. Subsequently, Ruiz and Viruel, and others, found further exotic fusion systems at odd primes, but unlike for the Solomon fusion system, the classification of finite simple groups is required to show that these are indeed exotic.

The theory of fusion systems took a new turn with groundbreaking work of Broto, Levi, and Oliver in 2003, laying the foundations of the homotopy theoretic side of fusion systems. One of the key results in this work is a cohomological criterion for when there is a topological space $B\mathcal{F}$ associated with a fusion system $\mathcal{F}$ on a finite p-group P such that $B\mathcal{F}$ coincides with the p-completed classifying space of any finite group G having P as a Sylow-p-subgroup and $\mathcal{F}$ as its fusion system on P. Oliver used this in conjunction with the classification of finite simple groups to prove the Martino–Priddy conjecture: two finite groups have homotopy equivalent p-completed classifying spaces if and only if they have isomorphic fusion systems. Very recently, Chermak has shown—also using the classification of finite simple groups—that every fusion system does indeed have such a space, and Oliver has given a proof recasting this in terms of the aforementioned cohomological criterion.

As a common link between finite group theory, modular representation theory, and homotopy theory, fusion systems have become a vibrant research topic in their own right. There is a rapidly growing body of work by numerous authors developing this theory. The two books under review give a remarkably comprehensive snapshot of the status quo of this area as of 2010, complemented by extensive lists of references for further reading.

Craven's book *The theory of fusion systems* starts with a detailed motivation, describing in three preliminary chapters how fusion systems arise in the three areas above, before moving on to their internal theory. A significant amount of this theory consists of extending some of the by now classical methods and notions from finite groups to fusion systems. This includes normal subsystems, simple fusion systems, normalisers, centralisers, morphisms between fusion systems, control of fusion results, and soluble fusion systems, for instance. The book finishes with considerations of exotic and block-exotic fusion systems—that is, fusion systems which do not arise from finite groups or blocks, respectively—and describes Oliver's p-group conjecture which, if true, would imply Chermak's theorem, bypassing the classification of finite simple groups. Except for the definition of fusion systems, much of this material covers research which is barely a decade old, and which is still in the process of standardising its terminology. The author takes great care when it comes to comparing various notions by different authors. The book is further complemented by extensive lists of exercises at the end of each chapter. This book is an ideal point of entry to the general theory of fusion system needed for contemporay research in this area, and indeed, for pursuing the subject in the second book under review, *Fusion systems in algebra and topology*, by Aschbacher, Kessar, and Oliver. Unlike Craven's textbook, this is more of a survey of what is the forefront of research in the three main areas mentioned above—and as such, the reader is referred to the literature for proofs for many of the more advanced results which would be beyond the scope of a book.

The first part of the book of Aschbacher, Kessar, and Oliver is a short, but detailed, review of the terminology and basic results on fusion systems needed in the sequel. It differs from Craven's exposition in that it is not intended as a motivational chapter, but as a point of reference for what follows.

Part II of this book, written by Aschbacher, is an account on extending local finite group theory to fusion systems. It moves rapidly into intricate and technical territory, leading to a Jordan–Hölder theorem for fusion systems, fusion systems of simple groups in relation to simple fusion systems, ending with a speculation on possibly classifying simple fusion systems at the prime 2 and the use of this for the classification of finite simple groups. A solid grounding in some of the methods from finite group theory is helpful for reading Part II. Some of Aschbacher's results appear in Chapter 8 of Craven's book, which would indeed be an excellent preparation for this part.

Part III, written by Oliver, contains a wealth of material around the homotopy theoretic aspects of fusion systems. After some general homotopy theoretic background and an introduction to the geometric realisation of a category, the core notion of a centric linking system is described, first for finite groups and then for abstract fusion system. This notion, due to Broto, Levi, and Oliver, is in this context the key link between algebra and topology: the p-completion of the geometric realisation of the centric linking system of a fusion system $\mathcal{F}$ is a topological space which turns out to be the p-completed classifying space of a finite group G whenever $\mathcal{F}$ is isomorphic to the fusion system of G on one of its Sylow-p-subgroups. At the time of the writing of both books under review, the existence of centric linking systems for abritrary fusion systems was still unknown (a fact which is mentioned in both books), but has since been settled in the positive by Chermak's theorem. Part III contains further a brief introduction to functor cohomology, including some computational methods on how to calculate certain higher limits, so that the cohomological criterion for the existence and uniqueness of centric linking systems can be stated properly and applied to constrained fusion systems and fusion systems of finite groups. All through Part III, the theory is illustrated by examples, such as fusion systems of finite groups of Lie type and exotic fusion systems.

Part IV, written by Kessar, describes how fusion systems arise in the context of block algebras of finite groups, and how these relate to the most prominent conjectures in this area. After a short introduction—mostly without proofs—to standard material on finite-dimensional algebras, the connection between fusion systems and certain algebras acted upon by a finite groups, is described in great detail and with proofs. This is done in a degree of generality which goes well beyond the traditional realm of fusion systems of blocks, including a criterion for when a p-permutation G-algebra yields a saturated fusion system. Much of Part IV is then devoted to the mysterious relationship between the structure of a block algebra and its fusion systems, illustrated by the finiteness conjectures of Donovan and Puig, some counting conjectures such as Alperin's weight conjecture, as well as Broué's abelian defect conjecture. The rather complicated relationship between the fusion systems of blocks and covered blocks of normal subgroups is the topic of the last section; this material can be used to show (using the classification of finite simple groups) that the known exotic fusion systems are also block-exotic.

Together, the books by Craven and by Aschbacher, Kessar, Oliver make an excellent and timely tandem for anyone wishing to get acquainted with fusion systems and their relevance for finite group theory, modular representation theory, and homotopy theory.

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