BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 50, Number 3, July 2013, Pages 495–502
S 0273-0979(2013)01406-X
Article electronically published on April 4, 2013

How groups grow, by Avinoam Mann, London Mathematical Society Lecture Note Series, Vol. 395, Cambridge University Press, Cambridge, 2012, ix+199 pp., ISBN 978-1-107-65750-2

A group G is said to be generated by a set $S \subset G$ if every element $g \in G$ can be written as a product $g = s_1 s_2 \cdots s_k$ of elements of $S \cup S^{-1}$. Suppose that the set S is finite, and let $\gamma(n)$ be the number of elements of G that can be represented as a product $s_1 s_2 \cdots s_k$ of length $k \le n$. Then $\gamma(n)$ is called the *growth function* of G.

There is a natural geometric interpretation of the growth function. Denote for $g \in G$ by $l(g)$ the *length* of g with respect to the generating set S, i.e., the shortest length of a representation of g as a product of elements of $S \cup S^{-1}$. It is easy to see that $d(g_1, g_2) = l(g_1^{-1} g_2)$ is a metric on G, invariant with respect to the action by left multiplication $h : g \mapsto hg$. Then $\gamma(n)$ is equal to cardinality of any ball of radius n in the metric space (G, d).

Of course, the metric d and hence the growth function depends on the choice of the generating set S. (Consider, for example, $\mathbb{Z}$ with respect to the generating sets $\{1\}$ and $\{2, 3\}$.) It is easy to see, however, that if l_1 and l_2 are lengths defined by two finite generating sets, then there exists a constant $C > 1$ such that $C^{-1} l_1(g) \le l_2(g) \le C l_1(g)$ for all $g \in G$. Namely, C is equal to an upper bound of the length of elements of one generating set with respect to the other. It follows that if γ_{G,S_1} and γ_{G,S_2} are growth functions of G defined using different finite generating sets, then there exists a constant $C > 1$ such that

$$\gamma_{G,S_1}(n) \le \gamma_{G,S_2}(Cn), \qquad \gamma_{G,S_2}(n) \le \gamma_{G,S_1}(Cn).$$

for all n.

Note that the last condition is an equivalence relation on the set of growth functions. More generally, we write $\gamma_1 \preceq \gamma_2$, for positive nondecreasing functions $\gamma_1, \gamma_2 : \mathbb{R} \longrightarrow \mathbb{R}$, if there exists a constant $C > 1$ such that $\gamma_1(n) \le C\gamma_2(Cn)$ for all sufficiently big n. We write $\gamma_1 \sim \gamma_2$ (and say that *growth rates of* γ_1 *and* γ_2 *are equivalent*) if $\gamma_1 \preceq \gamma_2$ and $\gamma_2 \preceq \gamma_1$. (For a discussion of different definitions of equivalence of growth rates, see [Har00, Section IV.B].) In particular, if γ_1, γ_2 are growth functions of a group G defined with respect to different generating sets, then $\gamma_1 \sim \gamma_2$.

It is easy to see that the growth rate of the infinite cyclic group $\mathbb{Z}$ is equivalent to n. In fact, the growth function $\gamma(n)$ for the standard generating set $\{1\}$ is equal to $2n + 1$. The free abelian groups $\mathbb{Z}^r$ have growth $\sim n^r$.

The free group F_k on k generators (with respect to the free generating set) has growth function $\gamma(n) = 2k(k-1)^{n-1} \sim e^n$. This is obviously the maximal possible growth function of a group generated by k elements. Consequently, $\gamma(n) \preceq e^n$ for any growth function $\gamma(n)$ of a group.

Explicit computation of the function $\gamma_{G,S}(n)$ is rarely possible. But it is natural in some especially nice cases to consider the (*cumulative*) *growth generating function* $S(x) = \sum_{n \ge 0} \gamma(n) X^n$, or the *strict* growth generating function $A(x) =$

2010 *Mathematics Subject Classification.* Primary 20F69; Secondary 20F05.

$\sum_{n\geq 0}(\gamma(n)-\gamma(n-1))X^n = (1-X)S(X)$. Note that $a(n)=\gamma(n)-\gamma(n-1)$ is cardinality of the sphere of radius n in the metric space (G,d).

One can ask, for instance, for which groups the growth generating function is rational (for some or for all generating sets). There are some classes of groups for which the growth generating functions are rational for all generating sets; e.g., virtually abelian groups [Ben83], Gromov hyperbolic groups [Can84], see also [Man12, Chapter 15]. It is shown in [Sto96] that there exist nilpotent groups that have generating sets for which the growth generating function is rational, as well as generating sets for which it is transcendental. There are many examples of groups for which growth generating functions are transcendental for all choices of the generating set.

Interest in systematic study of growth of groups was kindled by the discovery by A. Švarc [Šva55] and J. Milnor [Mil68a] of a connection between growth of groups and geometry. Namely, there is the following relation between the fundamental group and the universal covering of a manifold.

Theorem 1. *Let $\mathcal{M}$ be a compact Riemannian manifold, and let G be its fundamental group. Then growth of G is equivalent to the volume growth of the universal covering of $\mathcal{M}$.*

Here volume growth of a Riemannian manifold is the function $v(n)$ equal to volume of the sphere of radius n with center at a point x_0 of the manifold.

We have seen above that there exist obvious examples of groups of growth equivalent to n^r for any $r\in\mathbb{N}$, and of growth $\sim e^n$. In fact, for a long time no other growth rates of groups were known.

Polynomial growth of abelian groups can be generalized to finitely generated nilpotent groups. It was shown in [Gui70, Wol68, Bas72] that finitely generated groups containing a nilpotent subgroup H of finite index (*virtually nilpotent groups*) have polynomial growth, and moreover, their growth is equivalent to n^r, where r is a positive integer that can be found from the formula

$$r=\sum ir(i),$$

where $r(i)$ is the rank of the abelian group $\gamma_i(H)/\gamma_{i+1}(H)$. See a proof of this result in [Man12, Theorem 4.2].

J. Milnor conjectured in [Mil68b] that all groups of polynomial growth are virtually nilpotent. He asked at the same time whether all finitely generated groups are either of polynomial or exponential growth.

Groups of polynomial growth appeared naturally in the study of expanding endomorphisms of compact Riemannian manifolds. Here an endomorphism $f:\mathcal{M}\to\mathcal{M}$ is said to be expanding if there exist constants $C>1$ and $\lambda\in(0,1)$ such that

$$\|Df^{\circ n}(\vec{v})\|\leq C\lambda^n\|\vec{v}\|$$

for all tangent vectors $\vec{v}$ and all positive integers n. Here $Df^{\circ n}$ is the differential of the nth iteration of the map f.

It was shown in [Shu69, Shu70] that the fundamental group of a compact Riemannian manifold admitting an expanding endomorphism has polynomial growth. It was conjectured that such groups are virtually nilpotent, which, by results of M. Shub, would imply that all such Riemannian manifolds are *infra-nilmanifolds* and that the expanding map comes from an expanding automorphism of the nilpotent Lie group identified with the universal covering of the manifold.

J. Milnor's conjecture on groups of polynomial growth, and hence M. Shub's conjecture on expanding endomorphisms, was proved by M. Gromov in [Gro81a].

M. Gromov's proof is a very successful example of the geometric approach to group theory. The idea is to look at the metric space (G, d) from far away, i.e., to consider the metric spaces $(G, \frac{1}{n}d)$ and pass to some kind of a limit as n approaches infinity. One then shows that the limit is an arcwise connected, locally connected, homogeneous, finite dimensional, and locally compact metric space on which G acts by isometries. One then uses the action and the solution of Hilbert's Fifth Problem by A. M. Gleason, D. Montgomery, and L. Zippin to conclude that G is virtually nilpotent. Chapters 7 and 8 of *How groups grow* [Man12] contain an accessible and complete (modulo Hilbert's Fifth Problem and Tits' alternative for linear groups) exposition of this amazing proof.

A new proof of Gromov's theorem, not relying on the solution of Hilbert's Fifth Problem was recently found by B. Kleiner [Kle10]. Instead of using an action of G on a limit of metric spaces $(G, \frac{1}{n}d)$, B. Kleiner uses in his proof the space of Lipschitz harmonic functions to construct a linear representation of G.

The world of groups of intermediate growth, i.e., of growth strictly lower than exponential but higher than any polynomial, was discovered after the first such group was constructed by R. Grigorchuk in [Gri83]. The example was then generalized in [Gri85] to a family of groups with uncountably many different growth types.

The first example is a group of homeomorphisms of the space $\{0,1\}^{\mathbb{N}}$ of infinite binary sequences generated by transformations a, b, c, d whose action on $\{0,1\}^{\mathbb{N}}$ is defined recursively by the equalities

$$\begin{aligned} a(0w) &= 1w, & a(1w) &= 0w,\\ b(0w) &= 0a(w), & b(1w) &= 1c(w),\\ c(0w) &= 0a(w), & c(1w) &= 1d(w),\\ d(0w) &= 0w, & d(1w) &= 1b(w). \end{aligned}$$

It was shown by R. Grigorchuk that the growth function $\gamma(n)$ of this group satisfies the estimates $e^{\sqrt{n}} \preceq \gamma(n) \preceq e^{n^\beta}$, where $\beta = \log_{32} 31$. Later, these estimates were improved by Y. Leonov [Leo00] and L. Bartholdi [Bar01], who showed that $e^{n^{0.506}} \preceq \gamma(n)$ and $e^{n^{0.5157}} \preceq \gamma(n)$, respectively, and by L. Bartholdi [Bar98], who showed that $\gamma(n) \preceq e^{n^{\alpha_0}}$, where $\alpha_0 = \log 2/\log(2/\rho) \approx 0.7674$ for ρ equal to the real root of $x^3 + x^2 + x - 2$. The precise asymptotics of $\gamma(n)$ is still unknown.

It fact, for a long time no groups of intermediate growth with precise asymptotics of their growth function were known. The first such groups were constructed by L. Bartholdi and A. Erschler in [BE12]. They constructed two families of groups K_k and H_k such that

$$\gamma_{K_k}(n) \sim \exp\left(n^{1-(1-\alpha_0)^k}\right), \qquad \gamma_{H_k}(n) \sim \exp\left(\log(n) \cdot n^{1-(1-\alpha_0)^k}\right),$$

where $\alpha_0 \approx 0.7674$ is as in the previous paragraph.

The examples are constructed using permutational wreath products starting from the first Grigorchuk group of intermediate growth and from a torsion-free group of intermediate growth defined in [Gri85].

More examples of groups of intermediate growth with special restrictions on growth types were constructed in [Bri11] and [KP11].

The most general and complete result so far on groups with known intermediate growth rate is the following theorem by L. Bartholdi and A. Erschler [BE11]. Unfortunately, this result is too recent to have been included in [Man12].

Theorem 2. *Let $\eta \approx 2.4675$ be the positive root of $X^3 - X^2 - 2X - 4$. Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a function satisfying*

$$f(2R) \leq f(R)^2 \leq f(\eta R)$$

for all R large enough. Then there exists a finitely generated group with growth $\sim f$.

Note that $\alpha_0 \approx 0.7674$ is equal to $\log 2 / \log \eta$. Note also that the inequality $\gamma(2n) \leq \gamma(n)^2$ is satisfied for the growth function of any finitely generated group.

Theorem 2 provides many different examples of growth types. In particular, it shows that for every $\alpha \in [\alpha_0, 1]$, there exists a finitely generated group of growth $\sim e^{n^\alpha}$.

It is remarkable that even when such general theorems as Theorem 2 are now known, all examples of groups of intermediate growth (in particular, Theorem 2) are based on the Grigorchuk groups from [Gri85], or are very similar to them [FG91, BP06]. In some sense no essentially new examples of groups of intermediate growth have been constructed since 1985.

In particular, it is not known if there exists a finitely presented group of intermediate growth. This is an interesting question in relation with Theorem 1, since the class of finitely presented groups coincides with the class of fundamental groups of compact manifolds.

Similarly, it is not known if there exists a simple group of intermediate growth. The Grigorchuk groups are residually finite (and hence have many finite quotients), while all known methods of constructing simple groups from the Grigorchuk groups produce groups of exponential growth.

Another open problem, posed by R. Grigorchuk already in [Gri91], is about existence of a group of growth greater than any polynomial, but lower than $e^{\sqrt{n}}$. All known examples of groups of intermediate growth have growth rate strictly greater than $e^{\sqrt{n}}$. Moreover, it was proved by R. Grigorchuk in [Gri90] that if G is a finitely generated group that can be approximated by finite p-groups (i.e., for every $g \in G \setminus \{1\}$ there exists an epimorphism of G onto a finite p-group such that the image of g is nontrivial), then either G is virtually nilpotent (and thus has polynomial growth) or $e^{\sqrt{n}} \preceq \gamma_G(n)$. This fact can be generalized to groups that can be approximated by nilpotent groups [LM91]. See also [DdSMS99, Interlude E] and an overview of questions and results related to gaps in the set of possible growth types in [Gri12].

How groups grow by Avinoam Mann is the first book devoted to the subject of growth of groups. It gives a fairly complete overview of the main results in the area, and is accessible even for beginners in group theory. The two main theorems, Gromov's theorem on groups of polynomial growth and Grigorchuk's example of a group of intermediate growth, are described in great detail. Some additional topics, such as amenability and conjugacy growth, are also discussed.

The book starts with some basic Group Theory results used later in the book. In Chapter 3 groups of growth $\gamma(n) \sim n$ are described. Of course, it follows from Gromov's theorem and the formula for the degree of polynomial growth of virtually nilpotent groups that such groups contain a subgroup of finite index isomorphic

to the infinite cyclic group $\mathbb{Z}$. However, this result is much more elementary than Gromov's theorem, and its proof is a good introduction into the subject of growth of groups.

The next six chapters are devoted to Gromov's theorem on groups of polynomial growth and its ramifications. The first half contains the proof of the direction which was known before Gromov's paper—that nilpotent groups have polynomial growth (including the formula for the degree of growth), and that linear and soluble groups of polynomial growth are virtually nilpotent. The fact about linear groups follows from the fact about soluble groups due to Tits' alternative: every finitely generated linear group is either virtually solvable or contains a free subgroup.

In fact, soluble groups that are not virtually nilpotent have *uniformly exponential growth*, which is defined as follows. If $\gamma_{G,S}(n)$ is the growth function of a finitely generated group G defined for a finite generating set S of G, then we obviously have

$$\gamma_{G,S}(n_1 + n_2) \le \gamma_{G,S}(n_1)\gamma_{G,S}(n_2)$$

for all positive integers n_1, n_2. It follows that the limit $a_S = \lim_{n\to\infty} \sqrt[n]{\gamma_{G,S}(n)}$ exists. The group G is of exponential growth if and only if $a_S > 1$. If the group is not of exponential growth, then $a_S = 1$.

We say that G is of *uniformly exponential growth* if infimum of a_S over all finite generating sets S of G is greater than 1. By a result of D. Osin [Osi03], every soluble group of exponential growth has uniformly exponential growth; see [Man12, Section 5.2]. More generally, D. Osin has proved in [Osi04] that every *elementary amenable* group of exponential growth has uniformly exponential growth; see also [Man12, Section 12.1].

The question of existence of groups of exponential but not uniformly exponential growth was asked by M. Gromov in [Gro81b]. The first example of a group of nonuniform exponential growth was constructed by J. Wilson in [Wil04b, Wil04a].

There exists a natural relation between groups of nonuniform exponential growth and groups of intermediate growth. For every natural number k the set of all groups with k marked generators $s_1, s_2, \dots, s_k$ has a natural topology; see [Gri85, Cha00]. For instance, one can naturally identify this set with the set of normal subgroups of the free group F_k on k generators, and then consider it as a subset of the topological space 2^{F_k}.

Let G be a finitely generated group of exponential growth. Suppose that we can find a sequence of generating sets $S_i = \{s_{1i}, s_{2i}, \dots, s_{ki}\}$ of the same cardinality such that the sequence $(G, s_{1i}, s_{2i}, \dots, s_{ki})$ of groups with marked generators converges to a group of subexponential growth H. Then H has intermediate growth, and G has nonuniformly exponential growth. The latter follows from the fact that the map $(G, S) \mapsto a_S$ is upper semicontinuous on the space of marked finitely generated groups. See the papers [Bar03, Nek10], where this relation is explored explicitly.

In some cases uniform lower estimates for the exponent a_S of exponential growth can be found not just for one group, but for all groups in some class. Such estimates were obtained by M. Bucher, P. de la Harpe, R. Grigorchuk, J. O. Button, and A. Mann for free products, one-relator groups, and groups with positive deficiency (groups with a presentation with more generators than relators). These results are described in Chapter 16 of the book under review.

Chapter 7 describes *asymptotic cones* of finitely generated groups, which is the limit of the metric spaces $(G, d/n)$, used in the proof of Gromov's theorem. The

Gromov construction was interpreted in terms of nonstandard analysis (in particular, ultrafilters) by L. van den Dries and A. J. Wilkie [vW84], and it was generalized by M. Gromov to arbitrary metric spaces.

Chapter 8 then applies this construction to prove the M. Gromov's theorem on groups of polynomial growth. In the next chapter, a theorem on *infinitely generated* groups of uniformly polynomial growth, due to the author of the book, is proved.

The next four chapters of the book are devoted to groups of intermediate growth. It starts with a detailed analysis of the first Grigorchuk group. In particular, the original estimates on its growth due to R. Grigorchuk are proved. In Chapter 11 more examples are discussed: the uncountable family of Grigorchuk groups from [Gri85], groups acting on rooted trees and defined by finite automata, and examples of L. Bartholdi and A. Erschler.

As was mentioned above, it is natural to consider growth of groups modulo an equivalence relation. However, it makes sense, in some cases, to try to find the sequence $\gamma_G(n)$ or the generating function $\sum \gamma_G(n)x^n$ explicitly. This can be done only in some special cases. Such examples, and a general discussion of the growth generating functions can be found in Chapters 14 and 15 of this book.

The book ends with discussion of conjugacy growth, i.e., the number of conjugacy classes of a group G that intersect the set of elements of length at most n. The last chapter of the book contains a list of thirteen open research problems.

How groups grow is an excellent introduction to growth of groups for everybody interested in this subject. It also touches a variety of adjacent subjects (such as amenability, isoperimetric inequalities, groups generated by automata, etc.) It is written in a very accessible style, with very clear exposition of all main results.

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