Random perturbations of dynamical systems, 3rd ed., by Mark I. Freidlin
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Evolution of physical objects is often described by ordinary differential equations
having the form,

$$(1) \qquad \frac{dX_x(t)}{dt} = B(X_x(t)), \ X_x(0) = x,$$

where the solution $X_x(t)$ runs in $\mathbb{R}^d$ or on a smooth manifold. To ensure existence
and uniqueness in the equation (1), the right-hand side B is usually supposed to
be Lipschitz continuous. Then we arrive at a group of diffeomorphisms F^t, acting
by $F^t x = X_x(t)$, called a dynamical system or a flow to amplify the continuous
time here, taking into account that the theory of dynamical systems widely studies
also the discrete time case, i.e., iterates of one transformation. A description of
a physical evolution in the form (1) can be viewed as an idealization since real
motions are subject to various perturbations which can be considered random.

It seems that the problem of studying small random perturbations of dynamical
systems was first formulated explicitly by Pontrjagin, Andronov, and Vitt [9] in
1933. As a particular model of random perturbations we can consider a stochastic
process $X^\varepsilon(t) = X_x^\varepsilon(t)$ solving the equation

$$(2) \qquad X_x^\varepsilon(t) = x + \int_0^t B(X_x^\varepsilon(t))ds + \varepsilon w_t,$$

where w_t is the standard Brownian motion called also the Wiener process. More
generally, one of most studied models of random perturbations is represented by a
diffusion Markov process $X_x^\varepsilon(t)$, $X_x^\varepsilon(0) = x$, whose semigroup of transition operators
P_t^ε is generated by $L^\varepsilon = \varepsilon^2 L + (B, \nabla)$ where $L = \frac{1}{2}(a(x)\nabla, \nabla) + (b(x), \nabla)$ is a second
order elliptic (partial) differential operator. Usually, a more convenient way to deal
with $X_x^\varepsilon(t)$ is to represent it as a solution of the Itô stochastic differential equation

$$(3) \qquad dX_x^\varepsilon(t) = \varepsilon\sigma(X_x^\varepsilon(t))dw_t + B(X_x^\varepsilon(t))dt + \varepsilon^2 b(X_x^\varepsilon(t))dt,$$

where $\sigma(x)\sigma^*(x) = a(x)$. We arrive at the particular case (2) when $L = \Delta$ is the
Laplacian and $b \equiv 0$. As in (1), Lipschitz condition on coefficients in (3) ensure
existence and uniqueness of solutions there. It is clear that during a fixed bounded
time, $X_x^\varepsilon(t)$, with probability close to one, stays close to the unperturbed motion
$X_x(t)$ but when we are interested in events having very small probabilities or taking
place after a very long time, then randomness starts playing a crucial role.

Let φ_t, $t \in [0, T]$ be a continuous curve in the space $\mathbf{M}$ (either $\mathbb{R}^d$ or a Rie-
mannian manifold) where both $X(t)$ and $X^\varepsilon(t)$ live and we want to estimate the
probability that X^ε stays in a small neighborhood of φ, i.e., say $\rho_{0T}(X^\varepsilon, \varphi) \leq \delta$
where $\rho_{0T}(\varphi, \psi) = \sup_{0 \leq t \leq T} \rho(\varphi_t, \psi_t)$ and ρ is the distance on $\mathbf{M}$. The basic result
obtained by Freidlin and Wentzell in their seminal 1970 paper [10] says that the

above probability can be described by means of the functional (which they called an action functional) defined for absolutely continuous curves by

$$(4) \qquad\qquad S_{0T}(\varphi) = \int_0^T L(\varphi_t, \dot\varphi_t) dt.$$

Here $L(x, \beta) = \frac{1}{2}\big((a(x))^{-1}(\beta - B(x)), (\beta - B(x))\big)$ and $(\cdot, \cdot)$ denotes the inner product. If φ is not absolutely continuous, then $S_{0T}(\varphi)$ is set to be ∞. Now, the probabilities in question can be described in the following way. For any $s, \delta, \gamma > 0$,

$$(5) \qquad\qquad P\{\rho_{0T}(X^\varepsilon, \varphi) < \delta\} \geq \exp\{-\varepsilon^{-2}(S_{0T}(\varphi) + \gamma)\},$$

and for $\Phi(s) = \{\varphi : S_{0T}(\varphi) \leq s\}$,

$$(6) \qquad\qquad P\{\rho_{0T}(X^\varepsilon, \Phi(s)) \geq \delta\} \leq \exp\{-\varepsilon^{-2}(s - \gamma)\}.$$

These estimates have the form of the lower and upper bounds of the theory of large deviations. We see from (5) and (6) that on the time interval $[0, T]$, the perturbed motion X_x^ε stays close to X_x with overwhelming probability. Still with exponentially small in ε^{-2} probability other paths of X_x^ε are possible, as well.

It turns out that (5) and (6) enable us also to study the behavior of X^ε on very long time intervals. Namely, define

$$(7) \qquad\qquad V(x, y) = \inf_{\varphi, T \geq 0} \{S_{0T}(\varphi) : \varphi_0 = x, \ \varphi_T = y\},$$

which is a Lipschitz continuous function. Suppose that D is a relatively compact domain in $\mathbf{M}$ with a smooth boundary ∂D such that the flow F^t attracts $D \cup \partial D$ to a fixed point $\mathcal{O} \in D$ and at each $x \in \partial D$ the vector $B(x)$ points strictly inside D. Suppose that $\min_{y \in \partial D} V(\mathcal{O}, y) = V(\mathcal{O}, y_0)$ and that such y_0 is unique on ∂D. Let $\tau_x(D) = \inf\{t \geq 0 : X_x^\varepsilon(t) \in \partial D\}$. Then for any $x \in D$ and $\delta > 0$,

$$(8) \qquad\qquad \lim_{\varepsilon \to 0} P\{\rho(X_x^\varepsilon(\tau_x(D)), y_0) > \delta\} = 0,$$

i.e., with overwhelming probability X_x^ε exits from D near y_0. This result from [10] has a partial differential equations interpretation. Namely, let u^ε be the solution of the Dirichlet problem $L^\varepsilon u^\varepsilon = 0$ in D with continuous boundary conditions $u^\varepsilon|_{\partial D} = g$. Then in the above circumstances $\lim_{\varepsilon \to 0} u^\varepsilon(x) = g(y_0)$ for any $x \in D$. This result was new at the time and it was difficult to approach it by pure partial differential equations methods, which prompted development of the theory of viscosity solutions.

Among important questions on random perturbations of dynamical systems raised in [9] and later repeatedly advertized by Kolmogorov was the problem on stochastic stability of invariant measures of dynamical systems. Namely, suppose that both the flow F^t and its diffusion random perturbation $X^\varepsilon(t)$ live on compact Riemannian manifold $\mathbf{M}$. Many interesting classes of dynamical systems, especially so-called chaotic systems, possess abundance of invariant measures, i.e., probability measures μ such that $F^t\mu = \mu$, while a nondegenerate diffusion $X^\varepsilon(t)$ has a unique invariant measure μ_ε, i.e., $\int P_t^\varepsilon f d\mu_\varepsilon = \int f d\mu_\varepsilon$ for any bounded Borel function f. It is not difficult to see that all (weak) limit points of μ_ε as $\varepsilon \to 0$ are invariant measures of the flow F^t and, in particular, if the latter has a unique invariant measure μ (i.e., it is uniquely ergodic), then $\mu_\varepsilon \Rightarrow \mu$ as $\varepsilon \to 0$. On the other hand, if F^t possess a lot of invariant measures and still $\mu_\varepsilon \Rightarrow \mu_0$ as $\varepsilon \to 0$ for some particular μ_0, then it is natural to call the latter the most stable (and thus having physical meaning) invariant measure of the flow F^t. Initial results in this direction were obtained by Khasminskii (see [4]). Freidlin and Wentzell showed

in [10] that a part of the above problem can be studied relying on the functions $V(x,y)$ defined in (7). Namely, let the ω-limit set of the flow F^t consist of a finite number of disjoint compacts $K_1, \ldots, K_\ell$ such that $V(x,y) = V(y,x) = 0$ if and only if x and y belong to one compact. Then $V(x,y) = V_{ij}$ for $x \in K_i$ and $y \in K_j$ depends only on i and j. It was shown in [10] that the limiting as $\varepsilon \to 0$ behavior of the vector $(\mu_\varepsilon(K_1), \ldots, \mu_\varepsilon(K_\ell))$ can be described via invariant measures of certain Markov chains with ℓ states and transition probabilities determined by V_{ij}, $i, j = 1, \ldots, \ell$. If the flow F^t is topologically transitive on $\mathbf{M}$, i.e., it has a dense orbit, then $V(x,y) = 0$ for any pair $x, y \in \mathbf{M}$, which means that $\ell = 1$ and the above approach gives nothing. Even when $\ell > 1$, this method cannot give any information about μ_ε inside of each compact K_i. Then other machinery should be applied and, in the case of Anosov flows or, more generally, Axiom A flows, the problem was solved by the reviewer (see [6]) showing that μ_ε tends weakly as $\varepsilon \to 0$ to the so-called Sinai–Ruelle–Bowen invariant measure. A discrete time version of the above Freidlin–Wentzell theory appeared in the reviewer's paper [7].

Freidlin showed in [1] that a machinery similar to the above large deviations can be applied to some problems in the theory of averaging. The averaging setup can be viewed as a perturbation of a family of dynamical systems. Namely, consider a family of flows $F_x^t y = Y_{x,y}(t)$ determined by ordinary differential equations

$$(9) \qquad \frac{dY_{x,y}(t)}{dt} = b(x, Y_{x,y}(t)), \ Y_{x,y}(t) = y$$

parametrized by $x \in \mathbb{R}^d$. We may think about the flow $\Phi^t(x,y) = (x, F_x^t y)$ as describing an idealized physical motion where parameters $x = (x_1, \ldots, x_d)$ play the role of constants of motion. In a real system these parameters evolve slowly and we arrive at a "fully coupled" system

$$(10) \qquad \frac{dZ_{x,y}^\varepsilon(t)}{dt} = \varepsilon B(Z_{x,y}^\varepsilon(t), Y_{x,y}^\varepsilon(t)), \qquad \frac{dY_{x,y}^\varepsilon(t)}{dt} = b(Z_{x,y}^\varepsilon(t), Y_{x,y}^\varepsilon(t)),$$

where $Z_{x,y}^\varepsilon(0) = x$, $Y_{x,y}^\varepsilon(0) = y$. For obvious reasons Z^ε and Y^ε are called the slow and the fast motions, respectively. Sometimes, by a change of variables we can "decouple" (10), arriving at the equation

$$(11) \qquad \frac{dZ_{x,y}^\varepsilon(t)}{dt} = \varepsilon B(Z_{x,y}^\varepsilon(t), F^t y), \quad Z_{x,y}^\varepsilon(0) = x,$$

where now the fast motion does not depend anymore on the slow one. In place of the flow F^t in (11) we can consider a stochastic process ξ_t, which is the case studied both in [1] and in Chapter 7 of the book under review. So we proceed with the description of results in this "uncoupled" situation

$$(12) \qquad \frac{dZ_x^\varepsilon(t)}{dt} = \varepsilon B(Z_x^\varepsilon(t), \xi_t), \quad Z_x^\varepsilon(0) = x,$$

refering the reader to the study of the more difficult fully coupled case both in deterministic and stochastic setups to the reviewer's work [8]. The classical averaging principle yields that if $B(z,y)$ is Lipschitz continuous in z and the limit

$$\bar{B}(z) = \lim_{t \to \infty} \frac{1}{t} \int_0^t B(z, \xi_s) ds$$

exists, then the averaged motion $\bar{X}$ solving the equation

$$\frac{d\bar{X}_x(t)}{dt} = \bar{B}(\bar{X}_x(t)), \ \bar{X}_x(0) = x$$

approximates the time changed slow motion $X^\varepsilon(t) = Z^\varepsilon(t/\varepsilon)$ on any bounded time interval $[0, T]$, i.e., $\lim_{\varepsilon \to 0} \sup_{0 \le t \le T} |X_x^\varepsilon(t) - \bar{X}_x(t)| = 0$. Suppose that for any piecewise constant $\mathbb{R}^d$-valued functions φ_s and α_s on $[0, T]$,

$$(13) \qquad \lim_{\varepsilon \to 0} \varepsilon \ln E \exp\left(\frac{1}{\varepsilon} \int_0^T (\alpha_s, b(\varphi_s, \xi_{s/\varepsilon})) ds\right) = \int_0^T H(\varphi_s, \alpha_s) ds$$

for some function H on $\mathbb{R}^d \times \mathbb{R}^d$. Then $H(x, \alpha)$ is continuous and convex in α and its Legendre transform

$$(14) \qquad\qquad L(z, \beta) = \sup_\alpha ((\alpha, \beta) - H(z, \alpha))$$

is convex and lower semicontinuous. Now introduce the "action" functional by (4) only with L given by (14). If $H(x, \alpha)$ is differentiable in α, then according to [1] the slow motion X^ε satisfies the large deviations estimates (5) and (6) with this new "action" functional. Condition (13) can be verified (and the function H identified) for some stochastic processes ξ_t such as nondegenerate finite states Markov chains with continuous time (the case considered in [1]), nondegenerate diffusions on compact manifolds (more generally, for random evolutions), and for some classes of dynamical systems (see [8]). Assuming special structure of the flow $\bar{F}^t$ generated by the averaged vector field $\bar{B}$, we can obtain results about very long time behavior of $X^\varepsilon(t)$ similar to the random perturbations case described above. This concerns, in particular, exits of $X^\varepsilon(t)$ from a domain D attracted by the flow $\bar{F}^t$ to a fixed point $\mathcal{O} \in D$. A word of caution is due here: these results described in Section 6 of Chapter 7 in the book under review hold true if the function V defined by (7) (with the "action" functional obtained by L from (14)) is lower semicontinuous. This is not so obvious here as in the random perturbations case and requires some extra assumptions (see [8]). Furthermore, at the end of the same section there is a discussion on invariant measures of $X^\varepsilon(t)$, which is not quite clear since it is not a Markov process, in general. Thus this section should be read with a "grain of salt".

Up to now we discussed the theory developed in [10] and [1], which was the base for the whole first edition of the book as well as for Chapters 1–7 of the second and the third editions. The main addition in the latter editions is the study of random perturbations of Hamiltonian systems. Thus we return to the setup at the beginning of this review where now B in (1) is a vector field of the form $B = \bar{\nabla} H = (\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q})$ with a Hamiltonian $H = H(p, q)$. The unperturbed motion $X^0(t) = X(t)$ stays all the time on one surface of constant Hamiltonian (energy level) but the perturbed process $X^\varepsilon(t)$ can drift slowly across these surfaces. In the two dimensional case, i.e., for systems with one degree of freedom, when energy levels in a region form a family of closed orbits, the problem was studied by Khasminskii in [5], who showed that the drift across these orbits by the time rescaled process $Z^\varepsilon(t) = X^\varepsilon(t/\varepsilon^2)$ can be asymptotically described by a one dimensional diffusion process with characteristics obtained by averaging corresponding characteristics of Z^ε along closed orbits above. The motion across energy levels can be conveniently represented by the process $H(Z^\varepsilon(t))$, and the above result yields that the latter converges in distribution to a one dimensional diffusion. Suppose now that the Hamiltonian has several critical points, then the energy levels will have more complicated structures. Let, for instance, the Hamiltonian has two local minima and one local maximum so that the Hamiltonian flow has three families

of closed orbits separated by an ∞-shape curve (consisting of a saddle fixed point and two orbits). It turns out that in this and more general two dimensional cases the motion across constant energy curves can be asymptotically described by a diffusion on a graph obtained by identifying points on each such curve so that each edge will correspond to one connected component filled entirely by closed orbits. This diffusion is well defined in the interior of each edge of the graph which should be supplemented by appropriate boundary conditions at vertices to ensure correct transition laws from edge to edge. In dimension higher than two the global geometry of constant energy surfaces may become quite complicated and the description of the slow evolution of $X^\varepsilon(t)$ across them was obtained so far only in special cases described in Chapter 9 of the book under review. This part of the book is based on several more recent papers by Freidlin and Wentzell, in particular, [2], [3] and others.

The last two chapters are devoted to brief description of other related asymptotic problems, such as stability under random perturbations, asymptotics of transition densities for processes with small diffusion, reaction-diffusion equations with small parameters, and random perturbations of infinite dimensional semiflows generated by reaction-diffusion equations.

The book under review is neither a textbook nor a reference book devoted to a classical topic, but rather a research monograph that comprises the work of the authors (sometimes together with others) for about 40 years. It is quite rare that such a book would go already through its third edition, which demonstrates the high quality, originality, and importance of the theory developed there.

References

[1] M. I. Freĭdlin, *The averaging principle and theorems on large deviations*, Uspekhi Mat. Nauk **33** (1978), no. 5(203), 107–160, 238 (Russian). MR511884 (81k:60059)

[2] Mark I. Freidlin and Alexander D. Wentzell, *Random perturbations of Hamiltonian systems*, Mem. Amer. Math. Soc. **109** (1994), no. 523, viii+82. MR1201269 (94j:35064)

[3] M. I. Freidlin and A. D. Wentzell, *Diffusion processes on an open book and the averaging principle*, Stochastic Process. Appl. **113** (2004), no. 1, 101–126, DOI 10.1016/j.spa.2004.03.009. MR2078539 (2005e:60178)

[4] R. Z. Khasminskii, *Principle of averaging for parabolic and elliptic differential equations and for Markov processes with small diffusion*, Theor. Probab. Appl., 8 (1963), 1–21.

[5] R. Z. Khasminskii, *On the averaging principle for Itô stochastic differential equations,* Kibernetika (Prague), **4** (1968), 260–279 (in Russian).

[6] Yuri Kifer, *Random perturbations of dynamical systems*, Progress in Probability and Statistics, vol. 16, Birkhäuser Boston Inc., Boston, MA, 1988. MR1015933 (91e:58159)

[7] Yuri Kifer, *A discrete-time version of the Wentzell-Freidlin theory*, Ann. Probab. **18** (1990), no. 4, 1676–1692. MR1071818 (91i:60077)

[8] Yuri Kifer, *Large deviations and adiabatic transitions for dynamical systems and Markov processes in fully coupled averaging*, Mem. Amer. Math. Soc. **201** (2009), no. 944, viii+129. MR2547839 (2011c:60089)

[9] L. S. Pontryagin, A. A. Andronov, and A. A. Vitt, *On statistical consideration of dynamical systems*, J. Experiment. Theor. Phys., 3 no.1 (1933), 165–180 (in Russian).

[10] A. D. Ventcel′ and M. I. Freĭdlin, *Small random perturbations of dynamical systems*, Uspehi Mat. Nauk **25** (1970), no. 1 (151), 3–55 (Russian). MR0267221 (42 #2123)

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