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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
MLADEN BESTVINA

MR0648524 (83h:57019) 57N10; 20H15, 30F40, 57M35, 57M40, 57S17

Thurston, William P.

Three-dimensional manifolds, Kleinian groups and hyperbolic geometry.

Bulletin of the American Mathematical Society. New Series **6** (1982), *no.* 3, 357–381.

In this survey article the author announces his two now famous results: (1) The interior of a Haken 3-manifold M^3 admits a hyperbolic structure (=complete hyperbolic metric) if and only if M^3 contains no essential, non-boundary-parallel torus and is not homeomorphic to the I-bundle over the Klein bottle. (2) Suppose $L \subset M^3$ is a link (=system of differentiable embedded circles) such that the complement $M^3 - L$ admits a hyperbolic structure. Then almost all manifolds obtained from M^3 by Dehn surgery along L have hyperbolic structures.

After having stated these results, the author discusses some of their consequences: (a) The Smith conjecture is true. (b) Hyperbolic manifolds with finite volume (i.e. either closed manifolds or those whose boundary consists of tori only) are determined by their fundamental groups alone. (c) The mapping class group of hyperbolic manifolds is finite (the last two statements for all Haken 3-manifolds were proved by the reviewer without using hyperbolic structures for 3-manifolds [*Homotopy equivalences of 3-manifolds with boundaries*, Lecture Notes in Math., 761, Springer, Berlin, 1979; MR0551744 (82c:57005)]). (d) The class of hyperbolic 3-manifolds is strictly larger than the class of all atoroidal Haken manifolds. (e) The fundamental group of Haken manifolds is residually finite. (f) The order type of the set of all volumes of hyperbolic 3-manifolds is ω^ω.

The author mentions some of the ingredients obtained from Kleinian group theory (e.g. limits of Kleinian groups) for the proof of the existence theorem (1) above, and gives a conjectural picture of 3-manifolds as built up by geometric pieces (a picture which would include the Poincaré conjecture). The paper ends with a list of some open questions from the general area of hyperbolic 3-manifolds and surfaces.

From MathSciNet, October 2013

Klaus Johannson

MR1431827 (97m:57021) 57N10; 57M05, 57M10, 57N35

Cooper, D.; Long, D. D.; Reid, A. W.

Essential closed surfaces in bounded 3-manifolds.

Journal of the American Mathematical Society **10** (1997), *no.* 3, 553–563.

A long-standing question in the theory of 3-manifolds concerns the existence of subgroups of the fundamental group which are closed surface groups. That is, when M is a compact connected irreducible 3-manifold with infinite fundamental group, is there a map $i\colon S \to M$ of a closed surface into M such that $i_\#\colon \pi_1(S) \to \pi_1(M)$ is injective? When M has boundary, one requires that $i_\#(\pi_1(S))$ not be conjugate into the fundamental group of a component of ∂M. Such a map i is called essential. The authors give a complete resolution of this question when M has nonempty incompressible boundary, by proving that either M is covered by a product $F \times I$ or M contains an essential closed surface.

The paper is based on a clever idea for producing an essential closed surface, which the authors credit to a recent paper by B. Freedman and M. H. Freedman ["Haken finiteness for bounded 3-manifolds, locally free groups and cyclic covers", Preprint; per bibl.]. In the case when all components of M are tori, it may be assumed that its interior has a complete hyperbolic structure, since otherwise M contains essential tori. One first passes to a finite covering space of M that has at least three boundary components. For homological reasons, this cover must contain an imbedded incompressible surface that meets at most two boundary components. In the infinite cyclic covering determined by this surface, one takes two distant translates of the lifted surface, and tubes them together with annuli in the boundary. This yields an imbedded surface which may be compressible, but can be made incompressible by performing surgeries. Homological arguments show that if the translates were sufficiently distant, the resulting incompressible surface does not consist entirely of 2-spheres and surfaces parallel to boundary components. Applying the projection maps of the coverings to an essential component yields the essential map into M. If the boundary of M does not consist only of tori, the same basic idea can be used, but instead of tubing together the translates with annuli, it is necessary to use a portion of the boundary of the infinite cyclic covering to construct the initial closed surface. For this case, the argument that an essential surface remains after the compressions is rather more delicate.

Since the closed incompressible surface is imbedded in the infinite cyclic covering, it descends to an imbedded surface in a finite covering of M. This leads to another application. By finding two such nonparallel incompressible surfaces in a finite cover of M, the authors prove that either M is covered by a product $T^2 \times I$ or a finite-index subgroup of $\pi_1(M)$ maps onto a free group of rank two. The same kind of technique proves the existence of nonperipheral homology in finite covers of M. That is, if the interior of M has a complete hyperbolic structure, then given any integer K, either M is covered by a product $F^2 \times I$, or there is a finite sheeted cover $\widetilde{M} \to M$ such that $H_2(\widetilde{M})/j_*(H_2(\partial\widetilde{M}))$ has rank at least K, where $j\colon \partial\widetilde{M} \to \widetilde{M}$ is the inclusion.

The paper is very well written and clear.

From MathSciNet, October 2013

Darryl McCullough

MR2377497 (2009a:20061) 20F36; 20F55, 20F67

Haglund, Frederic; Wise, Daniel T.

Special cube complexes.

Geometric and Functional Analysis **17** (2008), *no.* 5, 1551–1620.

In the article under review the authors prescribe a set of conditions on cube complexes that guarantee any cube complex satisfying these conditions admits a local isometry to a finitely generated right-angled Artin group.

More precisely, the authors say that a cube complex X is A-special provided that

(1) every hyperplane of X embeds (does not "intersect itself"),
(2) every hyperplane is two-sided,
(3) no hyperplane directly self-osculates, and
(4) no two hyperplanes inter-osculate (both intersect and osculate).

(The "A" here stands for "Artin"; the authors also deal with C-special complexes, relating them to Coxeter complexes.) For such complexes X there is a local isometry $X \to A$, where A is the standard 2-complex of a right-angled Artin group.

The proof (spread across Sections 3 and 4 of the article) that these conditions are sufficient to guarantee the existence of the desired map is a straightforward combinatorial argument; the conditions have been chosen precisely in order to enable one to "type" the walls of X using some Artin group A.

As an immediate corollary of the main result, the fundamental groups of special complexes are linear. Moreover, there are other group theoretic consequences. For instance, the following are shown:

Theorem 1.3. If X is a compact special cube complex and its fundamental group $\pi_1 X$ is word-hyperbolic, then every quasiconvex subgroup is separable.

Theorem 1.5. If X is a compact special cube complex, then its fundamental group $\pi_1 X$ is residually torsion-free nilpotent (and is therefore indicable, assuming it is nontrivial).

This article is lengthy but entirely self-contained, including brief introductions to the needed theoretical underpinnings of cube complexes and combinatorial and geometric group theory.

From MathSciNet, October 2013

Patrick Bahls

MR2399130 (2009b:57033) 57M50; 57N10

Agol, Ian

Criteria for virtual fibering.

Journal of Topology **1** (2008), *no.* 2, 269–284.

An old conjecture of Thurston is that a hyperbolic 3-manifold has a finite sheeted covering which is a surface bundle over the circle. This is a provocative conjecture and it has attracted a good deal of attention by a variety of authors, notably by Leininger and Walsh, each of whom exhibited some infinite families of examples.

This paper introduces a new residual condition which is technical to state, but can be roughly thought of as (a little stronger than) a descending cofinal family of normal subgroups G_i where the quotients G/G_i are all soluble. The condition is denoted by RFRS: residually finite rational soluble. Commonly occurring groups which are at least virtually RFRS include surface groups, reflection groups, right angled Artin groups and arithmetic hyperbolic groups defined by a quadratic form.

The key result is that if M is a compact irreducible orientable 3-manifold with Euler characteristic zero and with fundamental group which is RFRS, then given any nonzero class in $H^1(M)$, there is a finite sheeted covering for which the pull back form lies in a fibred face. The method of proof is to begin with an embedded surface representing a nonzero (nonfibre) class and to reduce the complexity of the nonproduct piece of the JSJ decomposition using the RFRS condition.

The results can be assembled to produce many new examples of manifolds which virtually fibre, including the Seifert-Weber dodecahedral space and the Bianchi groups.

From MathSciNet, October 2013

Darren D. Long

MR2821442 (2012k:20084) 20F67; 20F65, 57M07

Sageev, Michah; Wise, Daniel T.

Periodic flats in CAT(0) **cube complexes.**

Algebraic & Geometric Topology **11** (2011), *no.* 3, 1793–1820.

The "flat torus" theorem asserts that if $H \cong \mathbb{Z} \times \mathbb{Z}$ is a subgroup of a group G acting "geometrically" on a CAT(0) space X, then there is an isometrically embedded plane in X stabilized by H (i.e. a "periodic flat plane"). An important open question asks: If there is a flat plane in X, then must G contain a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$? M. Gromov [in *Essays in group theory*, 75–263, Math. Sci. Res. Inst. Publ., 8, Springer, New York, 1987; MR0919829 (89e:20070)] proposed (via aperiodic Wang tiles) that there may be a negative example, even for 2-dimensional CAT(0) cube complexes. Still, there are positive results in many interesting settings. The main theorem of this paper is such a result.

If X is a CAT(0) cube complex, a "facing triple" is a set of three disjoint hyperplanes H_1, H_2, H_3 such that no H_i separates the other two in X. Let G act on the CAT(0) cube complex X. Then G has "cyclic facing triples" if for each facing triple H_1, H_2, H_3 the group $\bigcap_{i=1}^{3} \mathrm{Stabilizer}\, H_i$ is either finite or virtually cyclic.

Main Theorem. Let X be a cocompact cube complex with cyclic facing triples. If X contains a flat plane, then X contains a periodic flat plane.

This result generalizes a theorem of L. Mosher [Topology **34** (1995), no. 4, 789–814; MR1362788 (97i:57017)] where the result was proven when X is a 3-dimensional manifold with a nonpositively curved cubing (in which case it follows that X has cyclic facing triples). The result also generalizes a result of D. T. Wise [*Non-positively curved squared complexes: Aperiodic tilings and non-residually finite groups*, Ph.D. thesis, Princeton Univ., 1996; MR2694733], where the theorem was proven in the case that X is 2-dimensional.

The proof of the main theorem is fundamentally geometric. Induction on the dimension of the cube complex allows the authors to assume that hyperplanes are all δ-hyperbolic. Key ideas in the proof include a clever analysis of angles between flats and hyperplanes. The technically difficult work comes in understanding how the intersection of all half spaces that contain a flat approximates the flat and how this intersection contains periodic strips that can be pieced together to form a periodic flat.

From MathSciNet, October 2013

Michael L. Mihalik