BOOK REVIEWS

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Large networks and graph limits, by László Lovász, American Mathematical Society Colloquium Publications, 60, American Mathematical Society, Providence, RI, 2012, xiv+475 pp., ISBN 978-0-8218-9085-1, US $99.00

A classical mathematical theme is to estimate the growth of various parameters for structures whose size grows indefinitely. Number theory, analysis, geometry, and statistics (for example) all consider this as their realm, as the natural historical habitat of their investigations. The analysis of large objects is, after all, typical of all higher mathematical reasoning.

In this way combinatorics and graph theory also investigate the asymptotic behaviour of structural parameters (such as chromatic number, clique number, connectivity, hamiltonicity, to name just a few). This, closely related to the theory of random graphs and random processes, builds on the pioneering work of P. Erdős, A. Rényi, E. Gilbert, and others, which in the last decades has become one of the fastest growing areas of all discrete mathematics.

The book under review covers a qualitatively new level of investigation, which can be summarized by a simple question: Instead of studying individual parameters, say, of larger and larger graphs, can one devise a single object that would encode the asymptotic behaviour of "all essential" properties?

Viewing on the one side the huge variety of even the simplest finite combinatorial models (such as graphs) and on the other side the sheer detail and the wide spectrum of properties that are investigated, it is surprising that such a project could be realistic and have a positive and satisfactory answer. This book documents exactly this. And it is not just a documentation. Behind the rigorous mathematics lies a story that reads like a fairy tale. This reviewer cannot resist quoting the opening section of the introduction of the book under review. The author writes:

> Within a couple of months in 2003, in the Theory Group of Microsoft Research in Redmond, Washington, three questions were asked by three colleagues. Michael Freedman, who was working on some very interesting ideas to design a quantum computer based on methods of algebraic topology, wanted to know which graph parameters (functions on finite graphs) can be represented as partition functions of models from statistical physics. Jennifer Chayes,

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> who was studying internet models, asked whether there was a notion of "limit distribution" for sequences of graphs (rather than for sequences of numbers). Vera T. Sós, a visitor from Budapest interested in the phenomenon of quasirandomness and its connections to the Regularity Lemma, suggested to generalize results about quasirandom graphs to multitype quasirandom graphs. It turned out that these questions were very closely related, and the ideas which we developed for the answers have motivated much of my research for the next years.

This motivation is also a *leitmotiv* of the whole book, and the ordering of chapters reflects this. In fact, in many instances the ordering mirrors the chronology of discovery (which of course cannot be strictly kept in ongoing research). The path of the discovery is suggestive.

The book has five main parts, each of them subdivided into chapters. The introductory Part 1 with two chapters relates the three motivating problems to contemporary problems in the analysis of large networks.

The final Part 5 (Extensions: A brief survey) consists of a single chapter and an appendix. Chapter 23 (Other combinatorial structures) is particularly dear to this reviewer: possible generalizations to abstract algebras and category theory are included (if in fact, in most cases, sketched only). Appendix A contains eight sections, A.1–A.8, devoted to background material. The list of them indicates very well the scope of whole book:

A.1 Möbius functions
A.2 The Tutte polynomial
A.3 Some background in probability and measure theory
A.4 Moments and the moment problem
A.5 Ultraproduct and ultralimit
A.6 Vapnik–Cervonenkis dimension
A.7 Nonnegative polynomials
A.8 Categories

Yes, this book is by no means written and directed to combinatorists only. It is broadly based and covers material from probability, functional analysis, measure theory, and category theory, and of course, it is also an advanced combinatorial and graph theory book.

The remaining three parts, Part 2, Part 3, and Part 4, are the main sections wherein the theory is developed. Part 2 (The algebra of graph homomorphisms) is divided into four chapters. It includes a discussion of graph parameters and graph properties by means of homomorphisms and linear algebra. The notions of connection matrices, quantum graphs, contractors, and connectors are now studied on their own. The highlight of this part is the characterization of those graph parameters which correspond to partition functions (due to Freedman, Lovász, Schrijver). There are many variants of this result (for real weights, for complex weights, for randomly weighted graphs) but also for homomorphisms into graphons, dual form (i.e., counting subgraphs), morphisms in categories, and, last but not least, edge coloring models and its variants. Although some of these results have similar proofs and they are given at various places of the book, there seems to be no known "Master Theorem".

Part 2 (and particularly Chapter 5) is also one of the very few introductions to a systematic study of graphs and their homomorphisms (the others being a book by Godsil and Roy, and by Hell and this reviewer). Let us mention that this line of research started a long time ago with the seminal result of Lovász on the cancellation law (Tarski's problem) for finite structures, obtained in 1966. The fact that this was to become the cradle of developments related to partition functions must have been extremely pleasing to the author, and demonstrates well the persistence of old motivations (as nicely explained in Chapter 5).

Part 3 (Limits of dense graph structures) has eleven chapters, and with more than 200 pages it is a book in itself. Of course this part builds on results and concepts of Part 2, but to large extent it can be read independently. Here the author takes a new turn (and proceeds roughly in a reverse chronological order of the discoveries): First he defines graphons and kernels as symmetric measurable functions $[0,1]^2 \to \mathbb{R}$. Kernels and graphons are a generalization of weighted graphs which we get by sampling and lead to W-random graphs and their homomorphisms. The cut distance for graphs is defined next as induced by the cut norm of incidence matrices

$$\|A\|_\square = \frac{1}{n^2} \max_{S,T\subseteq[n]} \left| \sum_{i\in S, j\in T} A_{ij} \right|.$$

To define the cut distance, one needs to consider fractional overlays of graphs, which is thoroughly discussed. Some of these difficulties vanish when defining cut norm and cut distance for graphons. However for graphons one has to define isomorphism more carefully and there are several ways to do so. Thus the book contains three sections (7.3, 10.7, and 13.2) thoroughly discussing the problem. What follows is a detailed analysis of cut distance for (weighted) graphs as well as for graphons, which leads to the space of graphons denoted $(\tilde{\mathcal{W}}_0, \delta_\square)$ ($\tilde{\mathcal{W}}_0$ is set of all graphons factorized by cut distance 0). One of the highlights of this chapter (and one of the highlights of the whole book) is the fact that the space of all graphons is a compact space and that this is equivalent to the Regularity Lemma. The many-folded connections of the Regularity Lemma across all mathematics are truly amazing.

The book then takes a probabilistic twist by discussing sampling and related problems of counting ("counting lemma"), concentration, and isomorphisms. Chapter 11 then introduces (left- or shortly L-) graph convergence in various forms and proves that graphons serve as limit objects. This constitutes another highlight of the book: the fact that seemingly very different definitions of convergence (cut norm, homomorphism density, sampling) are in fact in one-to-one correspondence with graphons is remarkable and it has many applications and consequences. These are treated in the remaining chapters of Part 3. This is extensive and rich material and it will not be a purpose of this review to cover it all. Let us just mention that convergence may be defined in terms of homomorphisms and thus alternatively in terms of monomorphisms and induced subgraphs, and also in terms of colorings and epimorphisms. The connection between left and right convergence is important from the algorithmic point of view. Algorithms are of course an essential aspect of the theory and, while present in most of the book, there are two chapters (15 and 22) dealing exclusively with them. Chapter 15 deals with property testing (which is one of the motivations of the whole theory). The important Chapter 16 is devoted to extremal problems, where the use of limits has become a particularly effective tool (section 16.6 contains the solution, by Hatami and Norine, of Lovász's

17th Problem). Here progress has been very fast, and the whole area is blooming with activity; for a sample of recent developments, see e.g., [4], [7]. Yet, the theory of limits for dense graphs seems to be firmly based and, quoting the author, well understood.

On the other hand the same cannot be said about the corresponding questions (including the very definition of limits) for sparse graphs. Only particular instances are known, the most important being graphs with (uniformly) bounded degrees. This is the subject of Part 4. Here the notion of a graphing is introduced and studied in the context of measure-preserving, Borel graphs, and group theory. Local convergence (also called Benjamini–Schramm (or shortly BS-) convergence) is the key concept.

The questions are similar to the dense case, but (of course) the answers and techniques are quite different. For example, right convergence takes the form of entropy criteria and the structure of graphings is related to hyperfiniteness, isoperimetric problems, and expanders.

Merely the question of whether graphings are in correspondence with BS-convergence is a (difficult) open problem (Aldous–Lyons) related to geometric group theory as well as combinatorics. Perhaps it is true to say that present research activity is largely directed to the sparse case (see e.g. [5], [1]). The dichotomy of sparse vs. dense structures is one of the paradigms of contemporary mathematics and it is very much exemplified by this book.

Rarely does a monograph contain such an abundance of recent scientific activity of its author (in many cases unpublished elsewhere). At first glance it may seem to be an easy book that studies a simple concept: a particular instance of convergence. However this is an illusion as the theory of graph limits is the culmination of efforts which in many instances precede the formulation of the problem. Deeply rooted in asymptotic analysis and the study of random structures and their properties, and using some of the classical tools of functional analysis, measure theory, geometric group theory, and descriptive set theory, the theory of graph limits has produced a set of important notions (such as graphons, graphing, and modeling [5]) and revitalized the old sparse vs. dense dichotomy. The progress has been rapid and several old problems have been solved in this content. Here one should single out flag algebras with many applications to extremal problems (one can consult [6] for a recent survey). As with all important mathematics, the general theory finds its vindication in the solution of concrete problems.

A bibliography reflects the extensive research in this area up to 2012. Lovász is always punctilious in ascribing credits to others. This is a truly international all-mathematics endeavour. Here is a short list of some mathematicians and computer scientists who were involved in this project (any omissions are not intentional): M. Abért, D. Aldous, T. Austin, I. Benjamini, B. Bollobás, C. Borgs, S. Chatterjee, J. Chayes, F. Chung, P. Diaconis, G. Elek, M. Freedman, D. Gaborian, H. Hatami, S. Janson, J. Kahn, R. Kannan, D. Král′, R. Lyons, S. Norine, P. Ossona de Mendez, O. Pikurko, A. Razborov, G. Regts, O. Riordan, A. Scott, A. Shapira, 0. Schramm, L. Schrijver, V. T. Sós, B. Szegedy, T. Tao, S. Varadhan, K. Vesztergombi.

The book is written in the elegant style of a mature mathematician, giving essential details and being generous with insights and hints, yet not being overloaded with details. Lovász's research (and overall activity) has often been described as elegant. Yes, elegance is one of the prime qualities of this book.

In the introduction to his by-now classical book *Combinatorial Problems and Exercises* (North Holland Publishing Co., Amsterdam, 1979; AMS Chelsea Publishing, Providence, RI, 2007), the author speaks about combinatorics as emerging from the "slums of topology". This book provides further evidence in the same direction. In fact, I believe that hardly anybody will find it necessary to repeat this old anecdotal sentence. Here we are in the highest echelons of mathematics, rich with ideas and connections to other fields, aiming to be placed at the center of contemporary research in mathematics and theoretical computer science.

It is hard to be critical about this book. A few typos and inaccuracies which one can spot cannot mar the overall positive impression and, yes, pleasure provided by the book. Michel Mendès France once told me that one proper way to appreciate mathematics is to be jealous. Yes, then, this is a state of mind which some readers of this book may experience. The author maintains a webpage (`http://www.cs.elte.hu/~lovasz/book/homnotes.html`) for corrections and supplementary material.

This book is highly recommended to professional mathematicians and graduate students, and it will serve well for talented undergraduates. It is written in an accessible, non-technical style which will help and encourage students. It reflects and teaches what mathematical activity is at its finest.

References

[1] Ch. Borgs, J. T. Chayes, H. Cohn, Y. Zhao, *An L^p theory of sparse graph convergence I: limits, sparse random graph models, and power law distributions*, arXiv:1401.2906v2 [math.CO] (2014)

[2] S. Chatterjee, A. Dembo, *Nonlinear large deviations*, arXiv:1401.3495v3 [math.PR] (2014)

[3] R. Glebov, A. Grzesik, T. Klimoová, D. Král′, *Finitely forcible graphons and permutons*, arXiv:1307.2444v3 [math.CO] (2013)

[4] A. S. Kechris, *The spaces of measure preserving equivalence relations and graphs.*, preprint (2013)

[5] J. Nešetřil, P. Ossona de Mendez, *A unified approach to structural limits, and limits of graphs with bounded tree-depth*, arXiv:1303.6471v2 [math.CO] (2013)

[6] A. Razborov, *Flag algebras: an interim report*, In: Mathematics of Paul Erdős II, Springer 2013, pp. 207–232.

[7] Y. Zhao, *Hypergraph limits: a regularity approach* arXiv:1302.1634v2 [math.CO] (2013)

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