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Maximum principles and sharp constants for solutions of elliptic and parabolic systems, by G. Kresin and V. Maz'ya, Mathematical Surveys and Monographs, vol. 183, American Mathematical Society, Providence, RI, 2012, viii+317 pp., ISBN 978-0-8218-8981-7, US $96.00.

Everyone who has ever encountered either an undergraduate course on complex variables or an elementary course on partial differential equations is familiar with the maximum principle for harmonic functions. It stipulates that the funcions u that satisfy $\Delta u = 0$ in a bounded domain $\Omega \subset \mathbb{R}^n$,

$$\Delta := \sum_1^n \frac{\partial^2}{\partial x_i^2}$$

and are continuous in $\overline{\Omega}$ satisfy

$$\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u|. \tag{1}$$

Moreover, if Ω is connected and if for some $x_0 \in \Omega$

$$|u(x_0)| = \max_{\overline{\Omega}} |u|, \tag{2}$$

then $u \equiv u(x_0)$, i.e., is a constant.

We shall refer to (1) and (2) as the weak and strong maximal principles, respectively. Already, when one wants to extend the above principle to other differential equations, e.g., to a small perturbation of the Laplacian, $\Delta + C$, $C \in \mathbb{R}$, the Helmholtz equation, one must be cautious: (1), (2) hold for $C \leq 0$ but fail altogether for $C > 0$, even in dimension 1, as a moment's thought about a simple harmonic oscillator shows. Nevertheless, (1) and (2) hold, as is well known, for rather general elliptic operators

$$\sum_{j,k=1}^n a_{jk}(x) \frac{\partial^2}{\partial x_j \partial x_k} - \sum_1^n a_j(x) \frac{\partial}{\partial x_j} - a_0(x), \tag{3}$$

provided that the (bounded) coefficient matrix $(a_{jk}(x))$ is positive definite in $\overline{\Omega}$ and $a_0(x) \geq 0$. Moreover, (1) and (2) also hold for the parabolic equations (e.g., the heat equation):

$$\frac{\partial}{\partial t} - \sum_{j,k=1}^n a_{j,k}(x,t) \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^n a_j(x,t) \frac{\partial u}{\partial x} + a_0(x,t)\, u = 0 \tag{4}$$

with bounded coefficients that form a positive-definite matrix $(a_{jk}(x,t))$ with $a_0(x,t) \geq 0$ in the cylinder

$$Q_T = \Omega \times [0,T].$$

A good exposition of the current frontiers of this topic with a rich and diverse treasure box of examples and applications can be found in [Mir70, PW84, Spe81, Fra00, PS07] and in a forthcoming monograph [LG13].

The restrictive nature of the above-mentioned results, however, is clear: they all seem to mostly deal with scalar equations of the second order.

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A more general, although perhaps less precise principle, is the so-called Miranda–Agmon maximum principle (cf., e.g., [Mir70, Spe81, Fra00, PS07, LG13]), which holds for equations and systems of arbitrary order in smooth domains. For example, for solutions of the biharmonic equation $\Delta^2 u = 0$, this principle takes the form of the estimate

$$\max_{\overline{\Omega}} |\nabla u| \le C(\Omega) \max_{\partial\Omega} |\nabla u|, \tag{5}$$

with the constant $C := C(\Omega)$ depending on the domain. A weak version of the Miranda–Agmon maximum principle is (still for the biharmonic equation):

$$\max_{\overline{\Omega}} |\nabla u| \le K \max_{\partial\Omega} |\nabla u| + C\|u\|_{L^1(\Omega)}. \tag{6}$$

Equation (6) can be generalized to all elliptic operators of order 2ℓ, with $|\nabla u|$ replaced by

$$|\nabla_{\ell-1} u| := \left(\sum_{|\beta|=\ell-1} \frac{(\ell-1)!}{\beta!} \left|\mathcal{D}_x^\beta u\right|^2 \right)^{1/2},$$

cf., e.g., Chapter 5 of the book under review. (Here, following the authors, we use the standard multivariate notation of L. Schwartz; cf., e.g., [Joh82, Eva10].) However, in general, detailed expositions in the recent literature that deal with the equations of order higher than 2, and, especially, that deal with arbitrary systems, are quite scarce.

The monograph under review is perhaps the first detailed treatise in modern literature whose primary goal is to fill that gap. The authors offer a systematic study of a number of maximum modulus principles for elliptic and parabolic systems of arbitrary order. The results, most of which have been obtained by the authors themselves over the last three decades, are unified by combining two techniques stemming from classical potential theory and function theory: explicit representations and estimation of the norms of matrix integral operators similar to classical single and double layer potentials and, in addition, solutions of some finite-dimensional extremal problems (a.k.a. optimization problems).

Another theme explored in great detail in the book is devoted to the study of sharp pointwise estimates for derivatives of solutions of elliptic PDE. Perhaps, allowing oneself a slight oversimplification, one may consider those questions as stemming from the celebrated Schwarz Lemma in complex function theory, the foundation block and the starting point for most extremal problems in classical analysis. One of the problems the monograph addresses in depth is to obtain the sharp constants $C(\Omega)$ in (5) for solutions of the biharmonic equation in a geometrically simple situation, namely, in a half-space.

Another problem, studied in depth in the monograph, is in fact a straightforward extension of the Schwarz Lemma to the context of harmonic functions. More precisely, the authors study the problem of finding $C(x) := C(\Omega, x, \|\ \|_\Omega)$ such that

$$|\nabla u(x)| \le C(x)\|u\|_\Omega, \tag{7}$$

where $\|\ \|_\Omega$ can represent a variety of natural norms over the space of harmonic functions in Ω, e.g., L^p-norms in Ω, L^p-norms on $\partial\Omega$, L^∞-norms, etc. The problem (7) stems from the question considered in [Kha92] for the L^∞-norm, that stretches directly from the Schwarz Lemma for analytic functions in the plane to harmonic functions in higher dimensions. For a particular case,

$$\|\cdot\| = \|\cdot\|_{L^\infty(\Omega)},$$
$$\Omega = \{x : |x| < 1\},$$

it is believed that the extremal value is obtained for the radial derivative of an appropriate extremal harmonic function that takes two values 1 and -1 on the unit sphere. The problem is surprisingly tough and, at the present time, is still open.

However, the authors obtain an impressive number of precise estimates for the related extremal problems, e.g., those dealing with sharp pointwise estimates for directional derivatives of harmonic functions in the ball and half-space. These results are contained in Chapter 6.

A classical way of solving the Dirichlet problem for elliptic equations, or elliptic systems—in a domain Ω in $\mathbb{R}^n$—is to reduce the problem to studying the properties of explicit integral operators on the boundary of Ω (e.g., double layer potentials, single layer potentials, etc.) The key step in proving the solvability of the Dirichlet problem that goes back to Fredholm is then establishing the compactness of a relevant integral operator. When the boundary is irregular (e.g., has cusps or angular points in two dimensions, or edges and conic points in higher dimensions) the relevant integral operators fail to be compact and are merely bounded. One of the standard ways to "measure" the noncompactness is to evaluate the so-called essential norm of the operator, the distance to the space of compact operators in the operator norm. Radon [Rad19] was probably the first who used this idea to study the logarithmic potentials on curves with bounded rotation.

The explicit calculation of the essential norm is crucial in determining whether the general Fredholm theory applies to yield solvability of the integral equation and, hence, of the relevant Dirichlet problem. The authors give interesting illustrations of these strategies providing explicit estimates for boundary operators arising in the elasticity theory (Lamé systems) and hydrodynamics (Stokes system). The monograph studies the estimates both in the plane for domains with angular points and in $\mathbb{R}^3$ for domains with conic points and edges.

The second part of the book deals with equations and systems of parabolic type (cf. (4)). The authors carefully examine the limitations of the classical maximum principle for the uniformly parabolic (in the sense of Petrovskii) system

$$\frac{\partial \mathbf{u}}{\partial t} - \sum_{|\beta|\le 2\ell} A_\beta(x,t)\,\mathcal{D}_x^\beta \mathbf{u} = 0. \tag{8}$$

(Here, the vector-function $\mathbf{u} \in \mathbb{R}^m$.) It is shown that the classical maximum module principle fails in infinite slabs $\mathbb{R}^n \times [0,T)$ for $\ell \ge 2$, i.e., the first deviation from the parabolic equations of the second order. Some necessary and some sufficient conditions for the maximum modulus principle to hold are studied, and it is shown that these conditions merge when the coefficients in the system (8) are independent of the time variable t. The authors extend their investigations to the more general cylindrical domains.

The rest of Part II deals with a generalized notion of the maximum principle, christened as the maximum norm principle, where the norm in the finite-dimensional space $\mathbb{R}^m$ is understood in the most general sense, i.e., as the Minkowski functional of a given convex body that contains the origin.

The authors study such general maximum norm principles for uniformly parabolic systems and investigate a great number of particular examples for certain classes of norms, smooth norms, p-norms in $\mathbb{R}^m$, etc.

Although the reader may initially be overwhelmed by the notations and what looks like insurmountable technical difficulties, after an initial plunge, the book is surprisingly reader friendly. Every chapter starts with a careful and detailed preface, explaining the notations, background, the main problems considered therein and the most important results. Every chapter ends with helpful historical and bibliographical comments that are kept very much to the point but provide a good high-ground overview of the current state-of-the-art.

I think the monograph is unique in offering a novel and original point of view, mostly developed by the authors, and will, for many years to come, prove to be a useful and important resource for anyone working in linear partial differential equations, potential theory, elasticity theory, and hydrodynamics.

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