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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
DE PHILIPPIS AND FIGALLI

MR1440931 (98e:49102) 49Q20; 28A99, 35Q99, 60B99, 65K10

Gangbo, Wilfrid; McCann, Robert J.

The geometry of optimal transportation.

Acta Mathematica **177** (1996), *no.* 2, 113–161.

The Monge-Kantorovich mass transfer problem concerns the question of finding an optimal transportation between two mass distributions P_1 and P_2. In its original formulation due to Monge in 1782, the cost of transportation of unit mass between two domains in Euclidean space is given by the Euclidean distance. P_1, P_2 are the uniform masses on two subsets U_1, U_2 and admissible plans are those mappings between U_1 and U_2 which preserve uniform masses. It was conjectured by Monge that an optimal transport $S(x)$ satisfies the condition that $(S(x) - x)/|S(x) - x|$ is given by the gradient Du of a scalar potential. Appel (1928) gave a formal argument based on the corresponding Euler-Lagrange equation.

Real progress on the transportation problem is due to Kantorovich's 1940 relaxation of the transportation problem by allowing the class of all doubly stochastic measures as admissible transportation plans. This relaxation introduces convexity of the domain and linearity of the transportation functional. The basic result then is a duality theorem. A solution of the dual problem can be interpreted as Monge's transport potential and also as the Lagrange multiplier of the original mass transfer problem. Versions of the Kantorovich duality theorem for general cost functions $c(x, y)$ on abstract measure spaces and general probability distributions have been established by Kellerer (1984) and recently by Ramachandran and Rüschendorf (1994). As a consequence, characterizations of optimal transport measures in terms of subgradients of convex functions (equivalently, in terms of cyclically monotone support) in the case of Euclidean squared distance were given by Knott and Smith (1984), Rüschendorf and Rachev (1990) and Brenier (1991). This characterization of optimal solutions also implies existence of a Monge solution if one of the probability measures is Lebesgue-continuous. An extension of these results to general cost functions was given by Rüschendorf (1991, 1995). The characterization of optimal plans is obtained in terms of generalized c-subgradients of c-convex functions.

Gangbo and McCann work out a different approach to existence results as well as uniqueness results for optimal Monge functions. Their approach is no longer based on duality theory but uses directly the c-cyclical monotonicity condition of the support. Optimal maps are characterized by corresponding c-convex potentials. This leads to existence results without any moment conditions as in the duality approach. A related existence result was also obtained independently by Caffarelli (1996). To establish uniqueness, Aleksandrov's proof for surfaces of prescribed integral curvature is generalized to the transportation problem. Along the way the authors establish several important properties of c-concave functions, like the local Lipschitz property, and investigate the related Legendre transforms. The case of costs given by convex functions of the difference and by concave functions of the absolute difference is discussed in detail. In the strictly convex case the optimal transfer plan is uniquely determined by the potential; for radial costs of the form $h(|x-y|)$ the potential specifies the direction of transport but not the distance. Evans and Gangbo (1996) established in the case of the Euclidean cost $|x-y|$ an ordinary differential equation for the missing piece of information. The paper of Gangbo and McCann gives not only a new and sound basis for a general solution of the Monge-Kantorovich problem and finds a general solution in two important cases, but also contains some methodological insights and developments on the underlying class of c-concave functions and generalized Legendre transforms.

It should be mentioned that the problem treated in this paper is related to several important mathematical problems, and the methods developed in this paper are of interest for those problems, too. Brenier has discussed in several papers relations of the Monge-Kantorovich problem to rearrangement theorems for vector fields (polar decomposition) as well as to generalized solutions of a least action problem describing the motion of an ideal incompressible fluid. The Euler-Lagrange equation of this variational problem is identical to the Euler equations of incompressible flows in Lagrangian form. A discrete version of the least action principle leads to a Monge-Kantorovich-type problem with a fixed finite number of marginals.

It is also known that the solution of the transportation problem for the quadratic cost $c(x,y)=\|x-y\|^2$ is equivalent to generalized solutions of the Monge-Ampère PDE; for a general cost function c one obtains a correspondence with c-concave solutions of a generalized Monge-Ampère equation. Thus, numerical results and regularity theory can be transferred from one problem to the other. There are only partial results for these problems up to now.

Ludger Ruschendorf
From MathSciNet, June 2014

MR2079068 (2005f:35091) 35J60
Caffarelli, Luis A.
The Monge Ampere equation and optimal transportation.
Recent advances in the theory and applications of mass transport, 43–52, *Contemporary Mathematics*, 353, *American Mathematical Society, Providence, RI*, 2004.

With a brilliant informal style, the author reviews several results and techniques in the context of the Monge-Ampère equation.

After describing the motivation for the problem, he explains why this equation bridges the theory of divergence and non-divergence elliptic equations. Then, he deals with the existence and characterization of global solutions and with the existence of correctors for the periodic right-hand side (roughly speaking, in this case, global solutions are characterized to be in the form of a quadratic polynomial plus a periodic corrector).

The relation with optimal transportation is discussed at the end of the note and some open problems are also presented (with ideas on how to try to attack them).

Also, the classical theorems of Pogorelov and Jorgens and Calabi are outlined in a modern framework and some highlights of recent important results are given [see also L. A. Caffarelli and Y. Y. Li, Ann. Inst. H. Poincaré Anal. Non Linéaire **21**

(2004), no. 1, 97–120; MR2037248 (2004k:35104); O. V. Savin, Comm. Pure Appl. Math. **57** (2004), no. 1, 126–140; [MR2007358 (2004k:35428)].

Enrico Valdinoci

From MathSciNet, June 2014

MR2459454 (2010f:49001) 49-02; 28A75, 37J50, 49Q2, 53C23, 58E30

Villani, Cédric

Optimal transport. (English)

Old and new.

Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 338.

Springer-Verlag, *Berlin*, 2009, xxii+973 pp., $159.00, ISBN 978-3-540-71049-3

This book wins the challenge to give a new and broad perspective on the multifacet topic of the optimal mass transport. The general problem of optimal mass transport is as follows: given two probability measures μ and ν on a measured space M and a (cost) function $c\colon M \times M \to \mathbb{R}$, minimize the total cost

$$\int_M c(x, F(x))\, d\mu(x)$$

over all maps $F\colon M \to M$ pushing forward μ onto ν. In geometric situations, M is a metric space and the cost is taken to be the squared distance $d^2(x, y)$. In most cases, the total cost then coincides with the the relaxed minimization problem

$$\inf \iint_{M\times M} d^2(x, y)\, d\pi(x, y),$$

where the infimum is taken over all probability measures π on the product having μ and ν as marginals. The square root of this quantity defines a distance (called the Wasserstein or Kantorovich-Rubinstein distance) on the space of probability measures with finite second-order moment. This new metric space associated to M is referred to as the Wasserstein space (actually, when a reference measure is given, one may prefer to work with the subset of probability densities). If (M, d) is a geodesic space, then the associated Wasserstein space is also a geodesic metric space.

The present text builds on the modern theory of optimal transport initiated by Y. Brenier, R. J. McCann, F. Otto, and others. It is complementary to C. Villani's first book on the subject [*Topics in optimal transportation*, Amer. Math. Soc., Providence, RI, 2003; MR1964483 (2004e:90003)]. Indeed, the two books differ in the content and in the choices of presentation. In particular, two directions are emphasized in the present book. The first one is the connections with dynamical systems: Mather's theory is used to give new perspectives on classical properties of mass transport. The second one is geometry and curvature bounds. The book is divided into 30 chapters grouped into three parts. It is impossible to summarize all the content of the book but I shall try to present some of the questions addressed in each part.

The first part of the book is called "Qualitative description of optimal transport". It starts with basic properties of the Monge-Kantorovich duality (cyclical monotonicity) and of the Wasserstein distance. Geodesic curves in the Wasserstein space are shown to be action minimizing curves for a certain Lagrangian. This justifies studying interpolation along mass transport from a dynamical point of view.

A Monge-Mather shortening lemma is then stated from which many properties of optimal transport (previously obtained by other means) are derived. This part also contains the solution to Monge's problem for general costs, a change of variable formula for the optimal map (Jacobian equation) and observations on the smoothness of the optimal map.

The second part is devoted to "Optimal mass transport and Riemannian geometry". The author presents the connections between Ricci curvature lower bounds and optimal transport. This leads to differential inequalities for the Jacobian determinants of interpolated optimal maps and, through elementary comparison arguments, to integral inequalities. Then, the notion of displacement convexity is introduced and, with it, the notion of curvature-dimension of a manifold equipped with a reference measure. Roughly speaking, one asks that certain functionals on the Wasserstein space be convex along mass transport (i.e., geodesically convex on the Wasserstein space). Characterizations in terms of lower bounds of the (generalized) Ricci curvature tensor are given. Many consequences of the curvature-dimension bounds are then presented: Brunn-Minkowski, Bishop-Gromov, local Poincaré inequalities, and the so-called HWI inequalities from which log-Sobolev inequalities are derived. Concentration, transport and modified log-Sobolev inequalities are also discussed. This part closes with three chapters devoted to the study of gradient flows. Some important evolution equations are known to be gradient flows on the Wasserstein space, and some of the above-mentioned inequalities can be reinterpreted in terms of speed of convergence towards equilibrium of well-chosen entropy functionals. This requires discussion of the (non-smooth) differential structure of the Wasserstein space.

The third part deals with the new "Synthetic treatment of Ricci curvature". The general idea is to mimic the Riemannian situation and to define a notion of "lower bound on the Ricci curvature" for an abstract metric (geodesic) space in terms of displacement convexity of certain functionals on the associated Wasserstein space. A crucial property justifying this approach is that optimal mass transport is stable under the Gromov-Hausdorff convergence of metric spaces. After these definitions and properties are carefully stated, the author presents several applications of the Ricci bounds with emphasis on geometric (Bonnet-Myers, Bishop-Gromov) and functional (HWI, Poincaré) inequalities. Discussions on equivalent forms of the Ricci bounds in non-branching spaces and on the locality of the definition close this part.

This summary hardly reflects the richness of Villani's 1000-page text, which displays a countless number of ideas and embraces several areas of analysis (with special emphasis on the topic of functional inequalities). Villani's book is written with great mastery and with a strong pedagogical concern. Several sections and appendices are devoted to presenting non-expert notions and tools from non-smooth analysis and geometry. Special attention should be given to the extraordinary bibliographic notes ending each chapter. Besides extensive and accurate references, therein the reader will find comments on related questions barely touched upon in the main text, as well as lively presentations on how ideas and results have developed. This book should prove useful both to the expert and to the beginner looking for a reference text on the subject.

Dario Cordero-Erausquin

From MathSciNet, June 2014

MR2654086 (2011f:49071) 49Q20; 58E25

Kim, Young-Heon; McCann, Robert J.

Continuity, curvature, and the general covariance of optimal transportation.

Journal of the European Mathematical Society (JEMS) **12** (2010), *no.* 4, 1009–1040.

The paper under review deals with a very hot topic, namely regularity issues for optimal transport maps. Given are two smooth manifolds M_0, M_1, a pair of Borel probability measures $\rho_i \in \mathcal{P}(M_i)$, $i = 0, 1$, and a lower semicontinuous cost function $c\colon M_0 \times M_1 \to \mathbb{R} \cup \{+\infty\}$. Then one is concerned with the minimization of the total transportation cost, i.e.

$$W(\rho_0, \rho_1) = \inf_T \int_{M_0} c(x, T(x))\, d\rho_0(x), \tag{1}$$

among all maps T such that $\rho_1 = (T)_{\#}\rho_0$. Once we can prove the existence of an optimal transport map T_0 (this usually requires some assumptions on ρ_0, ρ_1 and the cost function c), a very natural and interesting question comes into play about the regularity of T_0. More precisely, since problem (1) admits a suitable relaxation to a linear programming problem, and this in turn possesses a dual formulation which involves pairs of potential functions, one is concerned with smoothness of optimal pairs of potentials (φ, φ^*) where, in particular, φ is a c-concave function and φ^* is its c-transform. A regularity theory for them is possible since they usually solve a (degenerate) elliptic equation of Monge-Ampère type, the form of the operator clearly depending on c. Back to the original question, one can recover the regularity of T_0 by appealing to the relation

$$D_x\varphi(x) = D_x c(x, T_0(x)),$$

linking φ and T_0.

When M_0, M_1 are bounded domains of $\mathbb{R}^N$ and $c \in C^4$, X. N. Ma, N. S. Trudinger and X. J. Wang [Arch. Ration. Mech. Anal. **177** (2005), no. 2, 151–183; MR2188047 (2006m:35105)] have shown that a sufficient condition for smoothness is "*a mysterious structure condition comparing third and fourth derivatives*" of c. Then G. Loeper [Acta Math. **202** (2009), no. 2, 241–283; MR2506751 (2010c:49084)] showed this condition (sometimes called the *Ma-Trudinger-Wang* condition; it is quite complicated to reproduce here) to be also necessary.

The aim of this paper is to provide an intrinsic reformulation (i.e. coordinate independent) in the general setting of manifolds of the Ma-Trudinger-Wang condition, allowing for a new deep insight into the regularity question. Indeed, the central core of this paper is the following idea: the Ma-Trudinger-Wang condition can be rephrased as a positivity condition on the sectional curvatures of certain null submanifolds of the product space $M_0 \times M_1$, once this is equipped with a pseudo-Riemannian metric h related to the cost function c and having signature (n, n) (n being the dimension of both M_0 and M_1). More precisely, the metric tensor h is defined as

$$h := \frac{1}{2}\begin{pmatrix} 0 & -\overline{D}Dc \\ -D\overline{D}c & 0 \end{pmatrix}$$

where D and $\overline{D}$ indicate differentiation with respect to x and y variables, respectively.

The main result of the paper (Theorem 3.1) is a new proof of the fact that the Ma-Trudinger-Wang condition implies the c-convexity of the c-superdifferentials of c-concave functions: this fact was first proved by Loeper (compare with Theorem 3.1, point (5), of [G. Loeper, op. cit.]), based on some regularity results by Trudinger and Wang [Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **8** (2009), no. 1, 143–174; MR2512204 (2011b:49121)], and this is a crucial ingredient for his regularity results. Theorem 3.1 of the present paper extends this result to more general situations (weakening the hypotheses even in the case of M_0 and M_1 subdomains of $\mathbb{R}^N$) and avoids the use of the results by Trudinger and Wang, thus making Loeper's theory self-consistent. Thanks to this new approach, the authors are also able to extend their investigation to the case, for example, of $M_0 = M_1 =$ product of spheres.

The paper is very well organized, with a nice introduction which smoothly takes the reader into the topic, and many clarifying remarks, examples and bibliographical references.

Lorenzo Brasco

From MathSciNet, June 2014

MR2883682 (Review) 49Q20; 53C23

Figalli, Alessio; Villani, Cédric

Optimal transport and curvature.

Nonlinear PDE's and applications, 171–217, *Lecture Notes in Mathematics*, 2028, *Springer, Heidelberg*, 2011.

This set of notes highlights the connections between optimal transport and geometry as of June 2008. A lazy description would be that these notes constitute a "Cliffs Notes" version of C. Villani's nearly 1000 page book [*Optimal transport*, Grundlehren Math. Wiss., 338, Springer, Berlin, 2009; MR2459454 (2010f:49001)] which was published around the same time. Some who are interested in the topic will find these notes one of the better introductions to the subject.

The focus is on two themes:

(a) stability of lower bounds on Ricci curvature under measured Gromov-Hausdorff convergence;

(b) smoothness of optimal transport in curved geometry.

In sections 2 and 4, basic results and definitions from Riemannian and metric geometry are outlined, including basic definitions for metric spaces, Jacobi fields, Hessian and second-order calculus, stability theorems for Ricci curvature bounds under measured Gromov-Hausdorff topology, Wasserstein space and Boltzmann entropy. Some detail is given, but someone seeking all the details will find them in [op. cit.].

Section 3 discusses the basic tools and structure of optimal transport, including the Kantorovich problem, existence, c-convexity, and concentration of the solution on a graph.

Section 5 goes into some detail discussing the smoothness issue and the Ma-Trudinger-Wang (MTW) condition. The discussion echoes most of the themes from G. Loeper's paper [Acta Math. **202** (2009), no. 2, 241–283; MR2506751 (2010c:49084)] and concludes with some regularity results demonstrating the interaction between properties of the cut locus and the MTW condition.

Sections 6 and 7 list some interesting open problems, other connections to problems in geometry and some references.

Micah Warren
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