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## 1. Some history

Harmonic analysis is such a vast and rapidly developing field that a new book, such as the one under review, is not only welcome but to some extent necessary. The field traces its roots to two centuries ago, in the work of Joseph Fourier. It is since that time that the idea of representing functions as combinations of sines and cosines has started to grow and has eventually led to what is nowadays referred to as Fourier analysis. The investigation of mapping properties of various operators by means of decompositions into Fourier series or their continuous analog—the Fourier transform—has always been a dominant theme in the field. In modern times, harmonic analysis has outgrown its initial Fourier-analytic foundation and has branched out to address fundamental problems in geometric measure theory and many other areas.

If the "Adam" of the field was the idea of Fourier decomposition, a good candidate for the "Eve" is the Hilbert transform, defined as a principal value distribution

$$\mathcal{H}f(x) = \lim_{\epsilon \to 0} \int_{|t| > \epsilon} f(x - t) \frac{dt}{t}.$$

The two volumes of the book under review are largely devoted to exploring the offspring of their marriage. Initially introduced by Hilbert in his work on what was to be called the Riemann–Hilbert problem on holomorphic functions, the operator $\mathcal{H}$ soon found a wide variety of applications in partial differential equations and complex analysis, well beyond its initial scope. In a nutshell, the first volume is centered around (but not restricted to) the linear theory associated with $\mathcal{H}$ and its cousins, while the second volume surveys the recent developments of its multilinear counterparts.

Any attempt to review thoroughly such a diverse field in just a few pages would be futile. However, the authors provide a good bird's eye overview in the preface of their book. Their insight is highly recommended to anyone trying to get a sweet glimpse into the last century of harmonic analysis, before embarking into deciphering the intricate mathematical content of the book. The next few paragraphs will nevertheless try to make an independent synthesis of some of the most important developments, with an emphasis on the topics that have found their way into the book in a more or less detailed form.

The Hilbert transform $\mathcal{H}$ is the simplest example of a *singular integral operator.* The latter term refers broadly to operators represented by a kernel, such as $\frac{1}{t}$ in the case of $\mathcal{H}$, that is not integrable but exhibits various cancellations that compensate for the lack of integrability. The initial investigation of $\mathcal{H}$ was via complex-analytic

techniques. It soon became apparent that higher-dimensional analogs of $\mathcal{H}$ play key roles in PDEs and other areas. The complex method lost traction in this new context, and real variable methods needed to be invented. A series of papers by Calderón and Zygmund, starting with [5], laid the foundation for a school of thought that, in addition to its remarkable initial success, has never been livelier than today. At its core lies a rather simple stopping-time–type technique that has since been called the *Calderón–Zygmund decomposition*. It is not easy to come up with another construction in mathematics that has found an equally gigantic number of applications.

In the 1960s Calderón initiated the study of a certain hierarchy of commutators in connection to the Cauchy integral on Lipschitz curves. While initially motivated by PDEs, his work would seed the field for future generations of harmonic analysts. We will follow some of the main threads of the story. Calderón managed to prove the boundedness of the first commutator in 1965, but new ideas were needed in order to address the higher-order ones. This was gradually achieved in the work of Coifman and Meyer throughout the mid-1970s. The multilinear perspective had started to take shape even as early as [9], but crystalized in a robust form in [11], with additional contributions from Kenig and Stein [19] and Grafakos and Torres [16].

The Cauchy integral along the graph of a Lipschitz function $A : \mathbb{R} \to \mathbb{R}$ is defined as

$$C_A f(x) = \lim_{\epsilon \to 0} \int_{|y-x|>\epsilon} \frac{f(y)}{(x-y) + i(A(x) - A(y))} dy.$$

It can naturally be decomposed as a sum of Calderón's commutators. To make the sum converge in the appropriate operator norm, some rather good bounds are needed on the commutators, ones that were not provided by the first round of arguments. The final resolution was provided a bit later by Coifman, McIntosh, and Meyer [10]. Slightly after this, David and Journé proved their celebrated "$T(1)$" theorem [12]. One of the principles of the Calderón–Zygmund theory is that if a standard singular integral operator is bounded on $L^2$, it is well behaved (in the expected sense) on all other $L^p$ spaces, $1 \le p \le \infty$. The $T(1)$ theorem resolved the remaining issue in a rather satisfactory manner, by providing easy-to-check necessary and sufficient conditions for $L^2$ boundedness. One of these conditions amounts to having the constant function 1 mapped to the space of functions with bounded mean oscillation. It was soon realized in [22] and [13] that 1 can be replaced with an *accretive* function $b \in L^\infty$, one such that $\inf_{x \in \mathbb{R}} \operatorname{Re} b(x) > 0$. The first immediate byproduct was a more streamlined proof of the boundedness of the Cauchy integral on Lipschitz curves, but the $T(b)$ theorem would soon turn into a powerful machine with spectacular, far-reaching consequences. One was the resolution of the Kato square-root problem for second-order elliptic operators [1].

An important extension of the $T(b)$ theorem was provided by Christ [7] in the context of homogeneous spaces. Introduced by Coifman and Weiss in [8], these metric spaces equipped with a doubling Borel measure proved to be a natural framework for expanding the classical Calderón–Zygmund theory beyond its traditional Lebesgue space habitat. The next logical step was to extend the $T(b)$ theorem even further to the non-homogeneous case, where the doubling condition is lost. The remarkable paper of Nazarov, Treil, and Volberg [25] develops this theory and introduces along the way the method of averaging over random grids.

The Nazarov–Treil–Volberg $T(b)$ theorem plays a critical role in the resolution by Tolsa [28] of Painlevé's problem. This classical question asked for a geometric characterization of those sets in the complex plane, such that every bounded analytic function on their complement is constant.

The idea of investigating the mapping properties of singular integral operators on weighted spaces has undergone a parallel thread of developments. Originating in the work of Muckenhoupt on the Hardy–Littlewood maximal function [23], the $A_p$ weights have proven to be the right tool to describe boundedness of arbitrary Calderón–Zygmund operators in euclidian space. In recent times, the more sophisticated question on the optimal dependence of the norm of the operator on the $A_p$ characteristic of the weight has gained prominence. The sharp weighted estimate for the Ahlfors–Beurling operator [26] was used by Petermichl and Volberg to establish borderline regularity for solutions of the Beltrami equation in the plane. The search for a similarly sharp estimate for arbitrary operators has revisited along the way variants of the $T(1)$ theorem and, finally, has culminated in the recent definitive work of Hytönen [18], who proved that each Calderón–Zygmund operator is a superposition of simpler operators called "dyadic shifts".

For those readers who are not yet deeply impressed by the impact of Calderón's commutators, here is the second part of the story. This is about how commutators have led to the development of the modern theory of multilinear singular integral operators. We have already mentioned the Coifman–Meyer theorem [11], covering Fourier multiplier operators for which the singularity of the associated symbol is zero dimensional (typically the origin). An important class of multilinear operators called *paraproducts* falls into this category, and they played a tremendous role in PDEs over the last decades. But what if the singularity is a line instead? Well, there is a very important such operator, and the credit for bringing it to light goes again to Calderón. It is called the *bilinear Hilbert transform* and is given by the formula

$$\mathcal{BH}_{\alpha,\beta}(f_1, f_2)(x) = \lim_{\epsilon \to 0} \int_{|t| > \epsilon} f_1(x + \alpha t) f_2(x + \beta t) \frac{dt}{t}.$$

What Calderón realized is that uniform estimates (over $\alpha, \beta$) for the norm of $\mathcal{BH}_{\alpha,\beta}$ would imply the boundedness of his first commutator. This observation was lost for many years due to the subsequent completion of his program via alternative approaches, as described above.

Interest in the bilinear Hilbert transform re-emerged in the early 1990s with the establishment of the Coifman–Meyer theory. It became apparent that Littlewood–Paley theory cannot by itself deal with the new type of singularity. The challenge posed by $\mathcal{BH}_{\alpha,\beta}$ was that it was very symmetric from a phase-space perspective, as will be discussed below. As a result, its analysis begged for a decomposition that was equally symmetric. Something similar had actually been done many years before and was about to be brought back to light again.

In 1915 Lusin conjectured that the Fourier series of an $L^2$ function on the torus converges almost everywhere. There was not a whole lot of evidence for why this might be true, and in fact only a few years later, Kolmogorov constructed an example of an $L^1$ function whose Fourier series diverges everywhere. The subsequent work of Paley and Zygmund confirmed Lusin's conjecture for the special case of random Fourier series, but the general form of the conjecture resisted for a long time. The seminal paper [6] by Carleson from 1966 put all speculations to rest.

His proof of Lusin's conjecture was a remarkable display of analytic and combinatorial techniques, well ahead of its time. A few years later, Fefferman produced an alternative argument in [15], which introduced the modern language of "tiles" and "trees". The key object in both proofs is *Carleson's operator*

$$Cf(x) = \sup_{\xi \in \mathbb{R}} |\lim_{\epsilon \to 0} \int_{|t| > \epsilon} f(x - t) \frac{e^{i\xi t}}{t} dt|.$$

While at first glance it seems like a rather benign alteration of the Hilbert transform, the operator $C$ possesses a new type of symmetry, called *modulation invariance.* This means that it does not distinguish between a function $f(x)$ and any of its modulations $f(x)e^{i\xi x}$. In addition to this new feature, the operator $C$ inherits the translation and dilation invariance from $H$. Fefferman's proof addresses these issues by producing a wave packet decomposition of $C$ which is symmetric on both the spatial and the frequency side. The argument becomes more geometric, as wave packets correspond to unit area rectangles in the plane called *tiles.* A cluster of such tiles forms a *tree*, which in turn is the geometric realization of a Hilbert transform. Delicate combinatorics and orthogonality arguments go into proving that trees do not interact too badly with each other. This marks the beginning of a new chapter in harmonic analysis, but at least twenty years passed before the next great page was written.

Drawing some inspiration from [6] and [15], Lacey and Thiele prove the boundedness of the bilinear Hilbert transform for a large range of $L^p$ spaces in [20] and [21]. The starting observation is that $\mathcal{BH}_{\alpha,\beta}$ and $C$ have the same group of symmetries. This breakthrough was followed shortly by a more general theorem due to Muscalu, Tao, and Thiele [24], and the window was opened to a multitude of questions. The time-frequency analysis of multilinear operators became a prominent subject and during the following years found noteworthy intersections with scattering theory and ergodic theory.

It turns out that the mid-1960s set the stage for yet another branch of harmonic analysis. It all started with an observation by Stein that the Fourier transform of a non-integrable function can sometimes be meaningfully restricted to a curved hypersurface such as the sphere. The circle of problems concerned with the range of $L^p$ spaces and the type of manifold for which this phenomenon persists would become known as *restriction theory.* Several papers by Fefferman in the early 1970s uncovered the first deep connections between the role of curvature and the hierarchy of "Kakeya conjectures". The central object in these conjectures are the Kakeya sets in $\mathbb{R}^n$, those which contain a unit line segment in every direction. Besicovitch proved in 1919 the counterintuitive fact that there exist Kakeya sets with zero Lebesgue measure. The *Kakeya set conjecture* asserts that all Kakeya sets must have full Hausdorff dimension. Proved in two dimensions by Davies in 1971, it continues to remain open in three and higher dimensions even to this day.

The even more difficult *restriction conjecture* took shape in the early 1970s. In broad terms it is concerned with restricting the Fourier transform in a meaningful way to zero Lebesgue measure subsets of the euclidean space. In dimension two, the restriction conjecture was also proved in the early 1970s, by Fefferman and by Zygmund. Around the same time, Stein and Tomas settled the $L^p(\mathbb{R}^n) \to L^q(S^{n-1})$ case of the restriction conjecture for the sphere, when $q \leq 2$. A similar result was proved by Strichartz in the case of the paraboloid. His work has led to important

developments in the theory of nonlinear evolution partial differential equations, with productive and still ongoing interactions with Fourier analysis.

A new phase opened in the early 1990s with several contributions by Bourgain, whose work improved the range of exponents for both the restriction and the Kakeya set conjectures. He brought to the subject a combinatorial perspective whose importance has grown ever since. Meanwhile, bilinear restriction theorems rose to prominence and culminated in the sharp results for the cone and the paraboloid, due to Wolff [29] and Tao [27], respectively.

A series of deep insights of Wolff in the mid-1990s through the early 2000s further transformed the subject. He used a bilinear approach to prove that Kakeya sets in $\mathbb{R}^n$ have Hausdorff dimension at least $\frac{n+2}{2}$. His work brought combinatorial techniques more systematically into the subject, and it crystallized a new perspective by posing the finite-field Kakeya conjecture. The next breakthrough came in 1999 [3] when Bourgain exploited the arithmetic structure of Kakeya sets together with a combinatorial insight, which originated in Gowers's improvement of the Balog–Szemerédi theorem, to improve on Wolff's bound in higher dimensions. This approach has inspired further explorations of the arithmetic structure of Kakeya sets and has in particular led to the sum-product theorem of Bourgain, Katz, and Tao [4]. One description of the sum-product phenomenon is that if $A$ is a small subset of a finite field whose sumset $A + A$ is small, then the product set $A \cdot A$ must be large.

A recent important development is the proof by Bennett, Carbery, and Tao [2] of the multilinear Kakeya and restriction theorems in the expected range of $L^p$ spaces, except for the endpoint. The proof by Guth [17] that followed shortly thereafter closed the gap and brought an infusion of algebraic topology to the subject, making crucial use of the polynomial "ham sandwich" theorem. The emergence of the polynomial method in euclidian harmonic analysis is paralleled by its successful use by Dvir to solve the finite field Kakeya conjecture of Wolff [14].

## 2. This book

We close with a few words about the material in the book under review. Most of the first volume covers the classical developments up to the 1970s related to Fourier series, Calderón–Zygmund theory, and Littlewood–Paley theory. In addition to these, it includes two chapters on topics that are rarely seen in other texts. Namely, a rather comprehensive chapter is devoted to the Weyl calculus, and another one to various quantitative forms of the uncertainty principle, such as the Amrein–Berthier and Logvinenko–Sereda theorems. These results are interesting in their own right, but additional effort is devoted to showing their applicability to solving constant coefficient linear PDEs. A few topics on restriction theory are also discussed, but the presentation is unfortunately limited to the initial developments from the early and mid-1970s.

The second volume is more specialized, focusing on the development of the multilinear theory of singular integral operators. A large part of the material here does not appear in other books. The first chapter provides part of the motivation for the later ones, and can be read independently of the rest of the book. Using the gKdV equation as a case study, it introduces the reader to key concepts, such as wave packets and phase-space portraits. The perspective on the asymptotics of the Airy function is quite unique. Chapters two, three and four offer a treatment of

the theory of paraproducts both in single and the multiparameter setting and also of Calderón's commutators, using discretized model sums. By the time the reader masters this approach in these simpler settings, he or she will be ready for the applications to more complicated operators, such as the bilinear Hilbert transform and Carleson's operator. The transition from classical to contemporary is mediated by chapter five, a pleasant, non-technical dissertation on how these operators are connected to the $n$-body problem from planetary motion.

The whole book is written in a very accessible style, with clear exposition of all main results and plenty of good exercises.

## Acknowledgment

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