PART 1. EXOTIC SPHERES

This section will consist of the following eight papers:

On manifolds homeomorphic to the 7-sphere, Annals of Mathematics **64** (1956) 399–405.

On the relationship between differentiable manifolds and combinatorial manifolds. Unpublished notes, Princeton University 1956.

Sommes de variétés différentiables et structures différentiables des sphères, Bulletin de la Société Mathematique de France **87** (1959) 439–444.

Differentiable structures on spheres, American Journal of Mathematics **81** (1959) 962–972.

A procedure for killing homotopy groups of differentiable manifolds, in "Differential Geometry," Proceedings Symposia in Pure Mathematics **III**, American Mathematical Society (1961) 39–55.

Differentiable manifolds which are homotopy spheres. Unpublished notes, Princeton University 1959.

Groups of homotopy spheres: I (with Michel Kervaire), Annals of Mathematics **77** (1963) 504–537.

Differential topology, in "Lectures on Modern Mathematics **II**," edited by T. L. Saaty; Wiley, New York (1964) 165–183.

Introduction: How these papers came to be written

During the 1950's, I worked on an ongoing project of trying to understand one particularly simple class of manifolds, namely $2n$-dimensional manifolds which are $(n-1)$-connected. Although my intended paper on this subject was never finished, the project none the less led to the eight papers which follow (as well as the paper "On Simply Connected 4-Manifolds," Milnor [1958][1]).

[1]Names in small caps refer to the bibliography at the end of this volume.

To understand how this came about, let me first describe the original plan. The homotopy theory of a closed manifold M^{2n} which is $(n-1)$-connected is relatively easy to describe. If the middle Betti number is β, then the manifold can be obtained (up to homotopy type) by attaching a $2n$-cell to a bouquet

$$W^n = S^n \vee \cdots \vee S^n$$

of β copies of the n-sphere. The attaching map can be described as an element of a homotopy group $\pi_{2n-1}(W^n)$ which is reasonably well understood. More explicitly, this group splits as the direct sum of β copies of the group $\pi_{2n-1}(S^n)$ together with $\beta(\beta-1)/2$ free cyclic groups which correspond to Whitehead products of distinct generators of $\pi_n(W^n)$.

Better still, this space can be described in terms of cohomology theory. The cohomology groups $H^k = H^k(M^{2n}; \mathbb{Z})$ are zero with the exception of H^0 and H^{2n} which are free cyclic and $H^n \cong H^n(W^n; \mathbb{Z})$ which is free abelian of rank β. The bilinear pairing

$$H^n \otimes H^n \ \to \ H^{2n}$$

is either symmetric or skew according as n is even or odd, and has determinant ± 1 by Poincaré duality. Finally the "stable" attaching map can be described by a cohomology operation

$$\psi \ : \ H^n \ \to \ H^{2n}(M^{2n}; \Pi_{n-1}) \ \cong \ \Pi_{n-1} \ ,$$

where Π_{n-1} is the stable homotopy group $\pi_{k+n-1}(S^k)$ for $k > n$. This operation can be described as follows. Any element $\eta \in H^n \cong H^n(W^n; \mathbb{Z})$ corresponds to a homotopy class of maps $W^n \to S^n$. Composing with the attaching map in $\pi_{2n-1}(W^n)$, we obtain an element of $\pi_{2n-1}(S^n)$ which stabilizes to the required $\psi(\eta) \in \Pi_{n-1}$.

The problem arises when one tries to flesh out this homotopy picture by constructing actual manifolds realizing specified homotopy invariants. In the simplest case $\beta = 1$, we must attach a $2n$-cell to a single n-sphere in such a way as to obtain a manifold. It seems that the best chance of carrying out this construction is to first "thicken" the n-sphere, replacing it by a tubular neighborhood in the hypothetical M^{2n}. In other words, we must form an n-disk bundle

$$D^n \ \hookrightarrow \ E^{2n} \ \twoheadrightarrow \ S^n \ .$$

In the hoped for situation, the boundary sphere bundle $\partial E^{2n} = \Sigma^{2n-1}$ will be homeomorphic to the $(2n-1)$-sphere. Hence we will be able to glue on a $2n$-disk by a boundary homeomorphism h so as to obtain a closed manifold $M^{2n} = E^{2n} \cup_h D^{2n}$ with the required homotopy type. Thus we are led to the following problem:

For which sphere bundles $S^{n-1} \hookrightarrow \Sigma^{2n-1} \twoheadrightarrow S^n$ is the total space Σ^{2n-1} homeomorphic to the sphere S^{2n-1}?

Three basic examples had been discovered by Heinz Hopf, namely the fibrations

$$S^1 \hookrightarrow S^3 \twoheadrightarrow S^2 \, , \quad S^3 \hookrightarrow S^7 \twoheadrightarrow S^4 \, , \quad \text{and} \quad S^7 \hookrightarrow S^{15} \twoheadrightarrow S^8 \ .$$

(In fact it is now known that such bundles can exist only in these particular dimensions. Compare "Some Consequences of a Theorem of Bott," pages 233–238

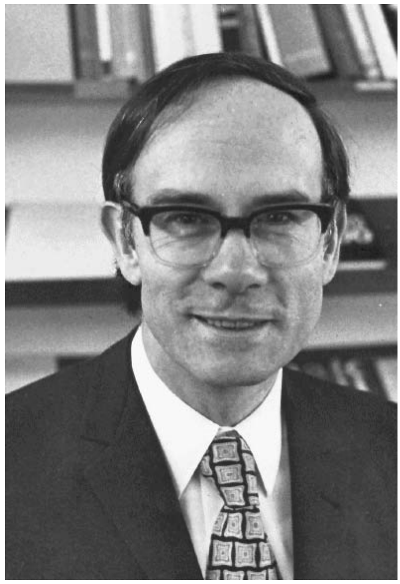


FIG. 1. *Hirzebruch at Erlangen in 1976.*

below.) The corresponding $(n-1)$-connected $2n$-manifolds $E^{2n} \cup_h D^{2n}$ were respectively the complex projective plane, the quaternion projective plane, and the Cayley projective plane. For circle bundles over the 2-sphere, classified by elements of the homotopy group $\pi_1(\mathrm{SO}(2)) \cong \mathbb{Z}$, the Hopf fibration was the only possibility, up to sign. However, for 3-sphere bundles over S^4, classified by elements of $\pi_3(\mathrm{SO}(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$, there are infinitely many bundles which at least have the homotopy type of the 7-sphere. More precisely, such a bundle is classified by two elements of $H^4(S^4;\mathbb{Z}) \cong \mathbb{Z}$, namely the *Pontrjagin class* p_1 and the the *Euler class* e (denoted by $\overline{c}$ in the first paper), subject only to the relation $p_1 \equiv 2\mathrm{e} \pmod{4H^4}$. It is not difficult to check that the total space Σ^7 has the homotopy type of a 7-sphere if and only if the Euler class generates $H^4(S^4;\mathbb{Z})$. Thus we potentially have infinitely many distinct 3-connected 8-manifolds (twisted versions of the quaternion projective plane), which are distinguished by their Pontrjagin classes, and in some cases by homotopy invariants as well.

It was natural to take a closer look at the structure of these hypothetical manifolds, using the signature theorem (or index theorem) which had recently been developed by René Thom and Fritz Hirzebruch. For any closed oriented $4m$-dimensional manifold M^{4m} we can form the signature (or "index") of the symmetric bilinear form

$$H^{2m} \otimes H^{2m} \to H^{4m} \cong \mathbb{Z}\,.$$

In the differentiable case, as a consequence of his cobordism theory, Thom had shown that this signature could be expressed uniquely as a rational linear combination of the Pontrjagin numbers $p_{i_1} \cdots p_{i_k}[M^{4m}]$ (where $i_1 \le \cdots \le i_k$ with

$i_1 + \cdots + i_k = m$). He worked out the precise formula only in dimensions 4 and 8; but around the same time Hirzebruch had conjectured such a formula and worked out the precise form which it would have to take in all dimensions. (Compare Hirzebruch [1956 or 1966, 1971].)

For an 8-dimensional manifold, the formula reads

$$\text{signature} = (7p_2 - p_1{}^2)[M^8]/45 \ .$$

In our case, the signature is ± 1, and we can choose the orientation so that it is $+1$. Since $p_2[M^8]$ is an integer, the first Pontrjagin class must satisfy the congruence

$$p_1{}^2[M^8] + 45 = 7\,p_2[M^8] \ \equiv \ 0 \qquad (\mathrm{mod}\, 7) \ .$$

> *Thus, for any allowable choice of p_1 which does not satisfy this congruence, we have constructed a homotopy 7-sphere Σ^7 which cannot be diffeomorphic to the standard 7-sphere.*

At this point, I believed that I had constructed a counterexample to the Poincaré conjecture in dimension 7. In other words, I assumed that Σ^7 could not even be continuously homeomorphic[2] to the standard S^7. Fortunately however, I did some experimentation, and discovered that this Σ^7 actually is homeomorphic to S^7. In fact, it can be obtained by pasting together the boundaries of two standard 7-disks under a boundary diffeomorphism. (I call such a manifold, a *twisted sphere*.) As an extra bonus, the proof also showed that there exist non-standard diffeomorphisms of the 6-dimensional sphere. These results are described in the paper **On Manifolds Homeomorphic to the 7-Sphere**, on pages 11–17 below. (For an analogous construction in dimension 15, see Shimada [1957].)

The next paper **On the Relationship between Differentiable Manifolds and Combinatorial Manifolds**, written in the same year but never published, carries the discussion further by tying it in with Henry Whitehead's theory of C^1-triangulation, and by describing Thom's proof, based on his theory of combinatorial Pontrjagin classes,[3] that the combinatorial manifold $E^8 \cup_h D^8$ described above has no compatible differentiable structure. (A few years later, Michel Kervaire [1960] constructed a topological 10-manifold which cannot be given any differentiable structure at all, and Steve Smale [1961] constructed an analogous example in dimension twelve. Still later, with the proof by Sergei Novikov [1965] that rational Pontrjagin classes are actually topological invariants, it followed that the manifold $E^8 \cup_h D^8$ above has no differentiable structure at all.)

The expository paper **Sommes de Variétés Différentiables et Structures Différentiables des Sphères**, presented (in English) at a conference in Lille a

[2]Actually, there is some question as to what Poincaré meant when he formulated his conjecture for 3-manifolds. The word "homeomorphism" did not have a universally understood meaning at the time, and he may well have intended that homeomorphisms were to be differentiable.

[3]Thom [1958]. For C^1-triangulations, see Whitehead [1940], Munkres [1963]. For combinatorial Pontrjagin classes, compare the presentation in Milnor and Stasheff [1974].

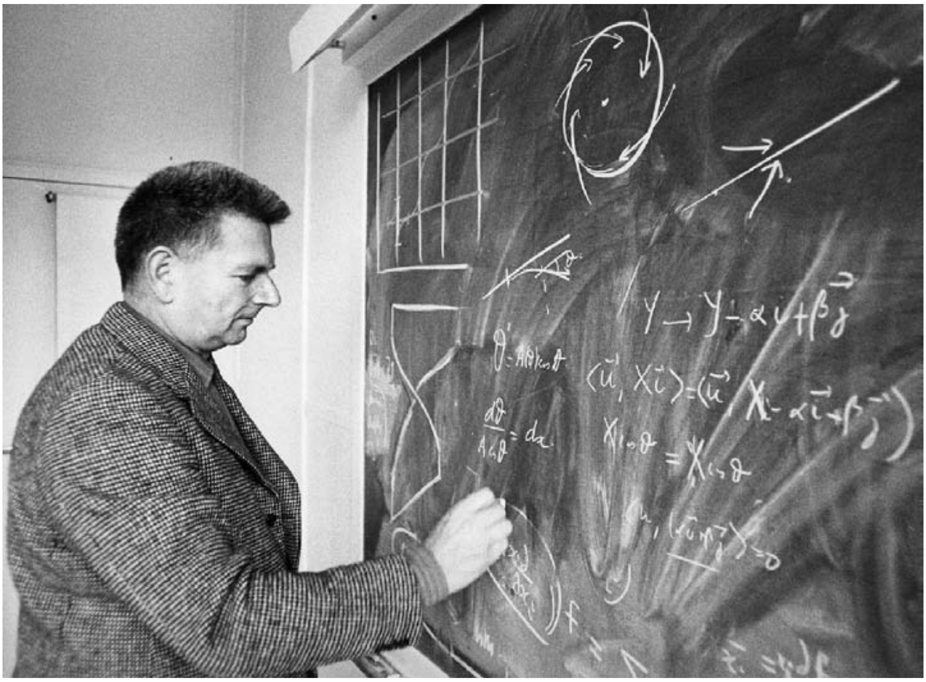


FIG. 2. *René Thom at the Institut des Hautes Études Scientifiques in the early 1970's.*

few years later, introduces this group Γ_n of oriented diffeomorphism classes of twisted spheres. Almost by definition, there is a homomorphism from the group $\pi_0\,\mathrm{Diff}^+(S^{n-1})$ of smooth isotopy classes of orientation preserving diffeomorphisms onto Γ_n. (In fact this homomorphism extends to an exact sequence

$$\cdots \to \pi_1\,\mathrm{Diff}^+(S^{n-1}) \to$$
$$\pi_0\,\mathrm{Diff}^+(D^n \,\mathrm{rel}\, S^{n-1}) \to \pi_0\,\mathrm{Diff}^+(D^n) \to \pi_0\,\mathrm{Diff}^+(S^{n-1}) \to \Gamma_n \to 1$$

of abelian groups. Compare the paper "Differential Topology" on pages 123–141 below.) A different characterization, due to THOM [1959] is that a smooth manifold is a twisted sphere if and only if it is combinatorially equivalent to the standard sphere.

This paper also introduced the group Θ_n of smooth oriented manifolds having the homotopy type of S^n, up to the relation which was then called "J-equivalence" but is now known as "h-cobordism". Evidently there is a natural homomorphism $\Gamma_n \to \Theta_n$. In fact we will see later (modulo the Poincaré Conjecture for the case $n = 3$) that Γ_n maps isomorphically onto Θ_n for all n.

The paper **Differentiable Structures on Spheres**, carries out a similar argument based on hypothetical manifolds of the form $(S^p \vee S^q) \cup D^{p+q}$, where the two sub-spheres intersect transversally, with normal bundles described by elements of $\pi_{p-1}(\mathrm{SO}_q)$ and $\pi_{q-1}(\mathrm{SO}_p)$ respectively. This construction is much more robust,

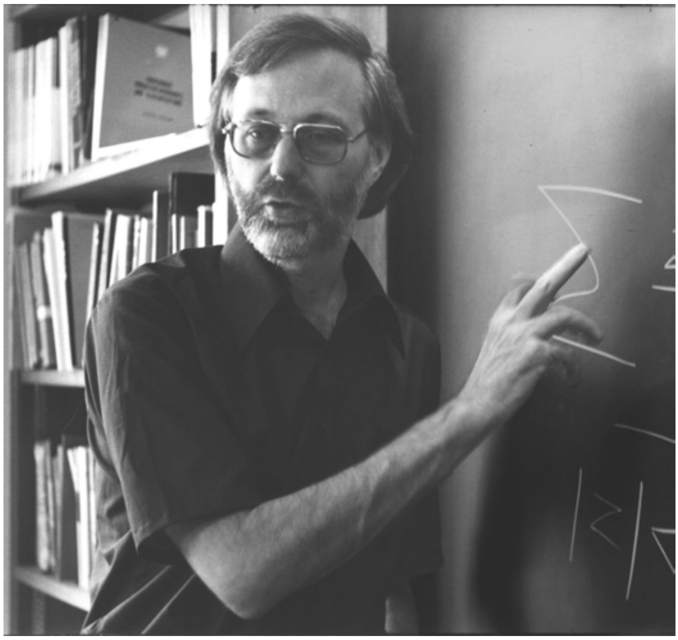


FIG. 3. *The author at the Institute for Advanced Study in 1981.*

not being limited to dimensions 7 and 15. (There is a closely related bilinear pairing

$$\pi_k \mathrm{SO}_\ell \otimes \pi_\ell \mathrm{SO}_\mathrm{k} \ \to\ \pi_0 \mathrm{Diff}^+(\mathrm{S}^{\mathrm{k}+\ell}) \ \to\ \Gamma_{\mathrm{k}+\ell+1}\ .$$

Compare the introduction to Part 2 on page 143.)

Although I talked with René Thom only a few times during the years when these papers were written, his influence was quite important. For example, his intuitive feeling for the structure of cobordism rings went far beyond his published work. During one particularly decisive conversation, he constructed an interesting example by a technique which I called *surgery*. (The same construction was introduced independently by Andrew WALLACE [1960], who called it *spherical modification*. Both terms have been frequently used in the literature.) I developed this idea in the paper **A Procedure for Killing Homotopy Groups of Differentiable Manifolds**. A fairly easy Morse theory argument shows that one manifold can be obtained from another by a sequence of spherical modifications if and only if the two belong to the same cobordism class. These ideas played a key role in further work on groups of homotopy spheres.[4]

The manuscript **Differentiable Manifolds which are Homotopy Spheres**, written in 1959, was never published since most of its results were absorbed into a larger paper **Groups of Homotopy Spheres I**, written in collaboration with

[4]For further developments in surgery theory, see for example, WALL [1970], BROWDER [1972], MADSEN AND MILGRAM [1979], and CAPPELL ET AL. [2000, 2001].

Michel Kervaire. These papers provided a much more systematic analysis of all possible homotopy spheres of any dimension (other than 3). In particular, they showed that Θ_n is finite for $n \neq 3$. The analysis is based on an exact sequence

$$0 \ \to \ bP_{n+1} \ \to \ \Theta_n \ \to \ \Pi_n / J\pi_n(\mathrm{SO}) \ .$$

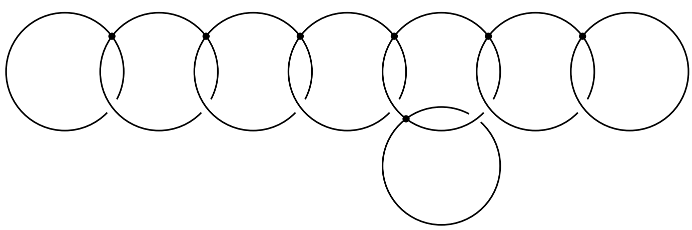
Here bP_{n+1} (denoted by $\Theta_n(\partial\pi)$ in the 1959 manuscript) is the subgroup consisting of those homotopy n-spheres which bound parallelizable manifolds. Using the Thom-Pontrjagin theory of cobordism for manifolds with framed normal bundle, the obstruction to bounding an n-manifold with framed normal bundle is measured by an element of the stable homotopy group Π_n. However, a change in framing will change this obstruction by an element in the image of the stable J-homomorphism $J : \pi_n(\mathrm{SO}) \to \Pi_\mathrm{n}$. (Compare the introduction to Part 3 on pages 223–227.) The subgroup bP_{n+1} is trivial for n even and has at most two elements for $n \equiv 1 \pmod 4$. However, this group is quite large when $n \equiv 3 \pmod 4$. In fact bP_{4m} is cyclic of order

$$a_m \, 2^{2m-2}(2^{2m-1} - 1) \ \mathrm{numerator}(B_m/m) \, ,$$

where a_m is one or two according as m is even or odd, and B_m is the m-th Bernoulli number. These numbers grow rapidly as $m \to \infty$, as one sees from the identity

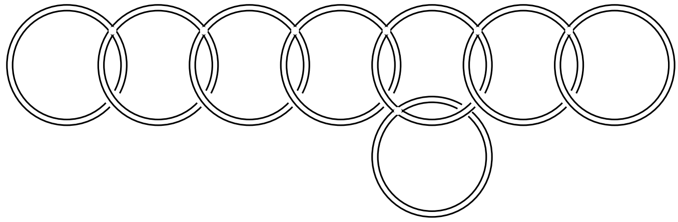
$$B_m = 2\left(1 + 2^{-2m} + 3^{-2m} + \cdots\right)(2m)!/(2\pi)^{2m} \ .$$

(Compare Milnor and Stasheff [1974, Appendix B].) An explicit generator can be constructed as follows.[5] Start with a $2m$-skeleton consisting of eight copies of the $2m$-sphere intersecting in seven points, indicated schematically as follows.



Now thicken each of these spheres by taking a $2m$-disk bundle isomorphic to its tangent disk bundle, with Euler number $+2$. The spheres are to cross each other transversally at each intersection point. (Here the normal bundles should all be non-trivial, but that is not shown in the very schematic picture on the next page.) The result will be a parallelizable $4m$-dimensional manifold W^{4m} of signature $+8$, having a homotopy sphere as boundary, provided that $m > 1$. It is shown that this boundary ∂W^{4m} is a generator for the finite cyclic group consisting of all homotopy spheres of dimension $4m - 1$ which bound parallelizable manifolds.

[5] This form of the construction, based on the E_8-lattice, was suggested by Hirzebruch. Compare the discussion in Hirzebruch [1987, pp. 673, 801]. My original construction was more complicated.



Clearly $\Theta_1 = \Theta_2 = 0$. Using these techniques, the first few groups Θ_n with $n \geq 4$ were described as follows. (Here, for example, the notation (8) stands for some abelian group of order 8.)

$n =$	4	5	6	7	8	9	10	11	12	13	14	15
$\Theta_n \cong$	0	0	0	$\mathbb{Z}/28$	$\mathbb{Z}/2$	(8)	$\mathbb{Z}/6$	$\mathbb{Z}/992$	0	$\mathbb{Z}/3$	$\mathbb{Z}/2$	(16256)

The projected "Groups of Homotopy Spheres: II" was never completed, although a very small part of it found its way into the expository paper **Differential Topology** (pages 123–141 below), which was published a few years later.

Meanwhile, the original project of publishing a paper on $2n$-manifolds which are $(n-1)$-connected, and a closely related project of studying "Spaces with a gap in cohomology" got lost in the shuffle. They were finally abandoned when Terry Wall published a beautiful exposition of the subject which made my attempts unnecessary. (See WALL [1962a], and compare WALL [1962b, 1964].)

Further Developments.

For the differential geometry of exotic spheres, see for example GROMOLL [1966], GROMOLL AND MEYER [1974], WRAITH [1997], GROVE AND ZILLER [2000], BOYER ET AL. [2005]. For exotic spheres via algebraic geometry, see BRIESKORN [1966], HIRZEBRUCH [1966/67], HIRZEBRUCH AND MAYER [1968], and compare MILNOR [1968].

The importance of the groups Γ_n of twisted spheres was brought out by Jim MUNKRES [1960a, 1964] and Moe HIRSCH [1963] who showed that they appear as the coefficient groups for obstructions to the existence and uniqueness of a compatible differentiable structure on a combinatorial manifold.

The more precise results of my paper with Kervaire seemed to be achieved at the cost of replacing these coefficient groups Γ_n by the cruder groups Θ_n. In other words, *twisted spheres* were replaced by *homotopy spheres*, and the relation of *diffeomorphism* by the apparently weaker relation of *h-cobordism*. However, Steve SMALE, in [1961, 1962], had proved that every homotopy sphere of dimension $n \geq 5$ is actually a twisted sphere.[6] In [1962] he proved that simply-connected h-cobordant manifolds of dimension at least five are actually diffeomorphic. (Compare

[6]More precisely, Smale proved this in [1961] for even n, while the general case with $n \geq 6$ follows by applying his h-cobordism theorem [1962] to the n-manifold with two open disks deleted. The 5-dimensional case depends on the h-cobordism theorem together with the additional information that $\Theta_5 = 0$.

the presentation in MILNOR, SIEBENMANN AND SONDOW [1965].) It followed that $\Gamma_n \xrightarrow{\cong} \Theta_n$ for $n \geq 5$.

The groups Γ_1 and Γ_2 are clearly trivial. SMALE [1959b] and MUNKRES [1960b] had shown that $\mathrm{Diff}^+(S^2)$ is connected which implies that $\Gamma_3 = 0$. Jean CERF showed in the [1962/63] Cartan Seminar that $\Gamma_4 = 0$, and indeed that $\mathrm{Diff}^+(S^3)$ is connected. (Compare CERF [1968]; and see ELIASHBERG [1992] for a different proof that $\Gamma_4 = 0$.) Combining these results with the statements above, it follows that Γ_n is trivial for $n < 7$. Thus the Munkres-Hirsch obstruction theory yields the following:

> *Every combinatorial manifold of dimension ≤ 7 possesses a compatible differentiable structure. Furthermore, in dimensions strictly less than 7 this structure is unique up to diffeomorphism.*

An analogous obstruction theory for passing from topological manifolds to combinatorial manifolds was constructed by Rob KIRBY and Larry SIEBENMANN [1969]. In this case, there is only one obstruction: in $H^4(M; \mathbb{Z}/2)$ for existence, and in $H^3(M; \mathbb{Z}/2)$ for uniqueness. Here the dimension of M must be at least 6 (or at least 5 if M has no boundary).

If we accept the announced proof of the Poincaré Conjecture by PERELMAN [2002, 2003a, 2003b], then it follows that $\Theta_3 = 0$, so that $\Gamma_n \xrightarrow{\cong} \Theta_n$ for all n. However, this simple statement conceals the fact that dimension 4 is a world by itself, different from all other dimensions. Topological 4-manifolds can be wildly non-differentiable, so it was necessary to introduce wildly non-differentiable methods in order to understand them. One key step was the introduction of "flexible handles" by Andrew Casson. (For a description of early steps see SIEBENMANN [1980] and MANDELBAUM [1980], as well as GUILLOU AND MARIN [1986].) The decisive breakthrough came with the work of Mike FREEDMAN [1982], who used Casson handles not only to prove the 4-dimensional topological Poincaré Hypothesis, but also to completely classify closed simply-connected topological 4-manifolds.

Completely different tools were needed to get a grip on the differentiable theory. If we consider only simply-connected 4-manifolds which are differentiable (or combinatorial) then Simon DONALDSON [1983, 1987] showed by gauge theory that there are very strong restrictions on the cohomology. It quickly became apparent to specialists that the contrast between Freedman's results and those of Donaldson led to very strange consequences. (Compare KIRBY [1989], GOMPF [1993].) For example Freedman showed that there exists a differentiable manifold homeomorphic to $\mathbb{R}^4$ which cannot be smoothly embedded in $\mathbb{R}^4$. Then Clifford TAUBES [1987] showed that in fact there are uncountably many such manifolds; while DEMICHELIS AND FREEDMAN [1992] showed that there are also uncountably many distinct examples which *can* be smoothly embedded in $\mathbb{R}^4$. The situation in other dimensions is quite different, since no exotic $\mathbb{R}^n$ can exist for $n \neq 4$. (See MOISE [1952] for $n < 4$ and STALLINGS [1962] for $n > 4$.)

The question of exotic structures on the 4-sphere seems particularly difficult. In fact one can ask whether there exists a non-standard differentiable structure on

S^4 which reduces to the standard structure on $S^4 \setminus$ (point), and one can also ask whether there exists one which does not. As far as I know, both questions remain open.

Since I cannot describe all the work in differential topology which has been done since these papers were written, let me simply list the titles of some books which help to fill in further history. (For complete citations, see the bibliography, starting on page 319.)

Munkres, "Elementary Differential Topology," 1963;
Milnor, Spivak and Wells, "Morse Theory," 1963;
Milnor, "Topology from the Differentiable Viewpoint," 1965;
Milnor, Siebenmann and Sondow, "Lectures on the h-Cobordism Theorem," 1965;
Stong, "Notes on Cobordism Theory," 1968;
Wallace, "Differential Topology, First Steps," 1968;
Guillemin and Pollack, "Differential Topology," 1974;
Milnor and Stasheff, "Characteristic Classes," 1974;
Hirsch, "Differential Topology," 1976;
Chillingworth, "Differential Topology with a View to Applications," 1976;
Kirby and Siebenmann, "Foundational Essays on Topological Manifolds, Smoothings, and Triangulations," 1977;
Bröcker and Jänich, "Introduction to Differential Topology," 1982;
Gauld, "Differential Topology, an Introduction," 1982;
Bing, "The Geometric Topology of 3-Manifolds," 1983;
Guillou and Marin (editors), "A la Recherche de la Topologie Perdue," 1986;
Dieudonné, "A History of Algebraic and Differential Topology, 1900-1960," 1989;
Kirby, "The Topology of 4-Manifolds," 1989;
Donaldson and Kronheimer, "The Geometry of Four-Manifolds," 1990;
Freedman and Quinn, "Topology of 4-Manifolds," 1990;
Akbulut and McCarthy, "Casson's invariant for oriented homology 3-spheres. An exposition," 1990;
Nash, "Differential Topology and Quantum Field Theory," 1991;
Kosinski, "Differential Manifolds," 1992;
Friedman and Morgan,"Smooth Four-Manifolds and Complex Surfaces," 1994;
McDuff and Salamon, "Introduction to Symplectic Topology," 1995.
Friedman and Morgan, "Gauge Theory and the Topology of Four-Manifolds," 1998;
Gompf and Stipsicz, "4-Manifolds and Kirby Calculus," 1999.