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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
JOHN MILNOR

MR0148075 (26 #5584) 57.10

Kervaire, Michel A.; Milnor, John W.

Groups of homotopy spheres. I.

Annals of Mathematics. Second Series **77** (1963), 504–537.

The authors aim to study the set of h-cobordism classes of smooth homotopy n-spheres; they call this set Θ_n. They remark that for $n \neq 3, 4$ the set Θ_n can also be described as the set of diffeomorphism classes of differentiable structures on S^n; but this observation rests on the "higher-dimensional Poincaré conjecture" plus work of Smale [Amer. J. Math. **84** (1962), 387–399], and it does not really form part of the logical structure of the paper. The authors show (Theorem 1.1) that Θ_n is an abelian group under the connected sum operation. (In § 2, the authors give a careful treatment of the connected sum and of the lemmas necessary to prove Theorem 1.1.)

The main task of the present paper, Part I, is to set up methods for use in Part II, and to prove that for $n \neq 3$ the group Θ_n is finite (Theorem 1.2). (For $n = 3$ the authors' methods break down; but the Poincaré conjecture for $n = 3$ would imply that $\Theta_3 = 0$.) We are promised more detailed information about the groups Θ_n in Part II.

The authors' method depends on introducing a subgroup $bP_{n+1} \subset \Theta_n$; a smooth homotopy n-sphere qualifies for bP_{n+1} if it is the boundary of a parallelizable manifold. The authors prove in § 4 that the quotient group Θ_n/bP_{n+1} is finite (Theorem 4.1). More precisely, they prove that bP_{n+1} is the kernel of a homomorphism $p'\colon \Theta_n \to \Pi_n/\mathrm{Im}J$, where Π_n is the stable group $\pi_{n+k}(S^k)$ and $\mathrm{Im}\, J$ is the image of the classical J-homomorphism. § 4 ends by giving (explicitly) the groups Θ_n/bP_{n+1} for $n \leq 8$ and the groups bP_{n+1} for $n \leq 19$, referring the reader to Part II for details.

The proof given in § 4 depends on results in § 3. In this section, Theorem 3.1 states that every homotopy sphere is S-parallelizable, that is, its tangent bundle is stably trivial. The proof uses previous work of the same authors, and involves quoting information about the J-homomorphism. The remaining lemmas in § 3 concern the stability of bundles.

It remains to prove that the groups bP_{n+1} are finite. The authors divide two cases. If n is even they prove that the groups bP_{n+1} are zero. That is, in §§ 5, 6 they prove (Theorem 5.1): If a smooth homotopy sphere of dimension $2k$ bounds an S-parallelizable manifold M, then it bounds a contractible manifold. The proof consists of simplifying M by surgery [J. Milnor, Proc. Sympos. Pure Math., Vol. III, pp. 39–55, Amer. Math. Soc., Providence, R.I., 1961; MR0130696 (24 #A556)]. The details are technical, and appear to be comparable with work of C. T. C. Wall, which also results in a proof of the same theorem [Trans. Amer. Math. Soc. **103** (1962), 421–433; MR0139185 (25 #2621)]. § 5 completes the proof for k even; the case in which k is odd is treated in § 6. Here the authors introduce the notion of a "framed manifold", that is, a smooth manifold M plus a given trivialisation of the stable tangent bundle of M. The authors arrange to carry this extra structure through the technique of surgery, making use of it as they go.

The case in which n is odd is treated in §§7, 8. It is shown that the groups bP_{2k} are finite cyclic, and for k odd they are either 0 or Z_2 (Corollary 7.6; Theorem 8.5). The case in which k is even is dealt with in § 7. Here the only obstruction to performing surgery on M is the signature or index $\sigma(M)$ (Lemma 7.3). This leads to the following result (Theorem 7.5). Let Σ_1 and Σ_2 be homotopy spheres of dimension $4m-1$ ($m>1$) which bound S-parallelizable manifolds M_1 and M_2. Then Σ_1 and Σ_2 and h-cobordant if and only if $\sigma(M_1)\equiv\sigma(M_2) \bmod \sigma_m$. Here σ_m is a certain positive integer. § 7 concludes by giving explicit information about the integer σ_m and the order of the groups bP_{4m} and Θ_{4m-1}. The reader is referred to Part II for details.

The cases $k=1,3,7$ are exceptional; the group bP_{2k} is then zero (Lemma 7.2). The case "k odd$\neq 1,3,7$" is studied in § 8. In this case the only obstruction to performing surgery on M is an "Arf invariant" lying in Z_2. The authors conjecture that in this case the group bP_{2k} is always Z_2 rather than 0; but this is known only for $k=5,9$.

J. F. Adams

From MathSciNet, June 2015

MR0802786 (87i:57031) 57R60; 57R65

Levine, J. P.

Lectures on groups of homotopy spheres.

Algebraic and geometric topology (New Brunswick, N.J., 1983), 62–95, *Lecture Notes in Math., 1126*, *Springer, Berlin*, 1985.

M. A. Kervaire and J. W. Milnor [Ann. of Math. (2) **77** (1963), 504–537; MR0148075 (26 #5584)] began the classification of homotopy spheres, smooth closed manifolds homotopy equivalent to spheres. It was to be the first of two papers, the second of which never appeared. The present paper is what the author believes would have been "Groups of homotopy spheres, II". This very valuable article is based on his 1969 lectures, later distributed as mimeographed lecture notes. We recall that the basic aim is to calculate θ^n, the group of h-cobordism classes of homotopy n-spheres, and bP^{n+1}, the subgroup of θ^n defined by those homotopy n-spheres which bound parallelizable $(n+1)$-manifolds. The goal of the lectures is to compute bP^{n+1} and $\theta^n/\mathrm{bP}^{n+1}$. The end result is expressed most elegantly by means of the Kervaire–Milnor long exact sequence $\cdots\to A^{n+1}\overset{p}{\to}P^{n+1}\overset{b}{\to}\theta^n\overset{i}{\to}A^n\overset{p}{\to}P^n\to\cdots$, where A^n can be calculated from the exact sequence $\cdots\to\pi_n(\mathrm{SO})\overset{J}{\to}\pi_n(S)\overset{t}{\to}A^n\overset{\mathrm{O}}{\to}\pi_{n-1}(\mathrm{SO})\overset{J}{\to}\pi_{n-1}(S)\to\cdots$, with J the stable J-homomorphism going from the homotopy groups of the infinite special orthogonal group into the stable homotopy groups of spheres—which can also be thought of as the framed cobordism group. A^n represents the group of "almost framed" cobordism classes of almost framed (i.e. framed at all but one point) closed n-manifolds. We also need to know that $P^n=0$ for $n\equiv 1,3 \pmod 4$, $P^n=\mathbf{Z}$ for $n\equiv 0 \pmod 4$, and $P^n=Z_2$ for $n\equiv 2 \pmod 4$.

Aside from basic techniques of algebraic and differential topology and the original Kervaire–Milnor article [op. cit.], the lectures assume familiarity only with Milnor's earlier paper [*Differential geometry*, 39–55, Proc. Sympos. Pure Math., III, Amer. Math. Soc., Providence, R.I., 1961; MR0130696 (24 #A556)] on surgery techniques.

Joel M. Cohen
From MathSciNet, June 2015

MR0620795 (82h:57027) 57R67; 18F25, 57-02

Ranicki, Andrew

Exact sequences in the algebraic theory of surgery. (English)

Mathematical Notes, 26.

Princeton University Press, *Princeton, N.J.; University of Tokyo Press, Tokyo*, 1981, xvii+864 pp., $16.50 paperbound, ISBN 0-691-08276-6

Surgery is concerned with two problems: When is a space homotopically equivalent to a manifold, and when are homotopically equivalent manifolds homeomorphic, diffeomorphic or PL equivalent? This theory has been vigorously developed during the last 20 years. Unfortunately, there are few books containing an exposition of this topic, and those are basically research monographs and hence make for rather heavy reading. In particular, the reviewer is familiar with three books, including the one here reviewed. The other two are *Surgery on simply-connected manifolds* by W. Browder [Springer, New York, 1972; MR0358813 (50 #11272)] and *Surgery on compact manifolds* by C. T. C. Wall [Academic Press, London, 1970; MR0431216 (55 #4217)]. These two books together with the one under review should be owned by everyone interested in surgery.

For someone who wants to learn this subject, the reviewer would suggest the following approach. First, read J. W. Milnor's paper [*Proceedings of the Symposia in Pure Mathematics, Vol. III*, pp. 39–55, Amer. Math. Soc., Providence, R.I., 1961; MR0130696 (24 #A556)]. It introduces the basic geometric construction of surgery. Then, read M. A. Kervaire and Milnor's paper [Ann. of Math. (2) **77** (1963), 504–537; MR0148075 (26 #5584)], which thoroughly examines the surgery obstruction in the simply connected case with the object of classifying the possible differential structures on the n-sphere, $n \neq 4$. After this, read Browder's paper [Proc. Cambridge Philos. Soc. **61** (1965), 337–345; MR0175136 (30 #5321)], which introduces codimension-one splitting problems and leads to S. P. Novikov's paper [*International congress of mathematicians* (Moscow, 1966), pp. 172–179, Amer. Math. Soc., Providence, R.I., 1968; MR0231401 (37 #6956)], which contains an important application of surgery, namely, the topological invariance of the rational Pontrjagin classes. This last paper also motivates the extension of the theory to non-simply connected manifolds, which is the principal focus of both Wall's and the present author's books. Next, read Browder's book, which develops surgery theory for simply connected manifolds and contains a very good discussion of the Kervaire-Arf invariant. Browder's book (together with Sullivan's lecture notes ["Triangulating and smoothing homotopy equivalences and homeomorphisms", Geometric Topology Seminar Notes, Princeton Univ., Princeton, N.J., 1967], which expounds the homotopy structure of G/PL) gives one a good picture of surgery theory for simply connected manifolds—at least, modulo some unsolved problems about the Kervaire-Arf invariant.

Next, one should read Wall's book. It extends the theory to non-simply connected manifolds and discusses many interesting applications, such as space-form problems. The crucial thing is to analyze the surgery map $\sigma\colon [M^n, G/\mathrm{Top}] \to L_n(\pi_1 M)$, where $[M^n, G/\mathrm{Top}]$ is the group of homotopy classes of maps of the manifold M^n to G/Top and $L_n(\pi_1 M)$ is the surgery obstruction group. Very important but only partially solved problems are to calculate $L_n(\pi_1 M)$ and σ.

The monograph under review recasts the foundational material in Wall's book in a more algebraic and functorial setting and answers many questions posed by Wall. For instance, it allows for a better understanding of the surgery map σ by taking care of the anomaly that $L_n(\ )$ is covariant while $[\ , G/\mathrm{Top}]$ is contravariant. It also gives more insights into calculating $L_n(\Gamma)$ via localization theorems and by giving algebraic proofs of most of the splitting theorems, thus allowing them to be applied inductively to larger classes of groups. (The details of some of these splitting results will be given in a later paper by the author ["Splitting theorems in the algebraic theory of surgery", to appear].) Also, the author has asked the reviewer to mention that the asserted "mild generalization of the splitting theorem of J. Shaneson [Ann. of Math. (2) **90** (1969), 296–334; MR0246310 (39 #7614)]" on p. 813 is wrong, and that "the discussion on pp. 812–814 should therefore be restricted to the case $\omega = +1$ only".

For the experts, a panorama of this book is best provided by a glance at its table of contents. Chapter 1. Absolute L-theory: 1.1 $\mathbf{Q}$-groups; 1.2 L-groups; 1.3 Triad $\mathbf{Q}$-groups; 1.4 Algebraic Wu classes; 1.5 Algebraic surgery; 1.6 Forms and formations; 1.7 Algebraic glueing; 1.8 Unified L-theory; 1.9 Products; 1.10 Change of K-theory. Chapter 2. Relative L-theory: 2.1 Algebraic Poincaré triads; 2.2 Change of rings; 2.3 Change of categories; 2.4 Γ-groups; 2.5 Change of K-theory. Chapter 3. Localization: 3.1 Localization and completion; 3.2 The localization exact sequence ($n \geq 0$); 3.3 Linking Wu classes; 3.4 Linking forms; 3.5 Linking formations; 3.6 The localization exact sequence ($n \in \mathbf{Z}$); 3.7 Change of K-theory. Chapter 4. Arithmetic L-theory: 4.1 Dedekind algebra; 4.2 Dedekind rings; 4.3 Integral and rational L-theory. Chapter 5. Polynomial extensions ($\overline{x} = x$): 5.1 L-theory of polynomial extensions; 5.2 Change of K-theory. Chapter 6. Mayer-Vietoris sequences: 6.1 Triad L-groups; 6.2 Change of K-theory; 6.3 Cartesian L-theory; 6.4 Ideal L-theory. Chapter 7. The algebraic theory of codimension q surgery: 7.1 The total surgery obstruction; 7.2 The geometric theory of codimension q surgery; 7.3 The spectral quadratic construction; 7.4 Geometric Poincaré splitting; 7.5 Algebraic Poincaré splitting; 7.6 The algebraic theory of codimension 1 surgery; 7.7 Surgery with coefficients; 7.8 The algebraic theory of codimension 2 surgery; 7.9 The algebraic theory of knot cobordism.

To summarize, this is a carefully written and lucid (but lengthy) account of an important topic in topology which the reviewer strongly recommends to anyone interested in the structure of manifolds.

F. T. Farrell

From MathSciNet, June 2015

MR2061749 (2005e:57075) 57R65; 19J25, 57R67

Ranicki, Andrew

Algebraic and geometric surgery. (English)

Oxford Mathematical Monographs.
The Clarendon Press, Oxford University Press, *Oxford*, 2002, xii+373 pp.,
ISBN 0-19-850924-3

Surgery theory, loosely speaking, refers to a variety of algebraic and geometric techniques used to classify manifolds, typically of dimensions 4 or greater. The theory encompasses topological classification within a homotopy type, existence and uniqueness of smooth or PL structures, and many other topics such as embeddings or automorphisms of manifolds. The term surgery itself refers to the process of cutting out a piece of a manifold (typically of the form $S^k \times D^{n-k}$) and replacing it with another (typically $D^{k+1} \times S^{n-k-1}$). This innocent-seeming operation becomes very powerful when combined with other tools such as bundle theory, handlebody theory (particularly the S-cobordism theorem) and the algebra of quadratic forms. The full complexity of the theory is seen when one is dealing with non-simply-connected spaces, and the calculation of the set $\mathcal{S}(X)$ of smooth n-manifolds homotopy equivalent to a space X is summarized in the "surgery exact sequence":

$$\cdots \to [\Sigma X, G/O] \to L_{n+1}(\mathbf{Z}[\pi_1 X]) \to \mathcal{S}(X) \to [X, G/O] \to L_n(\mathbf{Z}[\pi_1 X])$$

Of course, such an exact sequence per se is never enough to do real calculations; one must calculate and understand all of the terms in the sequence and the maps between them.

Ranicki's book provides an introduction to these ideas, pitched at a reader who knows the basics of algebraic topology and manifold theory. As such, it provides much more geometric background than the classic books on the subject [W. Browder, *Surgery on simply-connected manifolds*, Springer, New York, 1972; MR0358813 (50 #11272); C. T. C. Wall, *Surgery on compact manifolds*, Academic Press, London, 1970; MR0431216 (55 #4217)], from which at least two generations of practitioners have learned the subject. Wall's book, in particular, is a difficult (but rewarding) read that gives an impression of a subject just reaching its full power. (The second edition [*Surgery on compact manifolds*, Second edition, Amer. Math. Soc., Providence, RI, 1999; MR1687388 (2000a:57089)] has some useful commentary and updates by Ranicki.) The surgery sequence appears about halfway through Wall's book, and is followed by several dense chapters giving classification of manifolds in various homotopy types (tori, projective spaces, lens spaces) as well as applications to embeddings of manifolds. By contrast, the surgery sequence is the culmination of the book under review, which takes its time developing the geometric background and the algebra necessary to define the surgery groups, $L_n(\mathbf{Z}[\pi_1 X])$.

The treatment of the surgery groups, especially for n odd, differs from the original treatment in Wall's book. When n is even, the surgery groups are defined as equivalence classes of quadratic forms, which in turn are defined in terms of $\mathbf{Z}[\pi_1 X]$-valued intersection numbers. The detailed discussion of these intersection numbers (and the more subtle self-intersections) are an attractive feature of Ranicki's book. When n is odd, the definition of L_n is given in terms of pairs of Lagrangians in a standard $\mathbf{Z}[\pi_1 X]$-valued quadratic form. This definition, apparently suggested by S. P. Novikov [Math. USSR-Izv. **4** (1970), 257–292; ibid. **4** (1970), 479–505; translated from Izv. Akad. Nauk SSSR Ser. Mat. **34** (1970), 253–288; ibid. **34** (1970), 475–500.; MR0292913 (45 #1994)], is equivalent to that given in Wall's book but is somewhat easier to digest. Except for $\pi_1 = \{1\}$, there are no actual calculations of surgery groups in the book. However, some basic tools such as localization and the author's theory of "algebraic surgery" are briefly discussed, and references to the literature are given.

All in all, Ranicki's book is a readable introduction to this powerful theory that will be useful to a student or beginning user. One thing that such a reader should know, however, is that many of the proofs of the background results are sketched rather than being given in detail. The author's own advice (from the preface) is sound: start with the classic paper by M. A. Kervaire and J. W. Milnor [Ann. of Math. (2) **77** (1963), 504–537; MR0148075 (26 #5584)] and then move on to the more sophisticated versions. This book provides a very good next step.

Daniel Ruberman

From MathSciNet, June 2015

MR2874640 53D12; 57R17, 57R60, 57R90

Abouzaid, Mohammed

Framed bordism and Lagrangian embeddings of exotic spheres.

Annals of Mathematics. Second Series **175** (2012), *no.* 1, 71–185.

Recall that the cotangent bundle of a smooth manifold admits a standard symplectic structure, and that a diffeomorphism between two smooth manifolds induces a symplectomorphism of the corresponding cotangent bundles. From early on in the history of symplectic topology it was hoped that the symplectic structure on the cotangent bundle could be used as an effective invariant of smooth manifolds; for instance, one could try to show that two homeomorphic manifolds are not diffeomorphic by showing that their cotangent bundles are not symplectomorphic (it has long been known on the other hand that homeomorphic-but-not-diffeomorphic manifolds often have diffeomorphic cotangent bundles; see, e.g., [R. De Sapio, Math. Z. **107** (1968), 232–236; MR0238341 (38 #6617)] for the case of exotic spheres). However, by the middle of the last decade optimism regarding this program was beginning to fade in some quarters, partly because of the discovery of relations between certain symplectic invariants of the cotangent bundle with invariants arising from the string topology of the underlying manifold, which in turn depended only on the homotopy type of the manifold.

The appearance of the paper under review has restored such optimism, providing for the first time examples of pairs of homeomorphic smooth manifolds whose cotangent bundles are not symplectomorphic. Indeed, it is shown that for any integer k, a homotopy $(4k+1)$-sphere Σ can have cotangent bundle $T^*\Sigma$ symplectomorphic to T^*S^{4k+1} only if Σ is the boundary of a parallelizable compact manifold. Since by [M. A. Kervaire and J. W. Milnor, Ann. of Math. (2) **77** (1963), 504–537; MR0148075 (26 #5584)] for $k \geq 2$ there are several exotic $(4k+1)$-spheres that do not bound parallelizable compact manifolds, these exotic spheres cannot have cotangent bundle symplectomorphic to T^*S^{4k+1}.

More specifically, what is shown is that any Lagrangian homotopy sphere (or for that matter any compact Lagrangian submanifold, in view of progress on the nearby

Lagrangian conjecture) in T^*S^{4k+1} is the boundary of a parallelizable compact manifold. This obviously implies the aforementioned result, since a symplectomorphism $T^*\Sigma \to T^*S^{4k+1}$ would send the zero section of $T^*\Sigma$ to a Lagrangian submanifold of T^*S^{4k+1}. The starting point for the proof is the fact that the graph of the Hopf fibration embeds S^{4k+1} as a Lagrangian submanifold of $\mathbb{C}^{2k+1} \times \mathbb{CP}^{2k}$ (with the opposite of the standard symplectic form on the first factor). Consequently by the Weinstein neighborhood theorem any compact smooth manifold that embeds as a Lagrangian submanifold of T^*S^{4k+1} also embeds a Lagrangian submanifold L of $\mathbb{C}^{2k+1} \times \mathbb{CP}^{2k}$.

Like any compact submanifold of $\mathbb{C}^{2k+1} \times \mathbb{CP}^{2k}$, L can be disjoined from itself by the time-one map of some compactly supported Hamiltonian $H : [0,1] \times \mathbb{C}^{2k+1} \times \mathbb{CP}^{2k} \to \mathbb{R}$. One then considers, as $R \geq 0$ varies, solutions $u : \mathbb{R} \times [0,1] \to \mathbb{C}^{2k+1} \times \mathbb{CP}^{2k}$ to a Cauchy-Riemann-type equation

$$\begin{aligned} &\frac{\partial u}{\partial s} + J(s,t)\left(\frac{\partial u}{\partial t} - \lambda_R(s) X_H(t, u(s,t))\right) = 0, \\ &u(0,t), u(1,t) \in L. \end{aligned} \tag{1}$$

Here the compactly supported smooth functions $\lambda_R : \mathbb{R} \to [0,1]$ vary smoothly with R and have $\lambda_0 \equiv 0$ and, for $R \geq 1$, $\lambda_R|_{[-R,R]} \equiv 1$. Let $\mathcal{P}(L,0)$ denote the space of solutions to (1) which, after being compactified to maps of D^2 by the removable singularities theorem, represent the trivial homotopy class. As was observed in [Y.-G. Oh, Math. Res. Lett. **4** (1997), no. 6, 895–905; MR1492128 (98k:58048)], the fact that the time-one map of H disjoins L from itself implies that (1) has no homotopically trivial solutions when R is large; on the other hand for $R = 0$ the homotopically trivial solutions of (1) are just the constant maps to L.

As a result one obtains that, for generic auxiliary data, $\mathcal{P}(L,0)$ is a smooth manifold with boundary L. $\mathcal{P}(L,0)$ is not the desired parallelizable manifold, however, because it is demonstrably noncompact. Rather, the Gromov-Floer compactification of $\mathcal{P}(L,0)$ has a codimension-one stratum corresponding to homotopically nontrivial solutions of (1) with holomorphic disks attached to their boundaries, a codimension-two stratum corresponding to solutions of (1) with spheres attached to their interiors, and a codimension-three stratum corresponding to solutions of (1) with spheres attached to their boundaries. Using some ingenious constructions bolstered by quite refined gluing theorems for certain moduli spaces of holomorphic curves (the latter of which take up a substantial majority of the paper), the author nonetheless manages to construct the desired compact parallelizable manifold out of $\mathcal{P}(L,0)$. First he produces a compact manifold with corners whose corner strata consist of the codimension-one strata of the compactification of $\mathcal{P}(L,0)$ together with circle bundles over the original codimension-two and -three strata. This is then glued to another manifold with corners (obtained from a related moduli space of pseudoholomorphic disks) to yield a compact manifold $\widehat{W}(L)$ with boundary whose boundary components are given by L together with several copies of $S^2 \times S^{4k-1}$. Finally it is shown that $\widehat{W}(L)$ is stably parallelizable (it is important here that the dimension of the sphere is congruent to 1 mod 4 and not just odd), and that the $S^2 \times S^{4k-1}$ components of the boundary can be capped off in a way that yields the promised compact parallelizable manifold with boundary L.

Michael J. Usher

From MathSciNet, June 2015

MR0358813 (50 #11272) 57D65

Browder, William

Surgery on simply-connected manifolds. (English)

Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 65.

Springer-Verlag, *New York–Heidelberg*, 1972, ix+132 pp., \$13.40

The operation of surgery on a smooth compact n-manifold M consists of removing the interior of an imbedded solid torus $T = S^p \times D^{n-p} \subseteq \operatorname{interior} M$, $M_0 = M - \operatorname{interior} T$, and forming $M_1 = M_0 \cup D^{p+1} \times S^{n-p-1}$, making the natural identification on the boundaries. This operation can be used to alter the homotopy type of M while preserving its cobordism class and some tangential properties (as in J. Milnor's article [*Differential geometry* (Proc. Sympos. Pure Math., Vol. III, Univ. Arizona, Tucson, Ariz., 1960), pp. 39–55, Amer. Math. Soc., Providence, R.I., 1961; MR0130696 (24 #A556)]). Usually one tries to make M more highly connected. A natural generalization is the operation of surgery on a map $f\colon (M, \partial M) \to (A, B)$, where (A, B) is a pair of CW complexes satisfying Poincaré-Lefschetz duality $H_* A \simeq H_{m-*}(A, B)$, for some $m \geq 0$. In this case, one modifies f, usually trying to make it more highly connected, by performing surgery on suitably chosen $T \subseteq \operatorname{interior} M$ and extending $f|M_0$ to M_1. This procedure has had wide-ranging, deep applications in every area of the topology of smooth manifolds, including transformation groups, classification of manifolds, and imbedding and immersion theory. Analogous techniques and applications hold for PL and topological manifolds.

A systematic use of surgery operations requires more conditions on f, which we now describe. We restrict to the case $m = n$, which is representative and contains the most important applications. We require that f have degree ± 1 and that there exist a map of vector bundles $b\colon V \to E$ covering f, where V is the stable normal bundle of M. We then call f or (f, b) a normal map. Surgery is required to preserve these conditions, in a manner that we shall not make precise, and is called normal surgery. In this context, the fundamental problem of surgery theory is: Under what conditions can a normal map be modified by normal surgeries to produce a normal map (g, c) with g a homotopy equivalence? Without further conditions, one can show that a (g, c) may be produced such that g is $[n/2]$-connected; Poincaré duality implies that we need only raise the connectivity by one more to achieve a homotopy equivalence. The problem thus becomes one of describing the obstruction(s) to achieving this last step.

The case $(A, B) = (D^n, S^{n-1})$ was thoroughly analyzed and exploited by M. Kervaire and Milnor [Ann. of Math. (2) **77** (1963), 504–537; MR0148075 (26 #5584)]. This was generalized to the case of 1-connected A independently by the author [*Colloquium on Algebraic Topology* (Aarhus Univ., Aarhus, 1962), pp. 42–46, Mat. Inst., Aarhus Univ., Aarhus, 1962; see MR0146039 (26 #3565)] and S. Novikov [Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 365–474; MR0162246 (28 #5445); translated in Amer. Math. Soc. Transl. (2) **48** (1965), 271–396; see MR0189948 (32 #7366)]. Finally, the general case was developed by C. T. C. Wall [Ann. of Math. (2) **84** (1966), 217–276; MR0212827 (35 #3692); *Surgery on compact manifolds*, Academic Press, London, 1970]. We have deliberately omitted mention here of the operation of "surgery on the boundary", which leads to interesting variants of the theory.

The present book gives a comprehensive introduction to the theory of 1-connected surgery as well as some of the most important applications. It has a number of features that make it especially useful for anyone wishing to learn surgery theory. First, the author takes special pains with the basic algebraic topology and algebra involved in 1-connected surgery (e.g., cup and cap products, Poincaré duality spaces, intersection pairings, quadratic forms, etc.). Secondly, the book collects a great deal of diverse background information from the literature, giving relatively complete expositions (e.g., of spherical fibrations, Thom classes, Spivak normal fibrations, homotopy groups of Stiefel manifolds, Spanier-Whitehead duality, functional cohomology operations). This makes the book relatively self-contained. Of course, the abundance of the material needed forces the style to be somewhat compact; the book requires and merits careful reading. Finally, and perhaps most important is the way in which the author has organized the theory. After a chapter on preliminaries, he lists seven basic results of surgery theory and then proceeds to derive many important applications of the theory from these. The remainder of the book is devoted to a proof of the seven basic results. This procedure allows the student to arrive at interesting applications before becoming engulfed in the many technical details of the main proofs. It also focuses attention properly on the most significant tools provided by the theory for the study of 1-connected manifolds.

The most important of these results goes as follows: Let (f, b) be a normal map, as above, $n \geq 5$, with A 1-connected and $f|\partial M\colon \partial M \to B$ a homology equivalence; if n is odd, then (f, b) can be modified by normal surgeries so that the result g is a homotopy equivalence; if n is even, then (f, b) can be so modified if and only if a certain obstruction $\sigma(f, b) \in R$ is 0, where $R = \mathbf{Z}_2$ if $n = 4k + 2$ and $R = \mathbf{Z}$ if $n = 4k$.

Other basic results describe properties of σ: the possible values achieved by σ (all—using plumbing); the effect on σ of "summing" two normal maps (additive); a cobordism property ($\sigma(f, b) = 0$ if f "bounds"); an index property (if $n = 4k$ and $B = \emptyset$, then $8\sigma(f, b) = \operatorname{Index} M - \operatorname{Index} A$); a product formula.

We conclude with a list of the contents: (I) Poincaré duality: § 1: Slant operations, cup and cap products; § 2: Poincaré duality; § 3: Poincaré pairs and triads; sums of Poincaré pairs and maps; § 4: The Spivak normal fibre space. (II) The main results of surgery: § 1: The main technical results; § 2: Transversality and normal cobordism; § 3: Homotopy types of smooth manifolds and classification; § 4: Reinterpretation using the Spivak normal fibre space. (III) The invariant σ: § 1: Quadratic forms over $\mathbf{Z}$ and $\mathbf{Z}_2$; § 2: The invariant $I(f)$ (index); § 3: Normal maps, Wu classes, and the definition of σ for $n = 4k$; § 4: The invariant $c(f, b)$ (Kervaire invariant); § 5: Product formulas. (IV) Surgery and the fundamental theorem: § 1: Elementary surgery and the group $\mathrm{SO}(n)$; § 2: The fundamental theorem: preliminaries; § 3: Proof of the fundamental theorem for n odd; § 4: Proof of the fundamental theorem for n even. (V) Plumbing: § 1: Intersection; § 2: Plumbing disc bundles.

P. J. Kahn

From MathSciNet, June 2015

MR0212827 (35 #3692) 57.20; 57.10

Wall, C. T. C.

Surgery of non-simply-connected manifolds.

Annals of Mathematics. Second Series **84** (1966), 217–276.

The problem of classifying manifolds of a given homotopy type led W. Browder and S. P. Novikov (references are given in the preceding review [MR0212826 (35 #3691)]) to a surgery problem which is studied here in the non-simply connected case.

The author first generalizes the definition of a Poincaré complex X with non-trivial fundamental group π by requiring that the cap product with the fundamental class $[X] \in H_m(X)$ induces isomorphisms $H^i(X) \to H_{m-i}(X)$ and $H_i(X) \to H^{m-i}(X)$ for any kind of coefficient groups twisted by π.

Let M be a compact differentiable manifold, $\psi\colon M \to X$ a map of degree one and $\omega\colon X \to \mathrm{BO}$ a map such that $\omega \cdot \psi$ classifies the stable normal bundle of M. Such a datum is obtained from a normal invariant of the vector bundle on X classified by ω [see MR0212826 (35 #3691) above]. The fundamental problem is to replace ψ by a homotopy equivalence, more precisely, to construct an $(m+1)$-manifold W and a map $\Psi\colon W \to X$ such that the following conditions hold: the map $\omega \cdot \Psi$ classifies the stable normal bundle of W, the boundary of W is the disjoint union of M and M', the restriction of Ψ to M is ψ and its restriction to M' is a homotopy equivalence ψ'. The method, initiated by M. A. Kervaire and J. W. Milnor [Ann. of Math. (2) **77** (1963), 504–537; MR0148075 (26 #5584)] (see also W. Browder and P. S. Novikov [loc. cit.]), consists in modifying M and ψ by a sequence of spherical modifications.

As usual, there is no difficulty in making ψ connected below the middle dimension.

If $m = 2k$ and ψ is k-connected, the group $G = \pi_{k+1}(\psi)$ is a stably free Λ-module, where Λ is the group ring of the fundamental group π of X. The author associates to each element α of G a regular homotopy class of immersions of S^k in M. The element α can be killed by surgery if and only if this class contains an imbedding. The obstruction for that is defined by a self-intersection map μ of G in a quotient V of Λ depending on the parity of k. On the other hand, intersections of these immersions define a map $\lambda\colon G \times G \to \Lambda$. These maps μ and λ verify certain properties and define a kind of hermitian structure on the module G. The author constructs with such Λ-modules a Grothendieck group $L_m(\pi)$ and shows that the obstruction to making ψ a homotopy equivalence is the class of G in $L_m(\pi)$. This group depends only on π and on the residue class modulo 4 of m. For $\pi = 1$, it is isomorphic to $\mathbf{Z}$ for k even and $\mathbf{Z}_2$ for k odd, and corresponds to the index and Arf invariant studied by Kervaire and Milnor [loc. cit]. The author computes the group $L_m(\pi)$ when π is cyclic of prime order, using the work of G. Shimura [ibid. (2) **79** (1964), 369–409; MR0158882 (28 #2104)].

When m is odd, the setting is much more complicated and the author considers only the case where π is finite. He completely solves the case $\pi = \mathbf{Z}_2$.

In the last paragraph, the case of manifolds with boundary is studied.

{Reviewer's remarks: This important paper is very long and technical, and the results obtained are not always clearly expressed. We mention that the author has since written a new version giving a much more satisfactory account (unpublished), especially of the odd-dimensional case.}

A. Haefliger
From MathSciNet, June 2015

MR1747528 (2001c:57002) 57-03; 01A60
Milnor, John
Classification of $(n-1)$-connected $2n$-dimensional manifolds and the discovery of exotic spheres.
Surveys on surgery theory, Vol. 1, 25–30, *Ann. of Math. Stud., 145*, *Princeton Univ. Press, Princeton, NJ*, 2000.

This is a (very readable) short history of one of the most amazing discoveries of modern topology, namely the discovery of exotic spheres, and an interesting pendant to the famous paper by Milnor [Ann. of Math. **64** (1956), 399–405; MR0082103 (18,498d)] in which a 7-dimensional exotic sphere was described for the first time. In the present paper the author tells us how this discovery was made. The starting point was the problem of understanding the structure of closed, smooth $(n-1)$-connected $2n$-manifolds. The author describes the state of knowledge of topology during the 1950s (fibre bundles, obstruction theory, characteristic classes and Hirzebruch's signature theorem, early chapters of cobordism theory), and then shows that homotopy spheres bounding disc bundles over the ordinary n-sphere (normal neighbourhoods of the n-sphere in $2n$-manifolds) enter the theory of such manifolds in a natural way. In $7 = 2n-1$ dimensions such homotopy spheres can be easily described using quaternions, but only those diffeomorphic to the ordinary sphere can appear in this setting. Using Hirzebruch's signature formula, the author discovered that some of those homotopy spheres cannot bound (as smooth manifolds) the 8-disc. The author's first guess was that such a manifold must be a counterexample to the 7-dimensional Poincaré conjecture (any homotopy sphere is homeomorphic to the standard one), but next he found a real-valued smooth function on it with precisely two critical points, which proves that it must be homeomorphic to the standard sphere. This proves the manifold in question is the topological sphere, but with an exotic smooth structure.

Wiesław J. Olędzki
From MathSciNet, June 2015

MR1190010 (95b:57001) 57-02
Kosinski, Antoni A.
Differential manifolds. (English)
Pure and Applied Mathematics, 138.
Academic Press, Inc., *Boston, MA*, 1993, xvi+248 pp., ISBN 0-12-421850-4

The book under review takes the reader on a scenic tour of geometric topology from 1950–1970. The mathematics is beautiful and the exposition detailed and lively.

The starting point is differentiable structures, vector bundles and tubular neighbourhoods followed by transversality and Morse theory, leading to handle decompositions of smooth manifolds and the h-cobordism theorem. Other high points of the tour include inspection of the groups θ^m of homotopy spheres, operations on framed manifolds with their connection to homotopy theory and the beginnings of surgery theory.

The titles of the chapters are: I. Differentiable structures. II. Immersions, imbeddings, submanifolds. III. Normal bundle, tubular neighbourhoods. IV. Transversality. V. Foliations. VI. Operations on manifolds. VII. Handle presentation theorem. VIII. The h-cobordism theorem. IX. Framed manifolds. X. Surgery.

Each chapter concludes with historical remarks and comments, and there is an appendix containing consequences of the implicit function theorem, the Brown-Sard theorem, and some facts about the orthogonal group.

{Reviewer's comment: A slip occurs on p. 164 where the author asserts that "every simply connected 4-manifold is a connected sum of a certain number of copies of $S^2 \times S^2$, $\mathbf{C}\mathrm{P}^2$ and its conjugate $\overline{\mathbf{C}\mathrm{P}^2}$, and of two exotic manifolds". By Freedman's theorem, the classification of simply connected 4-manifolds is as complicated as the classification of unimodular symmetric bilinear forms over $\mathbf{Z}$, and there are many definite indecomposable forms.}

Ian Hambleton

From MathSciNet, June 2015