

## COMMENTARY ON THE Kervaire–Milnor Correspondence 1958–1961

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**ABSTRACT.** The extant letters exchanged between Kervaire and Milnor during their collaboration from 1958–1961 concerned their work on the classification of exotic spheres, culminating in their 1963 *Annals of Mathematics* paper. Michel Kervaire died in 2007; for an account of his life, see the obituary by Shalom Eliahou, Pierre de la Harpe, Jean-Claude Hausmann, and Claude Weber in the September 2008 issue of the *Notices of the American Mathematical Society*. The letters were made public at the 2009 Kervaire Memorial Conference in Geneva. Their publication in this issue of the *Bulletin of the American Mathematical Society* is preceded by our commentary on these letters, providing some historical background.

### LETTER 1. FROM MILNOR, 22 AUGUST 1958

Kervaire and Milnor both attended the International Congress of Mathematicians held in Edinburgh, 14–21 August 1958. Milnor gave an invited half-hour talk on *Bernoulli numbers, homotopy groups, and a theorem of Rohlin*, and Kervaire gave a talk in the short communications section on *Non-parallelizability of the  $n$ -sphere for  $n > 7$*  (see [2]). In this letter written immediately after the Congress, Milnor invites Kervaire to join him in writing up the lecture he gave at the Congress. The joint paper appeared in the *Proceedings of the ICM* as [10]. Milnor’s name is listed first (contrary to the tradition in mathematics) since it was he who was invited to deliver a talk.

### LETTER 2. FROM MILNOR, 8 SEPTEMBER 1958

Kervaire sent the first draft of [10] only two weeks after the invitation was issued, so they were both ready for the collaboration! At the time Milnor was a professor at Princeton University, and Kervaire was at the Battelle Institute in Geneva.

### LETTER 3. FROM MILNOR, 23 SEPTEMBER 1958

The Edinburgh Congress proceedings paper [10] was then submitted for publication.

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We are grateful to Mme. Aimée Kervaire and John Milnor for permitting us to publish the correspondence.

## LETTER 4. FROM Kervaire, 7 OCTOBER 1959

Kervaire had a copy of Milnor's notes [7]. These notes are dated January 23, 1959. This letter from Kervaire was written nine months later. In Autumn 1959 he moved to the United States, where he was a professor at NYU. He gets acquainted with surgery, and in this letter he asks Milnor specific questions and asks him to write a paper on the subject.

## LETTER 5. FROM MILNOR, 15 OCTOBER 1959

Milnor answers Kervaire's questions. He says that he will write a paper on the "foundations" of framed surgery: this was done a few months later in the paper as [9] in the Proceedings of the AMS Tucson meeting, held February 18–19 1960. He says also that in the situation investigated by Kervaire, the obstruction to surgery is proportional to the signature when  $n \equiv 0 \pmod{4}$ . Full details will appear in the Tucson paper. The answer to question 2) is the first occurrence of the signature (alias index) as the obstruction to surgery on  $4k$ -dimensional manifolds, and there is an anticipation of the Arf invariant as the  $(4k+2)$ -dimensional obstruction.

## LETTER 6. FROM MILNOR, 19 NOVEMBER 1959

In this letter Milnor corrects Kervaire's assertion [in a lost letter] that normal bundles of embeddings are additive for connected sums. (Pontrjagin had made a similar mistake 20 years earlier, so Kervaire was in good company.) The quadratic function  $\varphi : H_k(M^{2k}; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$  and its Arf invariant were defined in this letter, which sparked Kervaire's fascination with the applications of this algebraic invariant to the topology of manifolds—in this context it is known as the Kervaire invariant. In particular, as will be seen in Letter 10, Kervaire was instrumental in simplifying and extending the "rather involved" proofs of Milnor. This is the very beginning of the collaboration between Kervaire and Milnor which led eventually to [4]. The letter finishes with a question concerning the " $\chi$ -construction" (now known as surgery) on an odd-dimensional manifold.

## LETTER 7. FROM Kervaire, 22 NOVEMBER 1959

Kervaire answers the question, but is puzzled by the  $\chi$ -construction terminology.

## LETTER 8. FROM MILNOR, 15 DECEMBER 1959

In writing that " $\chi$  can be taken as an abbreviation for Chirurgie", Milnor was alluding to Thom's introduction of the modern surgery terminology, as acknowledged in [9].

The question at the end concerning the existence of  $8k+1$ -dimensional homotopy spheres (i.e., exotic spheres) which are not  $\pi$ -manifolds is cleared in Letter 9.

## LETTER 9. FROM KERVAIRE, 26 DECEMBER 1959

Kervaire answered Milnor’s question in Letter 8, in the sense of proving that if the image of  $J$  in dimension  $8k + 1$  is zero, then there exists an  $8k + 1$ -dimensional homotopy sphere which is not a  $\pi$ -manifold (also known as stably parallelizable). However, Adams’s work on the image of the  $J$ -homomorphism showed that  $J$  is injective, from which it followed that every homotopy sphere is a  $\pi$ -manifold—this was proved in [4, Chapter 3].

## LETTER 10. FROM KERVAIRE, 2 JANUARY 1960

This is the central letter of the correspondence, devoted to what is now called the Kervaire invariant, which provided the framework for the joint paper [4] on the surgery classification of homotopy spheres. We shall give a detailed account of its contents elsewhere.

## LETTER 11. FROM MILNOR, 15 MARCH 1960

The references are Aeppli [1], Wall [12], and Wallace [13]. Notwithstanding Milnor’s preference for Wallace’s “spherical modifications”, it is the “surgery” terminology which is in use now.

The simply connected surgery obstruction groups  $P_n$  were shown in [4] to be 4-periodic, with values

$$P_n = \begin{cases} \mathbb{Z} \text{ (signature/8)} & \text{if } n \equiv 0 \pmod{4}, \\ 0 & \text{if } n \equiv 1 \pmod{4}, \\ \mathbb{Z}_2 \text{ (Arf invariant)} & \text{if } n \equiv 2 \pmod{4}, \\ 0 & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

The result  $\Theta_{2k}(\partial\pi) = 0$  is equivalent to  $P_{2k+1} = 0$ .

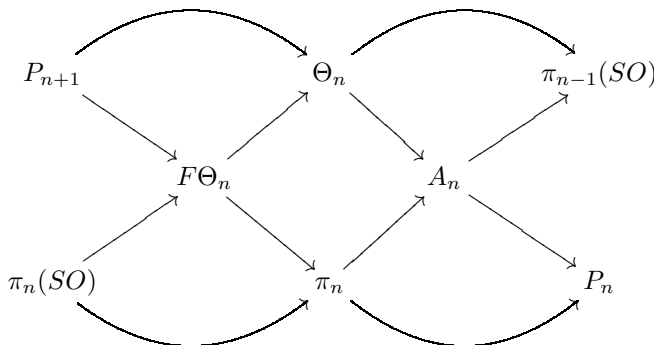
## LETTER 12. FROM MILNOR, 20 MAY 1960

Letter 12 is historically important since it shows that during the months of January and February 1960, Kervaire wrote the paper [3], using a 9-dimensional exotic sphere to construct a differentiable 10-dimensional PL manifold without a differentiable structure in a strong sense: Kervaire’s manifold does not have the homotopy type of a differentiable manifold. Milnor’s generous admission that he had himself tried to prove the existence of a such manifold is a measure the greatness of both.

In his early papers on the non-uniqueness of differentiable structures on spheres, Milnor’s exotic spheres were constructed first as boundaries of a single disk bundle over a sphere [6], and then as the boundaries of the plumblings of two such disk bundles [8]. In both cases, Milnor was able to construct a Morse function on the boundary with exactly two critical points, one minimum and one maximum. This implies that the boundary is homeomorphic to the sphere (in fact PL homeomorphic). Kervaire’s exotic 9-sphere was also constructed as the boundary of the plumbing of two disk bundles. But it was not known before the work of Smale [11] that such a Morse function exists on the boundary of the plumbing of  $> 2$  disk bundles, e.g., the boundary of the  $E_8$  plumbing of eight disk bundles.

## LETTER 13. FROM MILNOR, 29 JUNE 1961

This is the final letter, which makes clear that Kervaire and Milnor knew much more about the consequences of their classification of exotic spheres than they published in [4]. Indeed, this was billed as Part I of a paper, but Part II never appeared. The letter includes the first occurrence of a commutative braid of exact sequences:



The material which would have appeared in Part II of [4] was published 20 years later by Levine in [5].

AN INDEX FOR THE  $\Theta$ 'S.

- 1)  $\Theta_n$  is the additive group of  $h$ -cobordism classes of homotopy  $n$ -spheres, which for  $n \neq 3, 4$  are also the oriented diffeomorphism classes. See [4, §2].
- 2)  $\Theta_n(\pi)$  is the subgroup of  $\Theta_n$  which consists in homotopy  $n$ -spheres which are stably parallelizable. Thanks to Adams's work, it was proved that in fact  $\Theta_n = \Theta_n(\pi)$ .
- 3)  $\Theta_n(\partial\pi) = \text{im}(P_{n+1} \rightarrow \Theta_n)$  is the cyclic subgroup of  $\Theta_n$  consisting of the homotopy  $n$ -spheres which bound a stably parallelizable  $(n+1)$ -manifold. Later it was written  $bP_{n+1}$ .

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Photograph by Aimée Kervaire

Michel Kervaire on *Liberté*, 1956



Photograph by Jay R. Goldman

John Milnor at Princeton, 1965