BOOK REVIEWS

Methods of geometric analysis in extension and trace problems. Volume 1, by Alexander Brudnyi and Yuri Brudnyi, Monographs in Mathematics, Vol. 102, Birkhäuser/Springer Basel AG, Basel, 2012, xxiv+560 pp., ISBN 978-3-0348-0208-6, US$129

Methods of geometric analysis in extension and trace problems. Volume 2, by Alexander Brudnyi and Yuri Brudnyi, Monographs in Mathematics, Vol. 103, Birkhäuser/Springer Basel AG, Basel, 2012, xx+414 pp., ISBN 978-3-0348-0211-6, US$129

Extending maps defined on a set to a larger set while preserving some mapping structures or minimizing some quantities is a fundamental problem in pure and applied mathematics. For example, the classical Dirichlet problem can be viewed as an extension problem: given a real-valued function f on the boundary ∂D of a domain $D \subset \mathbb{R}^n$, one would like to extend f to a function F defined on D such that $\int_\Omega |\nabla F|^2 \, dx$ is as small as possible. The traditional plan of attack is to solve the Laplace equation $\Delta F = 0$ on D subject to the boundary constraint $F = f$ on ∂D; a very complete theory has been developed along this direction. However, considering the Dirichlet problem as an extension problem without looking at differential equations is a more intrinsic (and more challenging) way of studying the problem.

The study of the extension problems of various types was initiated in 1920s; after a spurt of foundational results, not much progress has been made until very recently. In the last three decades we have witnessed a plethora of breakthroughs. The two-volume book under review is a much welcome attempt to gather together cumulative efforts of many analysts and to bring to a wide audience the frontiers of current research on extension problems.

To illustrate the main themes of the subject, it will be useful to review one of the first fundamental results in the subject: the Tietze–Urysohn extension theorem. It states that continuous real-valued functions on a closed subset of a normal topological space can be extended to the entire space while preserving the boundedness. More precisely, let X be a normal topological space, and let S be a closed subset of X. If $f : S \to \mathbb{R}$ is a continuous function, then there exists a continuous function $F : X \to \mathbb{R}$ such that

(i) $F(x) = f(x)$ for all $x \in S$ and
(ii) $\|F\| := \sup_{x \in X} |F(x)| = \|f\| := \sup_{x \in S} |f(x)|$.

Historically, it was first proved by Brouwer in [3] and Lebesgue in [30] for $X = \mathbb{R}^n$. Tietze [34] showed the theorem for arbitrary metric spaces X; for normal spaces, it is due to Urysohn [35]. It is relatively easy to construct a function that satisfies condition (i). The real delicacy lies in requiring that the norm be preserved (i.e., condition (ii) holds).

The problem becomes more challenging when one demands that the extensions be smoother than just being continuous. In the setting of metric spaces, the next natural smooth class is *Lipschitz* functions. Let (M_1, d_1) and (M_2, d_2) be metric spaces. Let S be a subset of M_1. A map $f : S \to M_2$ is called *Lipschitz* if there exists $\lambda \geq 0$ such that $d_2(f(x), f(y)) \leq \lambda \cdot d_1(x, y)$ for all $x, y \in S$. The *Lipschitz constant* $L(f, S)$ of f on S is defined by

$$L(f, S) := \sup_{x, y \in S, x \neq y} \frac{d_2(f(x), f(y))}{d_1(x, y)}.$$

Given a Lipchitz function f on S, one would like to extend f to

$$F : M_1 \to M_2$$

while making the Lipschitz constant $L(F, M_1)$ as small as possible (and obviously no smaller than $L(f, S)$). For $M_1 = \mathbb{R}^n$ and $M_2 = \mathbb{R}$, McShane [31] showed that an extension can be constructed to have the same Lipschitz constant: Indeed, one can define such an extension $F : \mathbb{R}^n \to \mathbb{R}$ by

$$(1) \qquad\qquad F(x) = L(f, S) \cdot \mathrm{dist}(x, S) + \inf_{s \in S} f(s).$$

For $M_1 = \mathbb{R}^n$ and $M_2 = \mathbb{R}^m$, a component-wise application of (1) yields an extension $F : \mathbb{R}^n \to \mathbb{R}^m$ such that $L(F, \mathbb{R}^n) \leq \sqrt{m} L(f, S)$. As a pleasant surprise, Kirszbraun [26] showed that McShane's result still holds for $M_2 = \mathbb{R}^m$; in other words, there exists $F : \mathbb{R}^n \to \mathbb{R}^m$ such that $L(F, \mathbb{R}^n) = L(f, S)$. The known proofs of Kirszbraun's theorem appeal to Zorn's lemma. More generally, Valentine (see [36], [37]) proved that extensions preserving Lipschitz constants exist even if M_1 and M_2 are Hilbert spaces. Generalizations of the corresponding result to other spaces are more delicate. It is known that if either M_1 or M_2 is a Banach space, there may not exist an extension that preserves the Lipschitz constant. Generalizations to some Riemannian manifolds (with the geodesic metrics) are possible: for example, if both spaces M_1, M_2 are spheres of the same dimension, or spaces of constant curvature -1, the corresponding theorems have been established (see [37]). More generally, Lang and Schroeder [27] showed that metric spaces with upper or lower curvature bounds, in the sense of A. D. Alexandrov, admit extensions that preserve the Lipschitz constant.

The next natural question is to classify all pairs of metric spaces M_1 and M_2 such that every Lipschitz map f from a subset $S \subset M_1$ admits a Lipschitz extension $F : M_1 \to M_2$, not necessarily preserving the Lipschitz constant. One can quantify the problem by defining a Lipschitz constant $\Lambda(M_1, M_2)$ for a pair of metric spaces M_1 and M_2 as the least constant $C = C(M_1, M_2)$ such that every Lipschitz map f from a subset $S \subset M_1$ into M_2 admits an extension $F : M_1 \to M_2$ such that $L(F, M_1) \leq CL(f, S)$. In this terminology, Kirszbraun's theorem simply states that $\Lambda(\mathbb{R}^n, \mathbb{R}^m) = 1$ when $\mathbb{R}^n$ and $\mathbb{R}^m$ are equipped with the Euclidean metrics. When $\mathbb{R}^n$ is equipped with other metrics, the results may be different. The function

$d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given below defines a metric on $\mathbb{R}^n$:

$$d_p(x, y) = \begin{cases} \left(\sum_{i=1}^n |x_i - y_i|^p\right)^{1/p}, & 1 \le p < \infty, \\ \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}, & p = \infty, \\ \sum_{i=1}^n |x_i - y_i|^p, & 0 < p < 1. \end{cases}$$

The metric space $(\mathbb{R}^n, d_p)$ is denoted by l_n^p. We have the following results: $\Lambda(l_n^p, l_n^q) = 1$ whenever $q = \infty$, $(p, q) = (2, 2)$, or $0 < p \le 1/2$ and $q = 2$. For other values of p and q, the precise values (finite or infinite) for $\Lambda(l_n^p, l_n^q)$ have not been determined; however, we do know that $\Lambda(l_n^\infty, l_n^2) > 1$. For infinite dimensional Banach spaces, the story is even more delicate. Consider for example $L^p = L^p([0, 1])$ for $1 \le p \le \infty$. Johnson and Lindenstrauss [25] showed that for $1 \le p < 2$, $\Lambda(L^p, L^2) = \infty$. They conjectured that $\Lambda(L^p, L^2) < \infty$ for $2 \le p < \infty$. Partial results of their conjecture have been obtained: One breakthrough is due to Ball [1], who proved that $\Lambda(L^2, L^q) \le \frac{6}{\sqrt{q-1}}$ for $1 < q \le 2$. Another impressive result is obtained by Naor, Peres, Schramm and Sheffield [32], who showed that $\Lambda(L^p, L^q) \le 24\sqrt{\frac{p-1}{q-1}}$ for $1 < q < 2 < p < \infty$. They conjectured that the constant 24 is unnecessary. If true, this would yield a direct generalization of the Valetine's theorem (see above) which states that $\Lambda(L^2, L^2) = 1$. These results represent essentially the state of the art of the subject. For other values of p, q, nothing is known about $\Lambda(L^p, L^q)$.

Instead of focusing on the metric spaces M_1 and M_2 to understand the Lipschitz extension problem, when given a subset S of M_1 and a function $f : S \to M_2$, one can also examine the topological properties of S that obstruct the existence of extensions. Lang and Schlichenmaier [28] have made significant progress along this direction. Their results imply a Lipschitz analog of Hurewicz's theorem which states that the only obstruction to extending a map from a closed subset S of a metric space M into the n-sphere

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\}$$

is the topological dimension of M. In the Lipschitz setting, the substitute for the topological dimension is the Nagata–Assouad dimension. The Nagata–Assouad dimension of a metric space M, denoted by $\dim_{NA} M$, is the least integer n with the following property: for some constant $c > 0$ and every $t > 0$ there is a cover U of M such that $\operatorname{diam} U \le ct$ and every subset $S \subset M$ of diameter at most t meets at most $n + 1$ subsets of U. The topological dimension of a metric space does not exceed the Nagata–Assouad dimension. Some spaces with finite Nagata–Assouad dimension are doubling metric spaces and Gromov hyperbolic spaces of bounded geometry. One of the results from the Lang–Schlichenmaier theory states that if S is a nonempty closed subspace of a metric space M with either $\dim_{NA} S$ or $\dim_{NA} S^c$ finite, then a Lipschitz function f from S to a Banach space B can be extended to a Lipschitz function $F : M \to B$. Consequently, if M_1 is a doubling metric space or a Gromov hyperbolic space of bounded geometry and M_2 is a Banach space, then $\Lambda(M_1, M_2) < \infty$. However, the Lang–Schlichenmaier theory does not provide precise estimates on $\Lambda(M_1, M_2)$.

Another important aspect of the subject deals with extending real-valued functions or jets defined on closed subsets of $\mathbb{R}^n$ for spaces of differentiable and smooth functions. The Dirichlet problem discussed in the introduction is a typical example

of this type. Let $C^m(\mathbb{R}^n)$ denote the space of m-times continuously differentiable functions $F : \mathbb{R}^n \to \mathbb{R}$ for which the norm

$$\|F\|_{C^m(\mathbb{R}^n)} = \max_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} |\partial^\alpha F(x)|$$

is finite.

For $F \in C^m(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we write $J_x F$ to denote the mth degree Taylor polynomial of F at x and it is called the m-jet of F at x, i.e.,

$$(J_x F)(y) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} \partial^\alpha F(x) \cdot (y - x)^\alpha.$$

Suppose we are given a closed subset $E \subset \mathbb{R}^n$ and a function $f : E \to \mathbb{R}$. We are interested in the following fundamental questions:

<u>Problem 1:</u> How can we decide whether there exists $F \in C^m(\mathbb{R}^n)$ such that $F = f$ on E?

<u>Problem 2:</u> Compute the order of magnitude of

$$\|f\|_{C^m(E)} := \inf\{\|F\|_{C^m(\mathbb{R}^n)} : F = f \text{ on } E \text{ and } F \in C^m(\mathbb{R}^n)\}.$$

In other words, find a number X allowed to depend on m, n, E, and f such that $cX \leq \|f\|_{C^m(E)} \leq CX$ for some constants c, C depending only on m and n.

<u>Problem 3:</u> Consider the Banach space $C^m(E) = \{F|_E : F \in C^m(\mathbb{R}^n)\}$ equipped with the norm in the previous problem. Is there a bounded linear operator

$$T : C^m(E) \to C^m(\mathbb{R}^n)$$

such that $Tf = f$ on E for all $f \in C^m(E)$?

Collectively, these problems are called the Whitney extension problems. In his seminal paper [39] published in 1934, Whitney solved the problems for $C^m(\mathbb{R})$ and paved the way for studying problems in higher dimensions. The roman numeral "I" in the title of his paper [39] might have suggested that he intended to write a series of papers to prove similar theorems in higher dimensions. However, significant progress in this direction did not appear until some fifty years later.

In a different paper [38], Whitney also proved a related theorem, known as the Whitney extension theorem, which can be viewed as a partial converse to Taylor's theorem. It gives a necessary and sufficient condition to extend a given function on a closed subset E of $\mathbb{R}^n$ to have prescribed jets at the points of E. Notice that in Problems 1–3, we are only prescribed the function's values on E instead of their full derivatives. In the paper [38], Whitney introduced what is now known as the Whitney decomposition, an idea that has inspired many mathematicians for years to come.

It is worth mentioning that in 1958, using a geometric construction, Glaeser [23] solved Problem 1 for $C^1(\mathbb{R}^n)$.

From the 1980s to the early 2000s (see [4–10]), Y. Brudnyi and Shvartsman studied the problems for the space $C^{m,\omega}(\mathbb{R}^n)$ (the space of functions whose mth derivatives have modulus of continuity ω, e.g., the space of functions whose mth derivatives are Lipschitz continuous). They conjectured a finiteness principle, which in essence states that it suffices to understand the extension problems for finite subsets of $\mathbb{R}^n$. More precisely, it states that to decide whether a given function $f : E \to \mathbb{R}$ is extendable to a function $F \in C^{m,\omega}(\mathbb{R}^n)$, it suffices to consider all restrictions $f|_S$, where $S \subset E$ is a subset with at most $k^\#$ arbitrary points. Here,

$k^{\#}$ depends only on m and n. By Whitney's 1934 result, we already know that for $C^{m,\omega}(\mathbb{R})$ the finiteness principle holds with $k^{\#} = m + 2$. In [9], Y. Brudnyi and Shvartsman showed that for $C^{1,\omega}(\mathbb{R}^n)$ the finiteness principle holds with $k^{\#} = 3 \cdot 2^{n-1}$ as the optimal finiteness constant.

By finding an analogue of Glaeser's iterated paratangent space for $C^m(\mathbb{R}^n)$, Bierstone, Milman and Pawłucki in [2] were able to solve the first two problems for subanalytic sets E.

In a series of papers [11–13, 15] (from 2005 to 2009), Fefferman proved the finiteness principle conjectured by Y. Brudnyi and Shvartsman and gave complete answers to the problems for the spaces $C^{m,\omega}(\mathbb{R}^n)$ and $C^m(\mathbb{R}^n)$. Furthermore, Fefferman and Klartag [18, 19] have come up with algorithms that require $\kappa \cdot N \log N$ computer operations to solve the extension problems, where N is the number of points in E and κ is a constant depending only on m and n. Notice that just by examining the data set E (with N points) requires N computer operations. In view of this remark, we see that the Fefferman–Klartag algorithms are extremely efficient in theory.

The only shortcoming with their results is that the constants c and C that appear in Problem 2 may be enormous. To make c and C as close to 1 as possible is a challenging open problem. For $C^{1,1}(\mathbb{R}^n)$, LeGruyer [29] provided the solution; his result is a direct analogue of Kirszbraun's theorem for $C^{1,1}(\mathbb{R}^n)$. When we are given the full jets instead of just the function's values, Fefferman [14, 16] gave algorithms for computing the C^m-norm of a function $F : \mathbb{R}^n \to \mathbb{R}$ having prescribed m-jets at N given points within ϵ percent of the least possible; the computer operations involved are at most $\exp(C/\epsilon)N \log N$, where C is a constant depending only on m and n. Another interesting result by Fefferman [17] allows one to compute a function taking *prescribed values* at N points in $\mathbb{R}^2$, whose C^2-norm is within a factor of $(1 + \epsilon)$ of least possible with at most $C(\epsilon)N \log N$ computer operations. These partial results represent the current state of the art.

The extension problems for the function spaces $C^m(\mathbb{R}^n)$ and $C^{m,\omega}(\mathbb{R}^n)$ are well understood by now. It is natural to consider the extension problems in Sobolev spaces $L^{m,p}(\mathbb{R}^n)$ (the space of functions whose mth (distributional) derivatives are in $L^p(\mathbb{R}^n)$): they include the Dirichlet problem alluded to at the beginning as a special case. Significant progress has been made in the last few years. Important first steps were taken by Shvartsman [33] and Israel [24], who addressed the extension problems for $L^{1,p}(\mathbb{R}^n)$ (with $p > 1$) and $L^{2,p}(\mathbb{R}^2)$ (with $p > 2$), respectively. In [21], Fefferman, Israel, and Luli generalized the results to $L^{m,p}(\mathbb{R}^n)$ with $p > n$, and in [22] they made all the steps in [21] algorithmically effective and obtained the analogous Fefferman–Klartag algorithms for $L^{m,p}(\mathbb{R}^n)$ with $p > n$. Although a linear extension operator can still be constructed for $L^{m,p}(\mathbb{R}^n)$, the structure for the extension operator is fundamentally different from the one for $C^{m,\omega}(\mathbb{R}^n)$ or $C^m(\mathbb{R}^n)$. Specifically, the value of a bounded linear extension operator $T : L^{m,p}(E) \to L^{m,p}(\mathbb{R}^n)$ at a point $x \in \mathbb{R}^n$ may have to depend on the function's values at all the points in E (see [20]); whereas for $C^m(E)$ with E finite, Fefferman [13] showed that there exists a bounded linear extension operator $T : C^m(E) \to C^m(\mathbb{R}^n)$ that is sparse, in the sense that $Tf(x) = \sum_{y \in S_x \subset E} c(x,y)f(y)$, for all $x \in \mathbb{R}^n$, where $c(x,y) \in \mathbb{R}$ and $\#(S_x) \le k^{\#}(m,n)$. In other words, independent of the size of E (as long as it is finite), the number of nonzero coefficients in the linear expression for $Tf(x)$ is at most a universal constant (depending only m and n). This can be viewed as an effective version of the finiteness principle. By the Sobolev embedding theorem, we can make sense of an $L^{m,p}(\mathbb{R}^n)$-function's

pointwise value when $mp > n$; understanding the extension problems for $L^{m,p}(\mathbb{R}^n)$ when $mp > n$ remains elusive.

REMARKS ON THE BOOK

The two-volume book under review is enormous in scope and contains most of the old and current results on extension problems. Many of the theorems appear here for the first time in book form. The book is self-contained, and the detailed arguments make it accessible to a wide audience, especially graduate students interested in getting into the subject. The book covers topics beyond what is discussed above. For example, it includes several fundamental metric embedding theorems: Bourgain theorem for finite spaces, Assouad theorem for doubling metric spaces, and Bonk–Schramm theorem for Gromov hyperbolic spaces. The book is well written and well organized. It will become a standard reference in the subject, and it deserves a spot in the library.

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