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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
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MR0231827 (38 #155) 14.50

Tate, J. T.

p-divisible groups.

Proc. Conf. Local Fields (Driebergen, 1966), 158–183, *Springer, Berlin*, 1967.

In this article, the notion of p-divisible group is introduced and its fundamental properties are investigated.

(I) Let R be a commutative ring (or a prescheme), p a prime number and h an integer ≥ 0. A p-divisible group G over R of height h is an inductive system $G = (G_\nu, i_\nu)_{\nu\geq 0}$, where G_ν is a commutative finite R-group prescheme, locally free of rank $p^{\nu h}$ over R and where for each $\nu \geq 0$, G_ν is the kernel of multiplication by p^ν in $G_{\nu+1}$, i.e., $0 \to G_\nu \xrightarrow{i_\nu} G_{\nu+1} \xrightarrow{p^\nu} G_{\nu+1}$ is exact. A homomorphism $f\colon G \to H$ of p-divisible groups is defined as a system of homomorphisms $f_\nu\colon G_\nu \to H_\nu$ such that $i_\nu \cdot f_\nu = f_{\nu+1} \cdot i_\nu$ for all $\nu \geq 0$, where $G = (G_\nu, i_\nu)$ and $H = (H_\nu, i_\nu)$. A composition of closed immersions $i_{\nu+\mu-1} \cdot \cdots \cdot i_\mu$ ($\equiv i_{\mu,\nu}$) embeds G_μ into $G_{\mu+\nu}$ and has G_ν as its cokernel, $0 \to G \xrightarrow{i_{u,\nu}} G_{\mu+\nu} \xrightarrow{j_{\mu,\nu}} G_\nu \to 0$ exact, where $j_{\mu,\nu}$ factors the multiplication by p^μ in $G_{\mu+\nu}$, i.e., $i_{\nu,\mu} \cdot j_{\mu,\nu} = p^\mu$. Taking their Cartier duals, G' and $(j_{1,\nu})' \equiv i_\nu{}'\colon G_\nu{}' \to G'_{\nu+1}$ define another p-divisible group $G' = (G_\nu{}', i_\nu{}')$, which is called the (Serre) dual of G.

(II) Let R be a complete noetherian local ring with residue field k of characteristic $p > 0$. A p-divisible group $G = (G_\nu, i_\nu)$ is called connected if for all $\nu \geq 0$, G_ν is connected, i.e., the affine algebra A_ν of G_ν is local. Let $G = (G_\nu, i_\nu)$ be a connected p-divisible group, $G_\nu = \mathrm{Spec}(A_\nu)$, $\mu_\nu\colon A_\nu \to A_\nu \otimes_R A_\nu$ the comultiplication of G_ν, $A = \varprojlim_\nu A$ and $\mu = \varprojlim_\nu \mu_\nu\colon A \to A \mathbin{\hat{\otimes}_R} A$. Then A is isomorphic to the ring $R[[X_1, \cdots, X_n]]$ of formal power series in n variables for some integer $n \geq 0$ and μ defines a commutative formal group Γ over R in which the multiplication by p is an isogeny (such a formal group is called divisible). Conversely, if Γ is a divisible formal group over R, letting $G_\nu \equiv \Gamma_{p^\nu}$ = the kernel of the multiplication p^ν in Γ, G_ν and the natural immersion $i_\nu\colon G_\nu \to G_{\nu+1}$ define a connected p-divisible group. The first result says that the correspondences $G = (G_\nu, i_\nu) \rightsquigarrow \Gamma$ and $\Gamma \rightsquigarrow (\Gamma_{p^\nu}, i_\nu)$ define an equivalence between the category of connected p-divisible groups over R and the category of divisible commutative formal groups over R. In general, if $G = (G_\nu, i_\nu)$ is not necessarily connected, one defines the connected component containing the origin G^0 as a connected p-divisible group $(G_\nu{}^0, i_\nu{}^0)$, where $G_\nu{}^0$ is the connected component containing the origin of G_ν. Then $G_\nu/G_\nu{}^0$ is an etale group scheme, i.e., its affine algebra is separable over R and the $G_\nu/G_\nu{}^0$'s define a p-divisible group G^{et}. Then the dimension of a p-divisible group $G = (G_\nu, i_\nu)$ is defined as the dimension of the formal group corresponding to G^0. The second important theorem is that if n and n' are the dimensions of G and the dual G', respectively, then $n + n' = h$, the height of G and G'.

(III) Let R be a complete discrete valuation ring with residue field $k = R/\mathfrak{m}$ of characteristic $p > 0$ and let K be the field of fractions of R whose characteristic is assumed to be 0. Let L be the completion of an algebraic extension of K and let S be the ring of integers in L. For a p-divisible group $G = (G_\nu, i_\nu)$, define the group $G(S)$ of points of G with values in S by $G(S) = \varprojlim_i \varinjlim_\nu G_\nu(S/\mathfrak{m}^i S)$, where $\varinjlim_\nu \varprojlim_i G_\nu(S/\mathfrak{m}^i S)$ is the torsion subgroup $G(S)_{\text{tors}}$. $G(S)$ is a $\mathbf{Z}_p$-module, $\mathbf{Z}_p$ being the ring of p-adic integers. If the residue field k of R is perfect, then the sequence $0 \to G^0(S) \to G(S) \to G^{\text{et}}(S) \to 0$ is exact (more precisely $G \to G^{\text{et}}$ has a formal section). If L is algebraically closed, $G(S)$ is divisible.

Let t_G be the tangent space of G at the origin and let $t_G(L)$ be the group of points of t_G with values in L. Then the logarithm map $\log\colon G(S) \to t_G(L)$ is defined by $(\log x)(f) = \lim_{i\to\infty}((f(p^i x) - f(0))/p^i)$, where $x \in G(S)$, $f \in A^0$: the affine algebra of G^0. The kernel and cokernel of log are the torsion groups. Hence log induces an isomorphism $G(S) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \xrightarrow{\sim} t_G(L)$, $\mathbf{Q}_p$ being the field of fractions of $\mathbf{Z}_p$.

(IV) Let R, K, k and G be as in (III) and suppose k is perfect. Let $\overline{K}$ be the algebraic closure of K, C the completion of $\overline{K}$, D the ring of integers and $\mathcal{G} = \mathrm{Gal}(\overline{K}/K)$. Put $\Phi(G) = \varinjlim_\nu G_\nu(\overline{K})$ (with respect to $i_\nu\colon G_\nu \to G_{\nu+1}$) and $T(G) = \varprojlim_\nu G_\nu(\overline{K})$ (with respect to $j_{1,\nu}\colon G_{\nu+1} \to G_\nu$). Then $\Phi(G)$ and $T(G)$ are $\mathcal{G}$-modules which are isomorphic to $(\mathbf{Q}_p/\mathbf{Z}_p)^h$ and $(\mathbf{Z}_p)^h$ as $\mathbf{Z}_p$-modules, where $h =$ the height of G. Moreover, there exist canonical isomorphisms $\Phi(G) \xrightarrow{\sim} T(G) \otimes_{\mathbf{Z}_p} (\mathbf{Q}_p/\mathbf{Z}_p)$ and $T(G) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{Q}_p/\mathbf{Z}_p, \Phi(G))$. Therefore, knowledge of either $\Phi(G)$ or $T(G)$ is equivalent to knowledge of the general fibre $G \otimes_R K$ of G because in $G \otimes_R K = (G_\nu \otimes_R K, i_\nu)$, $G_\nu \otimes_R K$ is an etale K-group scheme and is determined by $\mathcal{G}$-module $G_\nu(\overline{K})$. If G' is the dual of G, one has, from the definition of the Cartier duality, a system of isomorphisms which are coherent for $\nu \geq 0$, ${G_\nu}'(D) \xrightarrow{\sim} \mathrm{Hom}_D(G_\nu \otimes_R D, G_m)$, whence an isomorphism $T(G') \xrightarrow{\sim} \mathrm{Hom}_D(G \otimes_R D, G_m(p))$, where G_m is the multiplicative group, $G_m(p) = (\mu_{p^\nu}, i_\nu)$, μ_{p^ν} being the kernel of multiplication by p^ν in G_m. From the last isomorphism, one deduces an exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Phi(G) & \longrightarrow & G(D) & \xrightarrow{L} & t_G(C) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \alpha_0} & & \big\downarrow{\scriptstyle \alpha} & & \big\downarrow{\scriptstyle d\alpha} & & \\
0 & \longrightarrow & \mathrm{Hom}(T', U_{\text{tors}}) & \longrightarrow & \mathrm{Hom}(T', U) & \longrightarrow & \mathrm{Hom}(T', C) & \longrightarrow & 0
\end{array}$$

where $T' = T(G')$, U denotes the group of units congruent to 1 in D, $U_{\text{tors}} = \Phi(G_m(p))$ is the group of roots of unity, Hom involves $\mathbf{Z}_p$-homs, L is the logarithm map defined above and where the right arrow in the bottom line is defined from the ordinary p-adic logarithm $U \to C$. The vertical arrows are $\mathcal{G}$-homomorphisms if $\mathcal{G}$ acts on a homomorphism f by $(sf)(x) = s(f(s^{-1}x))$, $s \in \mathcal{G}$. Then the results are as follows: (i) α_0 is bijective and α and $d\alpha$ are injective. (ii) The maps $\alpha_R\colon G(R)$ $(= G(D)^{\mathcal{G}}) \to \mathrm{Hom}_{\mathcal{G}}(T(G'), U)$ and $d\alpha_R\colon t_G(K)$ $(= t_G(L)^{\mathcal{G}}) \to \mathrm{Hom}_{\mathcal{G}}(T(G'), C)$ induced from α and $d\alpha$ are bijective. Therefore the dimension of G is determined by the $\mathcal{G}$-module $T(G)$, hence by the general fibre $G \otimes_R K$.

(V) Finally, the following theorem is proved: Let R be an integrally closed, noetherian integral domain whose field of fractions K is of characteristic 0. Let G and H be p-divisible groups over R. Then the natural map $\mathrm{Hom}(G, H) \to \mathrm{Hom}(G \otimes_R K, H \otimes_R K)$ is bijective, where Hom's are the homs of p-divisible groups.

Moreover, $\mathrm{Hom}(G \otimes_R K, H \otimes_R K) \to \mathrm{Hom}_{\mathcal{G}}(T(G), T(H))$ is bijective, where $\mathcal{G} = \mathrm{Gal}(\overline{K}/K)$.

To obtain the results of (IV) and (V), the Galois cohomologies of K (the same notation as above) are studied in detail in § 3 of this article.

M. Miyanishi

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MR0546620 (81i:10032) 10D20; 14D20, 14G25, 14K15

Deligne, Pierre

Varietes de Shimura: interpretation modulaire, et techniques de construction de modeles canoniques.

Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, 247–289, *Proc. Sympos. Pure Math., XXXIII*, *Amer. Math. Soc., Providence, R.I.*, 1979.

Let S_N be the modular curve of level N, equal to the quotient $\Gamma_N \backslash X^+$ of the upper half plane X^+ by the principal congruence subgroup Γ_N of $\mathrm{SL}_2(\mathbf{Q})$. The canonical model of S_N is a curve M_N over $\mathbf{Q}(e^{2\pi i/N})$ that can be characterized by either of the following two conditions: (a) the rational functions on M_N are the modular functions of level N whose Fourier expansions with respect to $e^{2\pi i/N}$ have coefficients in $\mathbf{Q}(e^{2\pi i/N})$; (b) M_N is the moduli variety over $\mathbf{Q}(e^{2\pi i/N})$ for elliptic curves with level N structure.

In a series of papers that appeared in the Annals of Mathematics between 1964 and 1970, Shimura attacked the problem of finding (and studying) canonical models for more general quotients $S_\Gamma = \Gamma \backslash X^+$ of bounded symmetric domains X^+ by congruence subgroups Γ. The first problem is to characterize them: if S_Γ is compact then there are no Fourier expansions, and (a) is inapplicable; in general, S_Γ will not be a moduli variety for abelian varieties and so (b) cannot be used. Shimura showed (in the cases he studied) that a complete family (M_Γ) of canonical models, in which the congruence subgroup Γ is allowed to vary but X^+ is fixed, can be characterized by specifying the fields in which certain special points lie. He also showed in several cases, most significantly one for which the variety is not a moduli variety, that the canonical models exist. Later K. Miyake and Shih proved the existence for other cases.

In his report [Séminaire Bourbaki, 23ème année (1970/71), Exposé No. 389, Lecture Notes in Math., 244, Springer, Berlin, 1971; MR0498581 (58 #16675)] on Shimura's work, the author made two important innovations. The first was to change the initial object of study from that of a connected quotient $\Gamma \backslash X^+$ to that of a finite disjoint union of such quotients (to which he gave the name Shimura variety). This has the technical advantage that each member of a family (M_Γ) of canonical models is defined over the same field. The second innovation was to give an axiomatic definition of a Shimura variety: whereas in Shimura's work one always begins with a concretely given reductive group and bounded symmetric domain and makes (superficially at least) ad hoc definitions and constructions, in the author's approach [op. cit.] one begins with a reductive group G and domain X satisfying a small number of axioms from which everything flows. In the report, he re-expressed Shimura's definition of a canonical model in terms of his axioms and gave a general proof of its uniqueness, but in the construction of canonical models he did not go in any essential way beyond Shimura's results.

The object of the paper under review is to motivate the axioms given in the report and to make an exhaustive analysis of the range of applicability of the methods initiated by Shimura for the construction of canonical models; in between, the paper provides a great deal of foundational material on Shimura varieties. The motivation for the axioms is (roughly speaking) that they are necessary and sufficient for the bounded symmetric domain X to parametrize a canonical family of Hodge structures having properties similar to those occurring naturally in algebraic geometry. (One can also simply accept that the axioms are sufficiently general to include all of Shimura's examples, and sufficiently special to provide a satisfactory theory.) The construction of the canonical models is based on the following steps: Mumford's work on moduli varieties allows one to construct a canonical model when the Shimura variety is a moduli variety for example when G is the group of symplectic similitudes and X is the Siegel double-space; if a Shimura variety can be embedded (in an appropriate sense) in a moduli variety, then the canonical model exists; such embeddings arise from maps from G into the symplectic group, and Satake has classified the symplectic representations of algebraic groups; finally, whether a canonical model exists for the Shimura variety defined by a pair (G, X) depends only on the derived and adjoint groups of G and on a connected component of X. The conclusion is that the canonical model exists (at least) when the semisimple part of G has simple factors only of type A, B, C, and (with some important restrictions) D.

J. S. Milne

From MathSciNet, August 2015

MR1047143 (91g:11066) 11G18; 11F32, 11F80, 11S37

Ribet, K. A.

On modular representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ arising from modular forms.

Inventiones Mathematicae **100** (1990), *no.* 2, 431–476.

In this paper the author proves a conjecture of Serre concerning the level of an irreducible modular Galois representation $\rho\colon \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}(2, \mathbf{F})$, where $\mathbf{F}$ is a finite field of odd characteristic l. Perhaps the most remarkable application of the paper is to Fermat's last theorem. A proof is sketched to show that if it is true, as the conjecture of Taniyama, Shimura, and Weil predicts, that every elliptic curve over $\mathbf{Q}$ is modular, then Fermat's last theorem is also true. That such a conclusion would follow from the main theorem of this paper was pointed out by G. Frey a number of years ago.

The representation ρ is said to be modular of level N if it arises from a weight 2 Hecke eigen-cusp form on $\Gamma_0(N)$. We say that ρ is "finite at p" if the group scheme over $\mathbf{Q}_p$ associated to the restriction of ρ to a decomposition group $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ extends to a finite flat group scheme over $\mathbf{Z}_p$. If $l \neq p$, this means simply that ρ is unramified at p. J.-P. Serre conjectured [Duke Math. J. **54** (1987), no. 1, 179–230; MR0885783 (88g:11022)] that if ρ is modular of level N and if ρ is finite at a prime p which divides N exactly once, then ρ is also modular of level N/p. Under the additional hypothesis that $p \not\equiv 1 \bmod l$, this was proved by Mazur. The main theorem asserts that Serre's conjecture is also true whenever l does not divide N.

Key to the proof of the theorem is a quite beautiful interchange principle—analogous to the Jacquet-Langlands correspondence—which compares certain data obtained from Shimura curves in characteristic p to corresponding data obtained from certain modular curves in characteristic q, where q is a prime different from p. More precisely, let C be the Shimura curve associated to an Eichler order of level M (prime to pq) in the quaternion algebra over $\mathbf{Q}$ with discriminant pq, and let J be the Néron model of the Jacobian of C. Also, let $J_0(pqM)$ be the Néron model of the Jacobian of the modular curve $X_0(pqM)$. It is proved that the character group of the toric part of $J_{/\mathbf{F}_p}$ is isomorphic as Hecke module to the "p-new part" of the character group of the toric part of $J_0(pqM)_{/\mathbf{F}_q}$ (note the interchange of $\mathbf{F}_p$ and $\mathbf{F}_q$).

The proof of the main theorem proceeds roughly as follows. Let ρ be a modular representation of level pM with p not dividing M which is finite at p. If $p \not\equiv 1 \bmod l$ the theorem follows from Mazur's result. So assume $p \equiv 1 \bmod l$. Then a "raising the level" principle is used to show that there are infinitely many primes $q \equiv -1 \bmod l$ such that ρ is pq-new of level pqM. Fix such a q and let J be as in the last paragraph. By the Jacquet-Langlands correspondence and the Eichler-Shimura relations it follows that ρ occurs as a Hecke submodule of $J(\overline{\mathbf{F}}_p)$. Now the interchange principle (together with some multiplicity one theorems and a theorem comparing the support of the group of connected components of $J_{/\mathbf{F}_p}$ with primes of fusion between p-oldforms and p-newforms of level pqM) is used to permute the primes p and q and to deduce that ρ is modular of level qM. Finally, since $q \not\equiv 1 \bmod l$, Mazur's result implies that ρ is modular of level M.

The paper makes use of some of the finer properties of Néron models of Jacobians (due to Raynaud) and of the reduction of modular curves (Deligne, Rapoport) and Shimura curves (Cherednik, Drinfel′d). This preliminary material is summarized in the early part of the paper.

Glenn Stevens

From MathSciNet, August 2015

MR1793414 (2002b:11072) 11F80; 11F11

Skinner, C. M.; Wiles, A. J.

Residually reducible representations and modular forms.

Institut des Hautes Etudes Scientifiques. Publications Mathematiques (1999), *no.* 89, 5–126.

This paper contains strong new evidence towards Fontaine-Mazur's conjecture. A form of this conjecture states that one can recognize the two-dimensional Galois representations ρ which come from a cusp form of weight $k \geq 2$ as those which are odd, absolutely irreducible, unramified outside a finite set of primes and potentially semistable at p with Hodge-Tate weights 0 and $w \geq 1$ (then, $k = w + 1$).

The particular case treated in the paper is when ρ is potentially ordinary at p (Theorem, p. 6), of weights 0 and $w \geq 1$. J.-M. Fontaine has proven that potential ordinarity (with weights 0 and w) implies potential semistability (with Hodge-Tate weights 0 and w). Details of the proof appeared in B. Perrin-Riou's paper [Astérisque No. 223 (1994), 185–220; MR1293973 (96h:11063) (Exposé IV)]. The extra assumption needed for the main theorem is that the residual representation is reducible, with its semisimplification given by two characters: 1 and χ where $\chi|_{D_p} \neq 1$. The main achievement in this paper is to treat residually reducible representations; the case of residually absolutely irreducible representations had been treated by R. Taylor and Wiles [Ann. of Math. (2) **141** (1995), no. 3, 553–572; MR1333036 (96d:11072)]. They treated the case of representations either

Barsotti-Tate at p or potentially ordinary. The first assumption has been relaxed by the work of C. Breuil [Ann. of Math. (2) **152** (2000), no. 2, 489–549; MR1804530 (2001k:14087)] and C. Breuil et al. [J. Amer. Math. Soc. **14** (2001), no. 4, 843–939 (electronic) MR2002d:11058] to reach the case of potentially Barsotti-Tate representations at p for a tame extension. This implies that any elliptic curve over $\mathbf{Q}$ is modular.

The method of the present paper does not consist in identifying a deformation ring with a Hecke algebra. Rather, it consists in restricting the deformation problem to a suitable totally real field F, in order to make larger the codimension of the locus of reducible deformations in the space of deformations of the residually reducible representation restricted to F. Since F can be chosen solvable over $\mathbf{Q}$, the result follows over $\mathbf{Q}$ by an application of Langlands' cyclic base change theorem.

It thus becomes sufficient to prove a version of the main theorem for an arbitrary totally real field $\mathbf{F}$, under some extra assumptions, which need to be verified for $\mathbf{F} = F$.

In Section 2, the authors define for any totally real field the notion of a deformation datum $\mathcal{D} = (\mathcal{O}, \Sigma, c, \mathcal{M})$. In particular, c is a Σ-ramified cohomology class, locally trivial at places above p, and which defines the extension of χ by 1 giving the residual representation

$$\overline{\rho}\colon \mathrm{Gal}(F_\Sigma/F) \to \mathrm{GL}_2(k).$$

They define the problem of deformations of type $\mathcal{D}$ of $\overline{\rho}$; it is representable by a complete Noetherian local $\mathcal{O}$-algebra $R_{\mathcal{D}}$. The first important results are: a lower bound on the Krull dimension of $R_{\mathcal{D}}$ and the comparison with the subring generated by $\mathrm{Im}(\mathrm{Tr} \circ \rho_{\mathcal{D}})$ (the main result is that if Q is a prime of $R_{\mathcal{D}}$ such that $\rho_{\mathcal{D}} \bmod Q$ is irreducible, then $\dim R_{\mathcal{D}^{\mathrm{ps}}}/Q^{\mathrm{ps}} \geq \dim R_{\mathcal{D}}/Q$ with equality if the right-hand side $= 1$), and an upper bound on the Krull dimension of the reducible locus in $\mathrm{Spec}\, R_{\mathcal{D}}$. Section 3 recalls (with proofs) the Hida theory of the nearly ordinary Hecke algebra $\mathbf{T}_\infty(U, \mathcal{O})$ of auxiliary level U, and the big nearly ordinary Galois representations ρ_Q associated to its irreducible components $\mathcal{C}_Q$ [H. Hida, in *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, 115–134, Johns Hopkins Univ. Press, Baltimore, MD, 1989; MR1463699 (2000e:11144)]. The notion of a permissible maximal ideal $\mathfrak{m}$, a special kind of maximal Eisenstein ideal, is introduced. The criterion for its existence (it is then necessarily unique) is as follows. Let $L_p(F, s, \chi\omega)$ be the Deligne-Ribet p-adic L-function of the character $\chi\omega$ over F (here, ω is the Teichmüller lifting of the character of action on μ_p and χ is identified to its Teichmüller lift). Then, if

$$(*) \qquad L_p(F, -1, \chi\omega) \text{ is not a } p\text{-adic unit,}$$

a permissible ideal exists. It is a variant of the standard result that this p-adic L-function is the constant term of an ordinary Eisenstein measure. The p-adic ordinary Eisenstein series for $s = -1$ becomes cuspidal mod p, hence its eigensystem mod p gives rise to the desired maximal ideal of the cuspidal Hecke algebra. This result allows the definition of the localized Hecke algebras $\mathbf{T}_{\mathcal{D}}$ and $\mathbf{T}_{\mathcal{D}}^{\min}$ associated to a given deformation datum $\mathcal{D}$. For $\mathbf{T}_{\mathcal{D}}$, one chooses a suitable auxiliary level group $U_{\mathcal{D}}$, and then one localizes $\mathbf{T}_\infty(U_{\mathcal{D}}, \mathcal{O})$ at a permissible maximal ideal (under the assumption $(*)$). $\mathbf{T}_{\mathcal{D}}^{\min}$ is a natural quotient of $\mathbf{T}_{\mathcal{D}}$.

One can then state the main theorems of the paper. A notion of a good pair $(F, \mathcal{D})$ is defined; this includes satisfying $(*)$ and having even degree d, large with respect to Σ and $\mathcal{M}$. A prime $\mathfrak{q}$ of $R_{\mathcal{D}}$ is promodular if there exists a homomorphism $\theta\colon \mathbf{T}_{\mathcal{D}} \to R_{\mathcal{D}}$ such that $\mathrm{Tr}(\rho_{\mathcal{D}}(\mathrm{Fr}_l)) \bmod q = \theta(T_l)$ for any l prime to Σ. Let $R_{\mathcal{D}_c}$ be the universal ring with Σ replaced by the minimum ramification set $\Sigma_c = \mathrm{Ram}(\overline{\rho}) \cup \{v \mid p \in F\}$. We say a prime $\mathfrak{q}$ of $R_{\mathcal{D}}$ is nice if it is the inverse image of a promodular prime of $R_{\mathcal{D}_c}$ and if $R_{\mathcal{D}}/\mathfrak{q}$ is dimension 1 of characteristic p and for any $v \mid p$, the quotient $\psi_{1,v}/\psi_{2,v}$ of the characters giving the semisimplification of $\rho_{\mathcal{D}} \bmod q$ has infinite order (there are also technical ramification conditions). The first key result (Proposition 4.1) states (in the following slightly vague form) that if $(F,\mathcal{D})$ is good, (P1) if any generalization of a nice prime of $R_{\mathcal{D}}$ is nice and (P2) if $R_{\mathcal{D}_c}$ admits a promodular nice prime, then all primes of $R_{\mathcal{D}}$ are promodular.

It makes use of a connectivity result due to Raynaud which implies that for any partition $(\mathcal{C}_1,\mathcal{C}_2)$ of $\mathrm{Spec}\, R_{\mathcal{D}}$, there exists $C_i \in \mathcal{C}_i$ such that $C_1 \cap C_2$ contains a prime of dimension $\geq B(d,\delta,\mathcal{M})$ (explicit bound, large when $d = [F\colon \mathbf{Q}]$ is large).

Then, after a criterion for (P2) come the two main theorems of the paper, Theorems A and B, presented as variants of the Main Theorem. The Main Theorem assumes the existence of a solvable totally real extension L/F of even degree, satisfying $(*)$ and such that $(F,\mathcal{D})\otimes L$ is good. The conclusion is that the odd irreducible representation $\rho\colon \mathrm{Gal}(F_\Sigma/F) \to \mathrm{GL}_2(E)$ we started with, such that $\det\rho$ is modular, $\overline{\rho}^{ss} = 1 \oplus \chi$ ($\chi|D_v \neq 1$ for any $v \mid p$) and which is nearly ordinary at $v \mid p$, is modular.

Theorem A assumes that F is abelian over $\mathbf{Q}$. One needs to check the existence of a good extension L/F as in the Main Theorem. For that purpose, the critical argument is a theorem of L. C. Washington [Invent. Math. **49** (1978), no. 1, 87–97; MR0511097 (80c:12005)] on the p-adic order of the χ-component of the class group in the cyclotomic $\mathbf{Z}_l$-extension of certain number fields, in relation with the p-adic L function (using an estimate on the Leopoldt defect by M. Waldschmidt [in *Topics in classical number theory, Vol. I, II (Budapest, 1981)*, 1617–1650, Colloq. Math. Soc. János Bolyai, 34, North-Holland, Amsterdam, 1984; MR0781200 (86h:11095)]).

In Theorem B, the extra assumption is the vanishing of an isotypical piece of the p-class group of some cyclic extension of F. There, a theorem of Iwasawa on the p-adic order of the χ-component of the p-class group in the cyclotomic $\mathbf{Z}_p$-extensions assures this. The next sections establish (P1) by a generalization of Taylor-Wiles systems. This paper is obviously a whole world of new devices and extremely skillful use of various fine number-theoretic techniques.

Jacques Tilouine

From MathSciNet, August 2015

MR1705837 (2000m:14024) 14F30; 14F20

Tsuji, Takeshi

p-adic etale cohomology and crystalline cohomology in the semi-stable reduction case.

Inventiones Mathematicae **137** (1999), *no.* 2, 233–411.

The theorem proved by the author, which resolves the so-called C_{st} conjecture, is the crowning of a long series of papers written over a period of almost forty years. The history of this process began in the sixties, with the following theorem of Tate: If A is an abelian variety defined over a local field K (of characteristic 0 and residue characteristic $p > 0$) and with good reduction over the ring of integers of K, and if $\widehat{\overline{K}}$ is the completion of the algebraic closure $\overline{K}$ of K, then the Galois module $A[p^\infty] = \lim_n A[p^n]$, where $A[p^n]$ is the kernel of the multiplication by p^n in $A(\overline{K})$, can be described, after extension of the scalars to $\widehat{\overline{K}}$, by the isomorphism

$$A[p^\infty] \otimes_{\mathbf{Z}_p} \widehat{\overline{K}} \simeq H^0(A, \Omega^1_A) \otimes_K \widehat{\overline{K}}(1) \oplus H^1(A, \mathcal{O}_A) \otimes_K \widehat{\overline{K}}.$$

The Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$ acts on the left-hand side of the isomorphism by the diagonal action $g(a\otimes b) = g(a)\otimes g(b)$, and on the right-hand side only through $\widehat{\overline{K}}$ with a Tate-type torsion (i.e., by the cyclotomic character) over $\widehat{\overline{K}}(1)$. This type of decomposition of $A[p^\infty]$ became known as the Hodge-Tate decomposition. During the same period, A. Grothendieck constructed a general theory for the cohomology of schemes and suggested a direction for generalizing Tate's theorem to general smooth and projective varieties by establishing a connection between p-adic étale cohomology and crystalline cohomology—the "mysterious functor" theory. In the early eighties, J.-M. Fontaine [Ann. of Math. (2) **115** (1982), no. 3, 529–577; MR0657238 (84d:14010)] made some precise conjectures about that connection, which he called comparison theorems between de Rham cohomology and p-adic étale cohomology. These comparison theorems involve a family of rings called p-adic period rings. Fontaine's construction of these rings was one of the crucial steps in the development of the theory. The first ring is $B_{\mathrm{HT}} = \bigoplus_{n\in\mathbf{Z}} \widehat{\overline{K}}(n)$, equipped with the action of the Galois group G_K. It arises in the so-called Hodge-Tate conjecture, C_{HT}: Let X be a proper smooth K-scheme; then there is an equivariant Galois isomorphism

$$H^n_{\text{ét}}(X \otimes \overline{K}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{HT}} \simeq H^n_{\mathrm{Hdg}}(X/K) \otimes_K B_{\mathrm{HT}},$$

where $H^n_{\text{ét}}(X\otimes\overline{K}, \mathbf{Q}_p)$ is the p-adic étale cohomology of $X \otimes \overline{K}$ and $H^n_{\mathrm{Hdg}}(X/K) = \bigoplus_{i+j=n} H^i(X, \Omega^j)$ is the Hodge cohomology of X.

Some years later, S. Bloch and K. Kato [Inst. Hautes Études Sci. Publ. Math. No. 63 (1986), 107–152; MR0849653 (87k:14018)] proved this conjecture in the case when X has ordinary good reduction and is of dimension $< p$. In parallel to this, G. Faltings [J. Amer. Math. Soc. **1** (1988), no. 1, 255–299; MR0924705 (89g:14008)] proved the Hodge-Tate conjecture with no condition on X.

The following conjecture, posed by Fontaine and known as the C_{crys}-conjecture, involves the de Rham period ring B_{dR}, which is a discrete valuation field, with residue field $\widehat{\overline{K}}$ equipped with a filtration and an action of the absolute Galois group of K, as well as a K_0-subalgebra B_{crys} of B_{dR} stable under Galois, called the crystalline period algebra, equipped with a semilinear endomorphism with respect to the Frobenius of K_0 and also called the Frobenius. Here K_0 is the maximal unramified extension of $\mathbf{Q}_p$ contained in K. For X a proper smooth scheme over the ring of integers of K, denote as usual by X_η, $X_{\overline{\eta}}$ and X_s the generic fiber, the geometric generic fiber and the special fiber of X, respectively. The C_{crys}-conjecture then states that there exist a Galois isomorphism and an invariant Frobenius,

$$H^n_{\text{ét}}(X_{\overline{\eta}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{crys}} \simeq H^n_{\mathrm{crys}}(X_s/K_0) \otimes_{K_0} B_{\mathrm{crys}},$$

where $H^n_{\mathrm{crys}}(X_s/K_0)$ is the nth crystalline cohomology group of X_s. The Galois group acts on the left-hand side of the isomorphism by the diagonal action $g(a\otimes b) = g(a)\otimes g(b)$ and on the right-hand side only through B_{crys}. Conversely, the Frobenius acts on B_{crys} only on the left-hand side and by the diagonal action on the right-hand side. Moreover, if this isomorphism is extended to B_{dR}, one has an isomorphism of filtered modules,

$$H^n_{\text{ét}}(X_{\overline{\eta}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}} \simeq H^n_{\mathrm{dR}}(X_\eta/K) \otimes_K B_{\mathrm{dR}}.$$

Here one uses the Berthelot-Ogus isomorphism

$$H^n_{\mathrm{dR}}(X_\eta/K) \simeq H^n_{\mathrm{crys}}(X_s/K_0) \otimes_{K_0} K,$$

and the filtration on $H^n_{\text{ét}}(X_{\overline{\eta}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}$ is derived from that of B_{dR}, whereas the filtration of $H^n_{\mathrm{dR}}(X_\eta/K) \otimes_K B_{\mathrm{dR}}$ is obtained by convolution of the filtration of B_{dR} by the Hodge filtration of $H^n_{\mathrm{dR}}(X_\eta/K)$.

The power of this conjecture is that one can use it to find the Galois representation $H^n_{\text{ét}}(X_{\overline{\eta}}, \mathbf{Q}_p)$ from $H^n_{\mathrm{dR}}(X_\eta/K)$ equipped with the Hodge filtration and from $H^n_{\mathrm{crys}}(X_s/K_0)$ equipped with the Frobenius.

Fontaine and W. Messing [in *Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985)*, 179–207, Contemp. Math., 67, Amer. Math. Soc., Providence, RI, 1987; MR0902593 (89g:14009)] proved this conjecture in the case where the relative dimension of the space X is strictly less than p. To that end they used a new topology, "syntomic topology", which established a link between the étale topology, which calculates the étale cohomology, and the crystalline topos, which calculates the crystalline cohomology. Soon thereafter, Faltings [in *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, 25–80, Johns Hopkins Univ. Press, Baltimore, MD, 1989; MR1463696 (98k:14025)] announced a proof of the crystalline comparison theorem without a condition on the dimension. His approach to the proof, based on the theory of "almost étale" extensions, was completely different from that of Fontaine and Messing.

The next step in the history of this topic occurred when U. Jannsen and Fontaine in their correspondence conjectured a theorem comparing étale cohomology and de Rham cohomology in the case where the scheme X over the ring of integers of K is not smooth but has semistable reduction. This conjecture, called C_{st}, states that the de Rham cohomology $H^n_{\mathrm{dR}}(X_\eta/K)$ has a K_0-structure D equipped with a semilinear Frobenius $\Phi : D \to D$ and a linear nilpotent endomorphism $N : D \to D$ called the monodromy (or logarithm of monodromy) and satisfying $N\Phi = p\Phi N$. Moreover, there exists an ad hoc "semistable" period ring B_{st}, equipped with an action of G_K, a Frobenius Φ and a monodromy N (with the property $B_{\mathrm{st}}^{N=0} = B_{\mathrm{crys}}$; B_{st} is in fact a ring of polynomials over B_{crys}), as well as a G_K-equivariant injection $B_{\mathrm{st}} \to B_{\mathrm{dR}}$. Using these data, one has an isomorphism of G_K-modules,

$$H^n_{\text{ét}}(X_{\overline{\eta}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{st}} \simeq D \otimes_{K_0} B_{\mathrm{st}},$$

that is compatible with the actions of Φ and N (with N acting only on B_{st} on the left-hand side and by $N(x \otimes y) = N(x) \otimes y + x \otimes N(y)$ on the right-hand side). Moreover, the extension of the scalars to B_{dR} provides an isomorphism of filtered modules as in the crystalline case. Also, this conjecture allows one to find the p-adic representation $H^n_{\text{ét}}(X_{\overline{\eta}}, \mathbf{Q}_p)$ from $H^n_{\mathrm{dR}}(X_\eta/K)$ equipped with the Hodge filtration and from D equipped with the Frobenius and monodromy. This is the conjecture that the author proves in the article under review. His great achievement is that, without introducing major new ideas, he skillfully combines the sophisticated techniques of Fontaine and Messing (syntomic cohomology), Bloch and Kato (the theory of symbols), Faltings (rings of relative periods), and Kato (logarithmic structures on schemes), the outline of the proof having already been more or less established [see, in particular, K. Katō, Astérisque No. 223 (1994), 269–293; MR1293975 (95i:14020)]. One hopes for a future generalization of this method to the case of open varieties and/or nonconstant coefficients.

Abdellah Mokrane

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Ramakrishna, Ravi

Deforming Galois representations and the conjectures of Serre and Fontaine-Mazur.

Annals of Mathematics. Second Series **156** (2002), *no.* 1, 115–154.

In this paper the author achieves a major breakthrough in the theory of deformations of mod p representations of the absolute Galois group $G_{\mathbf{Q}}$ of $\mathbf{Q}$. He proves that a continuous representation $\rho\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F})$, with $\mathbf{F}$ a finite field of characteristic $p > 5$, and with some mild assumptions on the behaviour of ρ when restricted to a decomposition group at p (we refer to this as the local behaviour of ρ at p), lifts to a representation $\tilde{\rho}\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(W(\mathbf{F}))$, with $W(\mathbf{F})$ the Witt vectors of $\mathbf{F}$, that is ramified at finitely many primes. For the precise assumptions on the local behaviour of ρ at p we refer to Theorem 1 of the paper: for instance, when ρ is odd, if ρ is ramified at p, then the assumptions on ρ in Theorem 1 of the paper are fulfilled. In this case the author in fact shows that there is a lift $\tilde{\rho}$ as above that is ramified at finitely many primes and is potentially semistable at p in the sense of Fontaine. Such p-adic representations $\tilde{\rho}$, with these 2 additional properties, are said to be geometric in the sense of J.-M. Fontaine and B. C. Mazur [in *Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993)*, 41–78, Internat. Press, Cambridge, MA, 1995; MR1363495 (96h:11049)]. The paper builds on earlier work of the author [see Invent. Math. **138** (1999), no. 3, 537–562; MR1719819 (2000j:11167)], but there are new ideas in this paper which are needed to get the results here that there exist *geometric* lifts for a wide class of 2-dimensional odd mod p Galois representations of $G_{\mathbf{Q}}$.

To put this paper in context it is useful to recall a very important conjecture of J.-P. Serre [Duke Math. J. **54** (1987), no. 1, 179–230; MR0885783 (88g:11022)] which was perhaps the only reason why, before Theorem 1 of this paper was proven, one would have expected a result of this kind, as from the viewpoint of pure Galois theory, there is no compelling reason to expect that ρ should lift to a p-adic representation.

Consider a representation ρ as above, and assume further that ρ is absolutely irreducible and odd: we say that such a ρ is of Serre type. Serre conjectured that ρ arises from a newform on a congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$. This means that there is a newform $f \in S_k(\Gamma_1(N))$ of weight k and level N such that the continuous representation $\rho_f\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathcal{O})$ associated to f by the work of Eichler, Shimura and Deligne, with $\mathcal{O}$ the ring of integers of a finite extension of $\mathbf{Q}_p$, when reduced modulo the maximal ideal of $\mathcal{O}$, gives a representation of $G_{\mathbf{Q}}$ that is isomorphic to ρ. It is known that such a ρ_f is ramified at finitely many primes, and is potentially semistable at p, i.e., is geometric. Serre's conjecture is still wide open, although it is widely believed to be true. The paper under review gives significant evidence in favor of the conjecture.

The fact that in the cases mentioned above the lifting $\tilde{\rho}$ that is constructed is geometric has led to several applications, of which we will mention a few. (1) R. L. Taylor [J. Inst. Math. Jussieu **1** (2002), no. 1, 125–143; MR2004c:11082] has used the results of the paper to provide further important evidence towards Serre's conjecture. He proved that for many ρ of Serre type, the lifts $\tilde{\rho}$ that the paper under review constructs, and hence ρ itself, arise from the p-power torsion of an abelian variety A defined over $\mathbf{Q}$ which has endomorphisms by a field K such that $\dim(A) = [K\colon \mathbf{Q}]$, something which is implied by Serre's conjecture. (2) The reviewer [Math. Res. Lett. **7** (2000), no. 4, 455–462; MR1783623 (2001k:11095)] has used the existence of the lifts $\tilde{\rho}$ to prove results that allow one to verify Serre's conjecture for many ρ of Serre type assuming that there is a totally real solvable extension F such that $\rho|_{G_F}$ arises from a Hilbert modular form for F. (3) In turn, Taylor [Amer. J. Math. **125** (2003), no. 3, 549–566] has devised a strategy that uses the previous application as an ingredient to prove Serre's conjecture for many ρ when the finite field $\mathbf{F}$ involved is small; for instance, $|\mathbf{F}| = 5$ [R. Taylor, op. cit.], $|\mathbf{F}| = 7$ [J. Manoharmayum, Math. Res. Lett. **8** (2001), no. 5-6, 703–712; MR1879814 (2002k:11091)], $|\mathbf{F}| = 9$ [J. S. Ellenberg, "Serre's conjecture over $\mathbf{F}_9$", preprint, `arXiv.org/abs/math/0107147`]. (4) The reviewer ["On isomorphisms between deformation rings and Hecke rings", Invent. Math., to appear] has used the results of the paper to give new proofs of the results of A. J. Wiles [Ann. of Math. (2) **141** (1995), no. 3, 443–551; MR1333035 (96d:11071)] and Taylor and Wiles [Ann. of Math. (2) **141** (1995), no. 3, 553–572; MR1333036 (96d:11072)].

The method by which the author constructs the lifting $\tilde{\rho}$ is ingenious and intricate. Let us focus on the case when ρ is odd. The author in fact constructs not just one geometric lifting $\tilde{\rho}$ but infinitely many such liftings. The liftings are indexed by very carefully chosen finite sets of auxiliary primes Q. The lifting corresponding to such an auxiliary set Q, let us denote it by ρ_Q, is unramified outside the set of primes at which ρ ramifies, p and the primes in Q. For a prime $q \in Q$, $\rho_Q|_{D_q}$ is "special" where D_q is a decomposition group at q, i.e., up to twist is an extension of the trivial representation by the p-adic cyclotomic character. The author uses the technique of introducing auxiliary sets of primes Q, a technique that has played a prominent role in number theory dating back at least to Chebotarev's proof of his density theorem by "crossing with auxiliary cyclotomic extensions", and subsequently used by Artin in the proof of his reciprocity law, and more recently by Wiles in his celebrated work [op. cit.]. We will not go into the very delicate considerations involved in the choice of the auxiliary sets Q; suffice it to say that the Q's are chosen to annihilate a certain dual Selmer group, using the Poitou-Tate duality theorems and the Chebotarev density theorem. Once the set Q is chosen, the method consists of lifting ρ in stages to mod $p^2, p^3, \ldots$ representations; by this we mean that for each n the author constructs a representation $\rho_n\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(W(\mathbf{F})/(p^n))$ such that ρ_1 is ρ and ρ_n mod p^{n-1} is ρ_{n-1}. These representations are unramified outside $S \cup Q$, where S is the set consisting of p and all the primes at which ρ is ramified, and have severe restrictions on the inertial behaviour at primes in S and on the local behaviour at primes in Q.

The author's method can be summarised by saying that a well-chosen auxiliary set Q entails that if there is a mod p^n lift of ρ that satisfies certain specified local conditions at primes in $S \cup Q$, then the mod p^n lift in fact has a mod p^{n+1} lift. Now this mod p^{n+1} representation may not satisfy the specified conditions at the primes in $S \cup Q$: the author adjusts this mod p^{n+1} lift so that it satisfies the local conditions at primes in $S \cup Q$, and at the same time mod p^n it remains the same. The artistry in the choice of Q is in ensuring that one can carry out these two steps. There have been technical improvements to this paper made by Taylor [see op. cit., 2003].

It is quite certain that the beautiful principle for lifting mod p Galois representations of this paper can be applied to many situations, and will continue to have a wealth of arithmetic applications in the future.

Chandrashekhar Khare

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Kisin, Mark

Overconvergent modular forms and the Fontaine-Mazur conjecture.

Inventiones Mathematicae **153** (2003), *no.* 2, 373–454.

Let E be a finite extension of $\mathbf{Q}_p$ and let $\rho\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(E)$ be a continuous representation of the absolute Galois group of $\mathbf{Q}$. The standard examples of such representations are those associated to elliptic curves and modular forms. These representations come from algebraic geometry in the sense that they occur as subquotients of the p-adic étale cohomology of some algebraic variety over $\mathbf{Q}$. A Galois representation coming from algebraic geometry has two important properties: it is ramified at only finitely many primes (which corresponds to the fact that an algebraic variety over $\mathbf{Q}$ has good reduction at almost all primes) and it is potentially semistable at p in the sense of Fontaine. (That the latter property holds for representations coming from algebraic geometry is a combination of a theorem of G. Faltings [in *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, 25–80, Johns Hopkins Univ. Press, Baltimore, MD, 1989; MR1463696 (98k:14025)] and a conjecture of J.-M. Fontaine; the latter conjecture was proved, although never published, by Fontaine himself for two-dimensional representations and in general by L. Berger [Invent. Math. **148** (2002), no. 2, 219–284; MR1906150 (2004a:14022)], Y. André [Invent. Math. **148** (2002), no. 2, 285–317; MR1906151 (2003k:12011)], Z. Mebkhout [Invent. Math. **148** (2002), no. 2, 319–351; MR1906152 (2003k:14018)], and K. S. Kedlaya ["Quasi-unipotence of overconvergent F-crystals", preprint, `arXiv.org/abs/math/0106193`, Ann. of Math. (2), to appear].) A striking conjecture of Fontaine and B. Mazur [in *Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993)*, 41–78, Internat. Press, Cambridge, MA, 1995; MR1363495 (96h:11049)] asserts that any p-adic Galois representation ρ satisfying these two conditions necessarily comes from algebraic geometry. Conjectures of Langlands then predict that, up to twist, any such ρ either is associated to a modular form or is even with finite image.

The first significant progress on the Fontaine-Mazur conjecture was a key portion of the celebrated work of A. J. Wiles [Ann. of Math. (2) **141** (1995), no. 3, 443–551; MR1333035 (96d:11071)] and R. Taylor and Wiles [Ann. of Math. (2) **141** (1995), no. 3, 553–572; MR1333036 (96d:11072)]. This work has been generalized by Taylor [J. Inst. Math. Jussieu **1** (2002), no. 1, 125–143; MR1954941 (2004c:11082)] and by F. Diamond, M. Flach and L. Guo ["Adjoint motives of modular forms and the

Tamagawa number conjecture", preprint, 2001; per bibl.] to verify the Fontaine-Mazur conjecture for many representations ρ such that the residual representation

$$\overline{\rho}\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(k)$$

(with k the residue field of the ring of integers of E) is ordinary in the sense that its restriction to a decomposition group G_p at p is reducible. The paper under review is instead concerned with the more difficult case of $\overline{\rho}$ with irreducible restriction to G_p (although the methods of the paper also yield interesting information in the ordinary case as well).

To describe the author's results we must first discuss the eigencurve of R. F. Coleman and Mazur [in *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, 1–113, Cambridge Univ. Press, Cambridge, 1998; MR1696469 (2000m:11039)]. Fix a modular residual representation

$$\overline{\rho}\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(k)$$

over some finite field k and fix some prime-to-p integer N divisible by the tame conductor of $\overline{\rho}$. (In fact, Coleman and Mazur consider only the case $N = 1$, but it is expected that the theory works in general.) Let $R_{\overline{\rho}}$ be the universal deformation space (for representations with tame conductor dividing N) of the pseudo-representation attached to $\overline{\rho}$ (as in [B. C. Mazur, in *Galois groups over* $\mathbf{Q}$ *(Berkeley, CA, 1987)*, 385–437, Springer, New York, 1989; MR1012172 (90k:11057)] and [A. J. Wiles, op. cit.]) and let Z denote the rigid analytic space associated to $R_{\overline{\rho}}[1/p]$. The eigencurve $\mathcal{C}$ is a rigid analytic space interpolating the set of points $(\rho_f, \lambda) \in Z \times \mathbf{G}_m$ where ρ_f is the Galois representation associated to a classical modular form f of level dividing Np^∞, $\overline{\rho}_f = \overline{\rho}$ up to semisimplification, and f has eigenvalue λ for the operator U_p. In particular, the Fourier coefficients a_n of the classical forms f as above interpolate to rigid analytic functions on $\mathcal{C}$. From this point of view, an overconvergent modular form is simply a p-adic modular form with q-expansion obtained by specializing these functions at a point of $\mathcal{C}$. It is expected that the Galois representations attached to those overconvergent forms which are not simply twists of classical forms (note that these representations are defined via the universal deformation of $\overline{\rho}$ and not via the cohomology of Kuga-Sato varieties as in the classical case) do not come from algebraic geometry. In particular, the Fontaine-Mazur conjecture would then predict that a point on the eigencurve corresponds to a twist of a classical modular form if and only if the associated Galois representation is potentially semistable.

In the paper under review, the author proves that this is indeed true (up to a certain exceptional case) for weights $k \neq 1$ by considering the problem in terms of crystalline periods. Although these methods do not yield results as strong as those yielded by the Taylor-Wiles method, they represent an extremely interesting and powerful new point of view on the problem. The author also constructs a rigid analytic space $X_{fs} \subseteq Z \times \mathbf{G}_m$ defined via representation theoretic conditions. He shows that X_{fs} contains $\mathcal{C}$ and that the resulting inclusion $\mathcal{C} \to X_{fs}$ is a local isomorphism at many classical points of $\mathcal{C}$. The author conjectures that in fact $\mathcal{C}$ equals X_{fs}. This conjectural isomorphism may be regarded as a generalization of the Taylor-Wiles isomorphism between a Hecke algebra and a deformation ring; its truth would be enormously useful for studying the eigencurve, and thus for studying p-adic modular forms.

As mentioned above, the unifying theme of the paper is the study of crystalline periods of p-adic Galois representations. To describe this, we must recall some of Fontaine's theory (see [J.-M. Fontaine and L. Illusie, in *Proceedings of the Indo-French Conference on Geometry (Bombay, 1989)*, 57–93, Hindustan Book Agency, Delhi, 1993; MR1274494 (95e:14013)] for a nice survey). Fontaine defines large rings B_{dR}, B_{cris} and B_{pst} possessing certain linear algebraic structures and an action of the absolute Galois group G_p of $\mathbf{Q}_p$. The ring B_{cris} in particular is equipped with a Frobenius semilinear operator φ. A p-adic representation $\rho\colon G_p \to \mathrm{GL}_2(E)$ with representation space V is said to be potentially semistable if the natural map

$$B_{\mathrm{pst}} \otimes (B_{\mathrm{pst}} \otimes V)^{G_p} \to B_{\mathrm{pst}} \otimes V$$

is an isomorphism respecting all linear algebraic structures induced from those on B_{pst}. (Roughly speaking, this implies that the Galois representation ρ can be recovered from the comparatively simple linear algebraic object $(B_{\mathrm{pst}} \otimes V)^{G_p}$.)

A crystalline period of V with φ-eigenvalue λ is a nonzero E-linear, G_p-equivariant map

$$h\colon V \to (B^+_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} E)^{\varphi=\lambda}.$$

That such a map exists when V is associated to a classical modular form f with U_p-eigenvalue λ is a deep result of T. Saito [Invent. Math. **129** (1997), no. 3, 607–620; MR1465337 (98g:11060)]. The first task in the present paper is to p-adically interpolate this result to prove the existence of crystalline periods for non-classical overconvergent eigenforms. This interpolation is far from trivial. The author begins by using S. Sen's theory of Hodge-Tate weights in families [Ann. of Math. (2) **127** (1988), no. 3, 647–661; MR0942523 (90b:11062); Bull. Soc. Math. France **121** (1993), no. 1, 13–34; MR1207243 (94e:11121)] to prove results about de Rham periods in families. (One says that ρ has $k \in \mathbf{Z}_p$ as a Hodge-Tate weight if there is a subspace of $\mathbf{C}_p \otimes V$ on which G_p acts via the $(-k)$th power of the cyclotomic character. The Galois representation associated to a p-adic eigenform of weight k has Hodge-Tate weights 0 and $k-1$.) Here a de Rham period is simply a map $V \to B^+_{\mathrm{dR}} \otimes E$. In particular, he finds that for representations which have 0 as a Hodge-Tate weight and which have no rational integer Hodge-Tate weights, the space of de Rham periods is essentially one-dimensional. (He also proves a partial result along these lines in the presence of integer Hodge-Tate weights; both cases are crucial to the later p-adic interpolation results.)

With this in hand, the author turns to the key technical result of the paper: he shows that for a separated rigid analytic space X equipped with a local system M of G_p-modules and a distinguished invertible function Y one can define a Zariski closed subspace X_{fs} essentially parameterizing those points $x \in X$ such that any de Rham period of the fiber of M at x factors through a crystalline period with φ-eigenvalue equal to the specialization of Y at x. (The subscript in X_{fs} stands for "finite slope".) Combined with the earlier results on existence of de Rham periods, it follows that for any $x \in X_{fs}$, the G_p-module M_x admits a crystalline period with φ-eigenvalue $Y(x)$. The proof that X_{fs} is closed is rather delicate: it is first necessary to use certain explicit elements in B_{cris} to reduce to a comparatively simple class of invertible functions Y. The key fact that X_{fs} is closed in X then follows from the fact that the corresponding inclusion

$$(B^+_{\mathrm{cris}})^{\varphi=Y} \to B^+_{\mathrm{dR}}/t^j B^+_{\mathrm{dR}}$$

has closed image for these Y and sufficiently large j, where t is a uniformizer of B_{dR}.

The author obtains the main results of the paper by applying this construction in two different settings. The above discussion implies that to show that overconvergent eigenforms admit crystalline periods it suffices to show that $\mathcal{C}_{fs} = \mathcal{C}$ (where the space $\mathcal{C}_{fs}$ is formed with respect to the canonical Galois representation on $\mathcal{C}$ and the invertible function a_p). Since $\mathcal{C}_{fs}$ is Zariski closed in $\mathcal{C}$ and by Saito we know that there exist crystalline periods for the subset of classical points of $\mathcal{C}$, one would hope that the desired equality follows immediately. In fact, there is a slight difficulty in that the one-dimensionality of de Rham periods may fail in the case of integer weights, so that it is first necessary to do a partial p-adic interpolation to overconvergent forms of p-adically integral weight via the results on de Rham periods in the presence of positive integer Hodge-Tate weights mentioned previously. The fact that $\mathcal{C}_{fs} = \mathcal{C}$ now follows from the earlier results applied to eigenforms with weight in $\mathbf{Z}_p - \mathbf{Z}$.

The proof that an overconvergent eigenform f with a potentially semistable Galois representation is necessarily classical is now an easy application. Specifically, we have seen above that an overconvergent eigenform f admits a crystalline period with φ-eigenvalue $a_p(f)$. Since V_f is potentially semistable, it is also Hodge-Tate, so that the weight k of f must be an integer. If $k \geq 2$, then the relation between the Hodge and Newton polygons of V_f provided by the assumption of potential semistability [see J.-M. Fontaine, Astérisque No. 223 (1994), 113–184; MR1293972 (95g:14024); Astérisque No. 223 (1994), 321–347; MR1293977 (95k:14031)] quickly yields that the p-adic valuation of $a_p(f)$ is between 0 and $k-1$. A theorem of Coleman [J. Théor. Nombres Bordeaux **9** (1997), no. 2, 395–403; MR1617406 (99g:11071); Invent. Math. **124** (1996), no. 1-3, 215–241; MR1369416 (97d:11090a)] now shows that f is indeed classical, except possibly when $a_p(f)$ has valuation $k-1$ and is the image of another overconvergent form under $k-1$ iterations of the theta operator. In fact, f is still known to be classical in the latter case by work of C. M. Skinner and Wiles [Inst. Hautes Études Sci. Publ. Math. No. 89 (1999), 5–126 (2000); MR1793414 (2002b:11072)] except if ρ_f is decomposable over G_p into characters with equal reductions. (This is the exceptional case mentioned previously.) The case of general weight $k \neq 1$ follows immediately by twisting. (The weight $k = 1$ presents special problems in that the representation then has both Hodge-Tate weights equal to 0, and thus is not approachable via these methods.)

The author also uses these methods to answer a question of F. Q. Gouvêa [*Arithmetic of p-adic modular forms*, Lecture Notes in Math., 1304, Springer, Berlin, 1988; MR1027593 (91e:11056) (Chapter II, Section 3.2)] on overconvergent modular forms generated by sums of Frobenius twists of arbitrary p-adic modular forms; he also re-proves the important result of Mazur and Wiles [Invent. Math. **76** (1984), no. 2, 179–330; MR0742853 (85m:11069)] that ordinary p-adic eigenforms have ordinary Galois representations.

To discuss the remainder of the paper, we fix a residual modular representation

$$\overline{\rho}\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(k)$$

with deformation space Z as before. Let Z^p denote the universal deformation space of the restriction of $\overline{\rho}$ to G_p. The preceding constructions are entirely local, so we define the closed rigid analytic subspace X_{fs} of $Z \times \mathbf{G}_m$ as the fiber product (over $Z^p \times \mathbf{G}_m$) of $Z \times \mathbf{G}_m$ and $(Z^p \times \mathbf{G}_m)_{fs}$ (where the latter space is formed with M the pullback of the universal deformation of $\overline{\rho}|_{G_p}$ and Y the canonical coordinate on $\mathbf{G}_m$). Let M now denote the restriction of the universal deformation on Z to X_{fs}. It follows from the finite slope construction that M_x admits a crystalline period with slope λ for any $(x, \lambda) \in X_{fs}$, and the author shows that the converse holds for most points of X_{fs}. In particular, it follows that the eigencurve $\mathcal{C}$ lies in X_{fs}. At this point it is natural to conjecture that this inclusion is in fact an equality, or, more conservatively, that it is an equality after deleting some spurious components of X_{fs} which may or may not actually exist.

Since $\mathcal{C}$ is a reduced curve, to prove that the inclusion $\mathcal{C} \hookrightarrow X_{fs}$ is a local isomorphism around a point (f, λ) it suffices to show that X_{fs} has dimension 1 at this point. Not surprisingly, the local dimension of X_{fs} can be studied deformation theoretically. The author begins by studying the local rings of the local model X^p_{fs}. In fact, the direct relation with deformation theory is somewhat surprising: while the local ring of Z^p at a point x is a localization of the universal deformation ring of $\overline{\rho}|_{G_p}$, the local ring of X^p_{fs} is also related to the universal deformation ring of the characteristic-zero representation M_x. The author nevertheless succeeds via an ingenious method in relating the characteristic-zero and characteristic-p deformation theory in order to gain some understanding of the local behavior of X^p_{fs}. In the global situation, the end result is that the tangent space at most points x of X_{fs} can be identified canonically with the dual of the group

$$\ker\bigl(H^1(G_{\mathbf{Q},S}, M_x \otimes M_x^*) \to H^1(G_p, B^+_{\mathrm{cris}} \otimes M_x^*)\bigr)$$

induced by the crystalline period of M_x. This is an interesting variant of the usual Bloch-Kato Selmer group [S. J. Bloch and K. Katō, in *The Grothendieck Festschrift, Vol. I*, 333–400, Progr. Math., 86, Birkhäuser Boston, Boston, MA, 1990; MR1086888 (92g:11063)]; the author is able to use his work ["Geometric deformations of modular Galois representations", preprint, 2002; per bibl.] as well as that of the reviewer [J. Reine Angew. Math. **543** (2002), 103–145; MR1887880 (2003d:11079); "Geometric Euler systems for locally isotropic motives", Compositio Math., to appear] to compute this tangent space and thus to conclude that the inclusion $\mathcal{C} \to X_{fs}$ is a local isomorphism at many classical points of $\mathcal{C}$.

The exposition of the paper is generally clear. The reader may find it easiest to skim the important Section 5 and to read carefully the applications in Sections 6 and 11 before confronting the more technical material that makes up the bulk of the paper. Even this technical material is surely of independent interest: there are many interesting new results on the ring B_{cris} and on deformation theory which are well worth the effort.

Thomas A. Weston

From MathSciNet, August 2015

MR2480604 (2009m:11077) 11F80; 11F11, 11R39

Khare, Chandrashekhar; Wintenberger, Jean-Pierre

On Serre's conjecture for 2-dimensional mod p representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Annals of Mathematics. Second Series **169** (2009), *no.* 1, 229–253.

The article under review marks a breakthrough in the quest for a proof of Serre's modularity conjecture [J.-P. Serre, Duke Math. J. **54** (1987), no. 1, 179–230; MR0885783 (88g:11022)].

The main results of the paper, which we will explain in more detail below, are:

- the existence in many cases of minimally ramified lifts of odd irreducible two-dimensional mod$p$ representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$;
- their embedding into compatible families of l-adic Galois representations;
- small level and weight cases of Serre's modularity conjecture;
- a strategy for reducing the odd level case of Serre's modularity conjecture to a modularity lifting conjecture (MLC).

Serre's modularity conjecture proposes a complete and very explicit classification of all odd irreducible two-dimensional mod p representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representations in terms of modular forms. For a summary of some of its important consequences, like the generalized Taniyama-Shimura-Weil conjecture and Artin's conjecture for odd two-dimensional complex representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the reader is referred to [C. Khare, J. Ramanujan Math. Soc. **22** (2007), no. 1, 75–100; MR2312549 (2008b:11062)].

In the meantime, Khare and Wintenberger have successfully carried through a strategy similar to the one outlined in Section 6.2 of the article under review and obtained a proof of most cases of Serre's modularity conjecture. Moreover, due to results of Kisin, the missing cases could also be settled, so that the conjecture has now become a theorem. Earlier on, Khare [Duke Math. J. **134** (2006), no. 3, 557–589; MR2254626 (2007e:11060)] had already proved the level one case using a modification of the strategy in Section 6.1.

It should be mentioned that some of the ideas of this article also appeared in work of L. V. Dieulefait [Rev. Mat. Iberoam. **23** (2007), no. 3, 1115–1124; MR2414504 (2009b:11095)] at around the same time. Much of the work culminating in the proof of Serre's modularity conjecture was initiated by A. J. Wiles [Ann. of Math. (2) **141** (1995), no. 3, 443–551; MR1333035 (96d:11071)] and R. L. Taylor and Wiles [Ann. of Math. (2) **141** (1995), no. 3, 553–572; MR1333036 (96d:11072)], who succeeded in proving the Taniyama-Shimura-Weil conjecture and thus Fermat's Last Theorem.

We will now first review Serre's modularity conjecture before describing the contents of the present article in more detail. In order to explain Serre's modularity conjecture, we first recall the theorem by Shimura, Deligne and Deligne-Serre associating with a newform a family of Galois representations. Let f be a cuspidal modular form which is an eigenfunction for all Hecke operators; assume that f is given by a Fourier series of the form $q + \sum_{n=2}^{\infty} a_n q^n$ with $q = q(z) = e^{2\pi i z}$ and that f has level N, weight k and nebentype character χ. For any prime p and any embedding $\iota\colon \mathbb{Q}(a_n; n \in \mathbb{N}) \hookrightarrow \overline{\mathbb{Q}}_p$ there is an odd (i.e. the determinant of the image of any complex conjugation is -1) irreducible Galois representation $\rho_{f,\iota}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ such that the characteristic polynomial at any prime $q \nmid Np$ is equal to $X^2 - \iota(a_q)X + \iota(\chi(q)q^{k-1})$. By choosing an integral model for $\rho_{f,\iota}$, reducing it modulo the maximal ideal and passing to the semi-simplification, one obtains a continuous Galois representation $\overline{\rho}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$. Note that this representation need not be irreducible.

A continuous Galois representation $\overline{\rho}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is said to be of S-type if it is odd and irreducible. Serre's modularity conjecture in its qualitative form claims that every $\overline{\rho}$ of S-type comes from some modular form by the theorem recalled above. The refined or quantitative form of the conjecture also specifies a weight $k(\overline{\rho})$ and a level $N(\overline{\rho})$ for a newform giving rise to $\overline{\rho}$ according to the principle that the ramification of $\overline{\rho}$ away from p is taken account of by the level ($N(\overline{\rho})$ is taken to be the Artin conductor of $\overline{\rho}$ outside of p), whereas the weight $k(\overline{\rho})$ contains information on the ramification at p. By work of, among others, Buzzard, Carayol, Coleman, Diamond, Edixhoven, Gross, Mazur, Voloch and Ribet, the equivalence of the qualitative and the quantitative form of the conjecture is known.

We now explain the results of the paper under review in more detail. Central to the whole approach is the existence of minimal lifts of mod p Galois representations. The notion of minimal lift is defined as follows. Let $\overline{\rho}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F})$ be an S-type representation, where $\mathbb{F}$ is a finite extension of $\mathbb{F}_p$. A lift ρ of $\overline{\rho}$ to the integers of a finite extension of the fraction field of the Witt vectors of $\mathbb{F}$ is called minimal if it is minimally ramified at all primes l. In order to define these notions, the cases $p \neq l$ and $p = l$ must be treated separately, according to the principle that the ramification at p is reflected in the weight $k(\overline{\rho})$ and the ramification at $l \neq p$ in the level $N(\overline{\rho})$. The minimality condition away from p is made so that $N(\rho) = N(\overline{\rho})$. At $l = p$ the recipe is the following: For $k(\overline{\rho}) \neq p+1$, ρ is minimally ramified at p, if the Hodge-Tate weights of ρ are $(0, k(\overline{\rho}) - 1)$. In the remaining case $k(\overline{\rho}) = p + 1$, one distinguishes minimal lifts of semi-stable type (then the Hodge-Tate weights are $(0, 1)$) and of crystalline type (with Hodge-Tate weights $(0, p)$).

The main result on minimal lifts of the paper under review (Theorem 3.3) asserts the existence of a minimal lift of an S-type $\overline{\rho}$ in odd residue characteristic p under a very weak assumption, whenever $2 \leq k(\overline{\rho}) < p$ or $k(\overline{\rho}) = p + 1$.

The important starting point for producing minimal lifts is Taylor's technique of potential modularity. Roughly speaking, it asserts the following. Suppose $\overline{\rho}$ is given with the assumption on $k(\overline{\rho})$ as above. Then there exists a totally real field F (upon which one may impose some additional properties) such that the restriction of $\overline{\rho}$ to (the absolute Galois group of) F arises from some Hilbert modular form over F satisfying certain local properties.

Having thus obtained the modularity of the restriction of $\overline{\rho}$ to F, results of, among others, Fujiwara, relating Hecke algebras and universal deformation rings, imply that the universal minimally ramified deformation ring of $\overline{\rho}$ is finite as a $\mathbb{Z}_p$-module. That it is also flat is a consequence of a theorem of G. Böckle [J. Reine Angew. Math. **509** (1999), 199–236; MR1679172 (2000b:11059)]. From this the existence of the minimally ramified lift follows.

In order to make use of the minimally ramified lifts in the context of Serre's modularity conjecture, it is necessary to embed the lift into a compatible family of Galois representations, which we describe now. For a number field E, an E-rational compatible system of two-dimensional Galois representations is a family (ρ_ι), indexed by the embeddings $\iota\colon E \to \overline{\mathbb{Q}}_l$ for all primes l, where the $\rho_\iota\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ are continuous semi-simple representations such that the Weil-Deligne representations at all primes $q \neq l$ are independent of ι and a certain property is satisfied at the prime l.

Under the same conditions as in the theorem on the existence of minimal lifts, the authors prove in Theorem 4.2 that there is a compatible family (ρ_ι) which is minimally ramified (in a suitable sense) such that a member at p reduces to the given $\overline{\rho}$. The strategy of proof follows along the lines of previous statements due to Taylor and Dieulefait.

Other very important ingredients are modularity lifting theorems. The results on the low level and weight cases of Serre's conjecture in the paper under review all use known modularity lifting theorems due to Skinner and Wiles. However, Khare and Wintenberger propose the following very general modularity lifting conjecture (MLC):

Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ and $\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O})$ be a continuous, absolutely irreducible odd representation which is ramified at only finitely many primes and de Rham at p with Hodge-Tate weights $(k-1,0)$ for $k \geq 2$. Assume that the reduction $\overline{\rho}$ is modular. Then ρ is isomorphic to an integral model of a p-adic representation ρ_f arising from a newform f.

By combining the fact that $\overline{\rho}$ can be embedded into a minimally ramified compatible system of Galois representations (Theorem 4.2) with the known cases of modularity lifting, nonexistence statements for certain abelian varieties (due to Fontaine, Brumer, Kramer and Schoof) as well as early results by Tate and Serre on the conjecture, the following theorems concerning low weight and low level cases of Serre's modularity conjecture are obtained.

- There is no S-type $\overline{\rho}$ with $N(\overline{\rho}) = 1$ and $k(\overline{\rho}) = 2$ for any p.
- There is no semi-stable S-type $\overline{\rho}$ with $N(\overline{\rho}) \in \{2,3,5,7,13\}$ and $k(\overline{\rho}) = 2$ for any $p > 2$.
- There is no S-type $\overline{\rho}$ with $N(\overline{\rho}) = 1$ and $2 \leq k(\overline{\rho}) \leq 8$ or $k(\overline{\rho}) = 14$ (this case under the extra assumption $p \neq 11$) for any p.
- If $\overline{\rho}$ is of S-type with $N(\overline{\rho}) = 1$ and $k(\overline{\rho}) = 12$, then $\overline{\rho}$ comes from the Ramanujan-Δ function for any p.

The article culminates in the outline of a strategy for proving the odd level case of Serre's modularity conjecture. Firstly, the case of level one is shown to be a consequence of Theorem 4.2 and the modularity lifting conjecture. The argument can be interpreted as 'changing the prime', and roughly proceeds as follows. Let $\overline{\rho}$ be of S-type. Embed it into a minimally ramified compatible family (ρ_ι). Take the reduction of any member at 3. The resulting mod 3 representation is known to be reducible and hence modular. Then the MLC implies that the whole family (ρ_ι) is modular, whence $\overline{\rho}$ is, too.

The strategy in the case of odd levels follows the principle of killing ramification by changing the prime. More precisely, it proceeds by induction on the number of prime factors dividing $N(\overline{\rho})$. The beginning of the induction is provided by the level one case. For the induction step, one again chooses some minimally ramified family (ρ_ι) by Theorem 4.2. If q is a prime dividing $N(\overline{\rho})$, then one passes to the reduction of a member of the family at q; denote this mod q representation by $\overline{\rho}_q$. If $\overline{\rho}_q$ is reducible, it is modular. If not, then it is of S-type; now $N(\overline{\rho}_q)$ is divisible by fewer primes than $N(\overline{\rho})$, whence it is modular by the induction hypothesis. In both cases, the MLC will imply that the whole family (ρ_ι) is modular, whence $\overline{\rho}$ is, too.

Gabor Wiese

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Scholze, Peter

Perfectoid spaces.

Publications Mathematiques. Institut de Hautes Études Scientifiques **116** (2012), 245–313.

In this fundamental paper, Peter Scholze introduces and develops the theory of perfectoid spaces. Meanwhile, he gives a wild generalization of G. Faltings' almost purity theorem [Astérisque No. 279 (2002), 185–270; MR1922831 (2003m:14031)] with a completely different proof. As a first application of his theory, he proves Deligne's weight monodromy conjecture for any geometrically connected proper and smooth variety over a locally compact field of characteristic 0 which can be embedded as a set-theoretic intersection in a projective and smooth toric variety.

The main techniques used by Scholze are almost étale mathematics of Faltings as developed by O. Gabber and L. Ramero [*Almost ring theory*, Lecture Notes in Math., 1800, Springer, Berlin, 2003; MR2004652 (2004k:13027)] and reduction to characteristic $p > 0$. This theory of perfectoid spaces is a very strong tool for reducing problems in mixed characteristic to problems in characteristic p. It has already quite a few important applications other than those given in this paper.

A *perfectoid field* is the fraction field K of a complete nondiscrete valuation ring K^0 of rank one whose residue field is of characteristic $p > 0$ and such that the Frobenius $x \mapsto x^p$ is onto on K^0/pK^0. If K is a perfectoid field, a perfectoid K-algebra R is a Banach K-algebra R such that the subring R^0 of the power bounded elements is bounded and that $x \mapsto x^p$ is onto on R^0/pR^0.

Generalizing a classical construction in p-adic Hodge theory, the author shows that the multiplicative monoid $\mathrm{Hom}(\mathbb{N}[1/p], R^\times)$ is the underlying multiplicative monoid of a well-defined topological ring, the *tilt* $R^\flat$ of R (to any norm $|\ |$ on R defining the topology there corresponds the norm $|\ |^\flat$ on $R^\flat$ defined by $|x|^\flat = |x^\sharp|$ (where $x^\sharp = x(1)$)). Tilting is functorial, $K^\flat$ is a perfectoid field of characteristic p and $R^\flat$ is a perfectoid $K^\flat$-algebra. Given K, the functor $R \mapsto R^\flat$ induces an equivalence of categories between perfectoid K-algebras and perfectoid $K^\flat$-algebras.

The fact that R and R^0 are almost never Noetherian leads to some technical difficulties and the most convenient way to define perfectoid spaces by gluing perfectoid K-algebras is to use R. Huber's theory of adic spaces [*Étale cohomology of rigid analytic varieties and adic spaces*, Aspects Math., E30, Friedr. Vieweg, Braunschweig, 1996; MR1734903 (2001c:14046)]. A *perfectoid affinoid K-algebra* is a pair (R, R^+) with R a perfectoid K-algebra and R^+ an integrally closed subring of R^0. To such a pair, Huber's construction associates a topological space $X = \mathrm{Spa}(R, R^+)$ (whose *rational subsets* form a basis for the topology) and presheaves of topological rings $\mathcal{O}_X$ and $\mathcal{O}_X^+$ on X. The author shows that X is an affinoid adic space, meaning that $\mathcal{O}_X^+$ and $\mathcal{O}_X$ $(= \mathcal{O}_X[1/\pi]$, if $\pi \in \mathfrak{m}$ is $\neq 0)$ are sheaves. He also proves that if i is a positive integer, then $H^i(X, \mathcal{O}_X^+)$ is almost zero (i.e. is annihilated by the maximal ideal $\mathfrak{m}$ of K^0) and therefore $H^1(X, \mathcal{O}_X) = 0$.

If K is of characteristic p, these results can be deduced by a limit process from Tate's acyclicity theorem. Done by reduction to the previous case, the characteristic 0 case is much more involved and Scholze proves much more: With obvious conventions, $X^\flat = \mathrm{Spa}(R^\flat, R^{+,\flat})$ is an affinoid space. There is a natural continuous map $X \to X^\flat$ and Scholze proves that this is a homeomorphism, inducing a bijection $U \mapsto U^\flat$ between the rational subsets of X and those of $X^\flat$. Moreover, if U is a rational subset, $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is a perfectoid affinoid K-algebra and $(\mathcal{O}_X(U)^\flat, \mathcal{O}_X^+(U)^\flat) = (\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))$. A key point of the proof is an approximation lemma showing that, for any $a \in R$, one can find $x \in R^\flat$ so that a is close enough to $x^\flat$ (though the image of the map $x \mapsto x^\sharp$ is not dense in R). If ϖ

is an element of the maximal ideal of $K^{\flat,0}$ such that $p \in \varpi^\sharp K^0$, one can identify $\mathcal{O}_X^+(U)/\varpi^\sharp$ to $\mathcal{O}_{X^\flat}^+(U^\flat)/\varpi$.

The category of *perfectoid spaces over* K is the full subcategory of the category of adic spaces over K which are locally of the form $\mathrm{Spa}(R, R^+)$ with (R, R^+) a perfectoid affinoid K-algebra. The *tilting functor* $X \mapsto X^\flat$ extends to an equivalence between perfectoid spaces over K and perfectoid spaces over $K^\flat$. Moreover, fiber products exist in these categories.

One of the most important results of this paper is the fact that tilting also induces an isomorphism of sites $X_{\text{ét}} \xrightarrow{\sim} X^\flat_{\text{ét}}$. One first needs to define the étale site $X_{\text{ét}}$. Scholze says that a morphism $Y \to Z$ of adic spaces is *étale* if, locally (for the given topology), it is an open embedding followed by a morphism of the form $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(C, C^+)$ with B a finite étale C-algebra and B^+ the integral closure of C^+ in B. The category underlying $X_{\text{ét}}$ is the category of étale morphisms $Y \to X$ and coverings are surjective families of étale morphisms. It is a site because Scholze proves that étale morphisms are closed by composition and under base change by a perfectoid space. To get these results, Scholze needs to prove a huge generalization of Faltings' almost purity theorem: If R is a perfectoid K-algebra and S a finite étale R-algebra, then S is a perfectoid K-algebra and S^0 is almost étale over R^0 in the sense of Faltings [op. cit.]. As in the previous parts of the paper, the author first proves the results in characteristic p and uses almost étale techniques [G. Faltings, op. cit.; O. Gabber and L. Ramero, op. cit.].

Let k be a locally compact non-Archimedean field whose residue field $\mathbb{F}$ is of characteristic p and let k^s be a separable closure of k. P. Deligne conjectured that, if Y is a geometrically irreducible proper and smooth variety over k, then, if $\ell \neq p$ is a prime and if $i \in \mathbb{N}$, the monodromy filtration on $H^i_{\text{ét}}(Y_{k^s}, \mathbb{Q}_\ell)$ is pure of weight i. When k is of characteristic p, this conjecture was proved by Deligne [Inst. Hautes Études Sci. Publ. Math. No. 52 (1980), 137–252; MR0601520 (83c:14017)] assuming that X comes by base change from a proper and smooth variety over a function field contained in k, and it was extended by T. Ito [Amer. J. Math. **127** (2005), no. 3, 647–658; MR2141647 (2006b:14039)] to the general case.

Assume now k is a finite extension of $\mathbb{Q}_p$ and let $\mathbb{C}_p$ be the completion of k^s. Let $\varpi = (\varpi^{(n)})_{n\in\mathbb{N}}$ be a sequence of elements of k^s such that $\varpi^{(0)}$ is a uniformizing parameter of k and $(\varpi^{(n+1)})^p = \varpi^{(n)}$ for all $n > 0$. The completion K of $K_\varpi := k((\varpi^{(n)})_{n\in\mathbb{N}})$ is a perfectoid field and $K^\flat$ is the completion of the radical closure of $E = \mathbb{F}((\varpi))$. With obvious conventions, we have identifications $H := \mathrm{Gal}(k^s/K_\varpi) = \mathrm{Gal}(K^s/K) = \mathrm{Gal}(K^{s,\flat}/K^\flat) = \mathrm{Gal}(E^s/E)$. If $Y \to k$ is proper and smooth, one sees immediately that, to check the weight monodromy conjecture for $H^i_{\text{ét}}(Y_{k^s}, \mathbb{Q}_\ell)$, it is enough to know the action of an open subgroup H' of H on this $\mathbb{Q}_\ell$-vector space. Hence, it suffices to show that there exists a finite separable extension E' of E contained in E^s and $Z' \to E'$ proper and smooth with an $H' = \mathrm{Gal}(E^s/E')$-equivariant isomorphism of $H^i_{\text{ét}}(Y_{k^s}, \mathbb{Q}_\ell)$ onto a direct summand of $H^i_{\text{ét}}(Z'_{E^s}, \mathbb{Q}_\ell)$.

Scholze shows that, if Y is a set-theoretic intersection in the toric variety $X_{\Sigma,k}$ over k associated to a fan Σ such that $X_{\Sigma,k} \to k$ is proper and smooth, one can find such a Z'. Roughly, there is a natural way to associate to Σ an adic space $\mathcal{X}_{\Sigma,K}$ equipped with a natural endomorphism φ. Iterating φ we get a natural "perfectoidization" $\mathcal{X}^{\mathrm{perf}}_{\Sigma,K}$ which tilts to $\mathcal{X}^{\mathrm{perf}}_{\Sigma,K^\flat}$ and morphisms of topoi

$$(\mathcal{X}^{\mathrm{perf}}_{\Sigma,K})^\sim_{\text{ét}} \to (\mathcal{X}^{\mathrm{ad}}_{\Sigma,K})^\sim_{\text{ét}} \quad \text{and} \quad (\mathcal{X}^{\mathrm{perf}}_{\Sigma,K^\flat})^\sim_{\text{ét}} \to (\mathcal{X}^{\mathrm{ad}}_{\Sigma,K^\flat})^\sim_{\text{ét}}$$

(the second one being an isomorphism). We also have similar continuous maps for the underlying topological spaces and, using tilting, we get "projections"

$$\pi_{\text{ét}} : (\mathcal{X}^{\mathrm{ad}}_{\Sigma,K^\flat})^\sim_{\text{ét}} \to (\mathcal{X}^{\mathrm{ad}}_{\Sigma,K})^\sim_{\text{ét}} \quad \text{and} \quad \pi : |\mathcal{X}^{\mathrm{ad}}_{\Sigma,K^\flat}| \to |\mathcal{X}^{\mathrm{ad}}_{\Sigma,K}|.$$

Choose an open neighborhood $\widetilde{Y}$ of the "adification" Y_K^{ad} of Y_K in $\mathcal{X}^{\mathrm{ad}}_{\Sigma,K}$. Using a variant of the approximation lemma mentioned above, Scholze shows that one can find a closed subvariety $Z \subset X_{\Sigma,K^\flat}$, defined over a finite extension of $\mathbb{F}(\varpi)$, such that $\dim Z = \dim Y = d$ and $Z^{\mathrm{ad}} \subset \pi^{-1}(\widetilde{Y})$. We take for Z' a projective smooth alteration of Z. The ℓ-adic étale cohomology does not change when we replace $Y_{\mathbb{C}_p}$ by $Y^{\mathrm{ad}}_{\mathbb{C}_p}$ [R. Huber, op. cit.] and, assuming we chose $\widetilde{Y}$ small enough, $Y^{\mathrm{ad}}_{\mathbb{C}_p}$ by $\widetilde{Y}_{\mathbb{C}_p}$ [R. Huber, J. Algebraic Geom. **7** (1998), no. 2, 359–403; MR1620118 (99h:14021)]. Using $\pi_{\text{ét}}$ we get a map $H^i_{\text{ét}}(Y_{k_s}, \mathbb{Q}_\ell) = H^i_{\text{ét}}(Y_{\mathbb{C}_p}, \mathbb{Q}_\ell) \to H^i_{\text{ét}}(Z'_{E^s}, \mathbb{Q}_\ell)$. One checks this is an isomorphism for $i = 2d$ and Poincaré duality implies the image is a direct summand for all i.

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