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*The defocusing NLS equation and its normal form*, by B. Grébert and T. Kappeler, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2014, x+166 pp., ISBN 978-3-03719-131-6, US$38.00

Until the end of the nineteenth century one of the major goals of differential equations theory and mechanics was solving equations of motion, i.e., finding formulae that describe the time dependence of the mechanical configuration. Surprisingly enough, this has been successfully achieved in many interesting cases, such as a planet under the action of gravitation (Kepler problem, two-body problem), the motion of a free rigid body, or the motion of a symmetric rigid body under the action of gravitation (Lagrange top). All these systems are now known to belong to the general class of *integrable Hamiltonian systems*, for which an explicit procedure of integration of the equations was found by Liouville.

Starting from the celebrated paper by Kruskal and Zabusky [ZK65], who discovered existence of solitons in the Kortweg–de Vries equation (KdV), mathematicians recognized that there are some partial differential equations that are infinite-dimensional analogs of the classical integrable systems. During the last fifty years, the techniques developed for the integration of classical finite-dimensional systems and for the study of their perturbations have been largely adapted to the case of partial differential equations (PDEs). The book by Grébert and Kappeler is a clear and complete exposition of the theory for the case of the defocusing Nonlinear Schrödinger equation (dNLS), one of the most important integrable PDEs.

Let me start by recalling the main result of the theory of integrable systems, namely the Liouville–Arnold–Jost theorem (usually known simply as Liouville–Arnold theorem). Such a theorem provides a very clean picture of the dynamics, indeed it ensures that, corresponding to any initial datum, the solution is described by a Kronecker flow on a torus.

In order to compare with the infinite-dimensional situation, it is worth giving a slightly more precise statement.

Consider a Hamiltonian system with Hamiltonian function $H = H(p, q)$ with $(p, q)$ canonical variables belonging to an open subset of $\mathbb{R}^{2n}$. Define the Poisson brackets of two function $F, G$ by

$$\{F; G\} := \sum_{k=1}^{n} \frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k} \equiv L_{X_G} F.$$

Then the system $H$ is said to be *integrable* if there exist $n$ integrals of motion which are independent and in involution. By this we mean that there exist $n$ functions $\Phi_1, \ldots, \Phi_n$ whose gradients are independent, which satisfy

$$\{\Phi_i; \Phi_j\} \equiv 0 \quad \forall i, j \ ,$$

and which are integrals of motion, namely

$$\{H; \Phi_j\} \equiv 0 \ , \quad \forall j = 1, \ldots, n \ .$$

In this case the Liouville–Arnold–Jost theorem ensures that if the surface $\Phi_i = c_i$, $i = 1, \ldots, n$, is compact, then it is diffeomorphic to an $n$-dimensional torus $\mathbb{T}^n$.

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Furthermore, this torus is invariant under the flow of the Hamiltonian vector field of $H$. Finally, this flow is equivalent to a linear flow on the torus (usually known as Kronecker flow).

The above statement is local, in the sense that, having fixed one level surface, there exists a neighborhood of that surface which is foliated into $n$-dimensional invariant tori on which the flow is the Kronecker flow.

There is also a second part of the Liouville–Arnold–Jost theorem, which is particularly useful in order to study perturbations of integrable systems (for example, in order to apply the Kolmogorov–Arnold–Moser (KAM) or the Nekhoroshev theorems). This second part ensures the existence of the so-called *action angle variables*, namely canonical variables $(I, \phi) \in \mathbb{R}^n \times \mathbb{T}^n$ in which the Hamiltonian turns out to depend on the actions $I$ only: i.e., one has $H = H(I)$, and thus the Hamilton equations are given by

$$\dot{I} = 0 \ , \quad \dot{\phi} = \frac{\partial H}{\partial I} = \text{constant}.$$

One has to remark that the proof of the theorem is constructive in the sense that it provides an algorithm to construct the action angle variables.

A good example to understand the construction is that of a mechanical system with one degree of freedom and Hamiltonian function given by

$$H(p,q) = \frac{p^2}{2} + V(q) \ ,$$

where $V$ is a function with a minimum at zero. Assume also $V(0) = 0$. In this case, for small positive values of the Hamiltonian, the invariant tori are one-dimensional curves and coincide with the level surfaces of $H$. The action variable is the area enclosed by the curve, and the angle variable is a rescaled polar angle. In particular, the angle variable has a singularity at the origin: $(p,q) = (0,0)$. This situation is typical: also in the general case the foliation in invariant tori has singularities and, correspondingly, the action angle variables have singularities of the same kind as the polar coordinates at the origin.

Consider now the case of PDEs and in particular the defocusing NLS equation (dNLS) with periodic boundary conditions, which is the object of the book:

$$i\partial_t u = -\partial_{xx} u + 2|u|^2 u, \ (x,t) \in \mathbb{R} \times \mathbb{R}, \quad u(x+1,t) = u(x,t).$$

It is convenient to enlarge the system by considering $u$ as independent of $\bar{u}$, so one studies the system

$$\begin{aligned} i\partial_t \phi_1 &= -\partial_{xx}\phi_1 + 2\phi_1^2\phi_2, \\ -i\partial_t \phi_2 &= -\partial_{xx}\phi_2 + 2\phi_2^2\phi_1, \end{aligned}$$

which is well known to be Hamiltonian on the Sobolev space $H^m(\mathbb{T},\mathbb{C}) \times H^m(\mathbb{T},\mathbb{C})$, $m \geq 1$, where $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. Such a system coincides with the dNLS on the subspace $H_r^m$ of the real states, namely the states satisfying $\phi_1 = \bar{\phi}_2 = u$.

The main observation is that dNLS admits a Lax pair formulation, which can be written in the form

$$\partial_t L = [B, L] \ , \quad [B,L] := BL - LB,$$

where $L = L(\phi)$ is the Zakharov–Shabat operator

$$L(\phi) := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & \phi_1 \\ \phi_2 & 0 \end{pmatrix},$$

and $B$ a suitable skew-symmetric operator (here $\phi \equiv (\phi_1, \phi_2)$).

As a consequence of the Lax pair formulation, one has that if $\phi(x,t)$ evolves according to the NLS equation, then the periodic spectrum of the operator $L(\phi(t))$ does not depend on time. In particular, each eigenvalue of $L(\phi(t))$ is an integral of motion which one would like to use in order to solve the NLS equations in the same way the functions $\Phi_j$ are used in the Liouville–Arnold–Jost theorem.

It is because of the Lax pair formulation that spectral theory enters in a prominent way in the theory of integrable systems, and a large part of the theory of integrable PDEs is now devoted to the study of spectral properties of some linear operators. This is why the first two chapters of the book are devoted to the study of the Zakharov–Shabat operator and its spectrum.

It turns out that a convenient set of integrals of motion for the dNLS is obtained by considering the periodic spectrum of $L(\phi)$ on the interval $[0, 2]$ (notice that we doubled the period!). When $\phi \in H^m_r$, $m \geq 0$, such a spectrum is a bi-infinite sequence satifying

$$\cdots < \lambda^-_{-1} \leq \lambda^+_{-1} < \lambda^-_0 \leq \lambda^+_0 < \lambda^-_1 \leq \lambda^+_1 < \cdots ,$$

where equality can occur only where the sign $\leq$ appears. Then a complete sequence of independent integrals of motion, which are also in involution, is given by the sequence of the *spectral gaps*

$$\gamma_n := \lambda^+_n - \lambda^-_n \ .$$

The reason why these objects are very useful for the study of the dNLS is that the map $\phi \mapsto \gamma^2_n(\phi)$ turns out to be a smooth function of the phase point $\phi$, a property which is not true for the periodic eigenvalues.

In the first two chapters of the book the spectral theory of the Zakharov–Shabat operator is developed in detail in an elementary way, so that the book can be read by graduate students without specific training in spectral theory or integrable systems.

The simplest type of solutions of the dNLS equation that one can consider are the so-called *finite gap solutions*, namely solutions corresponding to initial data with the property that, for some finite $N$, one has $\gamma_n = 0$ if $|n| > N$. The set of the states with such a property is invariant under the dynamics and is $(4N+2)$-dimensional. It is called the space of the $N$-gap potentials. The restriction of the system to this set is a finite-dimensional integrable system and the solution corresponding to an $N$-gap potential lie on an $(2N+1)$-dimensional torus. Explicit formulae for finite gap solutions were obtained first for the KdV equation in [DN74, Dub75, DMN76] in terms of theta functions.

Then one would like to introduce action angle variables in the space of $N$-gap solutions, and this requires studying the symplectic properties of the coordinates describing finite gap potentials and also obtaining formulae for action angle variables. Remarkable integral formulae for the action variables, in terms of the discriminant of the Lax operator $L$, have been obtained in [FM76] for the case of KdV and Toda lattice. In the case of NLS, similar formulae hold [MV97]. Exploiting such formulae, it is actually possible to introduce action angle variables in order to parametrize the space of $N$-gap potentials for any finite $N$.

The result however is far from being satisfactory, since it covers only data lying on finite-dimensional submanifolds. In order to extend such a result to open sets of initial data, one has to overcome a nontrivial difficulty: at the points where one of the gaps is closed, there is a singularity in the foliation in invariant tori similar to that met at the origin in the case of a one-dimensional mechanical system. Correspondingly, one has a singularity in the action angle coordinates. The situation is even more complicated due to the fact that the sequence of the gaps corresponding to a potential $\phi$ of class $H^0_r$ is square summable and thus always has zero as an accumulation point. Thus *the singularities of the foliation and of the action angle variables are dense in the phase space!*

The way out of this difficulty was found by Kappeler and coworkers [Kap91, BBGK95], and it consists in regularizing the variables at the singularity by introducing the analogs of Cartesian coordinates corresponding to the polar type coordinates given by the action angle coordinates. These Cartesian analogs of action angle coordinates are now called *Birkhoff coordinates.*

A further difficulty one has to face in order to get good coordinates is related to an inverse spectral problem: in order to be able to use spectral quantities as coordinates, one has to understand the sequences that appear as spectra of some potentials $\phi$.

The book under review provides all the tools leading to the complete solution of the previous problems, and it actually provides the construction of the Birkhoff coordinates for dNLS.

In order to state the final theorem proved in the book, we define the space of real sequences

$$h^m_r := \{(x,y) = (x_n, y_n)_{n\in\mathbb{Z}} \ : \ \|x\|_m + \|y\|_m < \infty\} ,$$

where

$$\|x\|^2_m := x_0^2 + \sum_{n\in\mathbb{Z}} n^{2m} x_n^2 \ ,$$

endowed by the standard symplectic form $\omega := \sum_n dx_n \wedge dy_n$. Then the main result of the book is the following theorem:

**Theorem.** *There exists a bi-analytic diffeomorphism $\Omega : H^0_r \to h^0_r$ with the following properties:*

1. *$\Omega$ is canonical, that is, it preserves the Poisson brackets.*
2. *The restriction of $\Omega$ to $H^m_r$ with $m \geq 1$ gives rise to a map $\Omega : H^m_r \to h^m_r$ that is again onto and bi-analytic.*
3. *$\Omega$ introduces global Birkhoff coordinates for NLS on $H^1_r$; that is, on $h^1_r$ the transformed NLS Hamiltonian is a real analytic function of the actions $I_n := (x_n^2 + y_n^2)/2$ with $n \in \mathbb{Z}$.*
4. *$d_0\Omega$ is the Fourier transform.*

As has already been mentioned, the coordinates introduced by this theorem are usually called Birkhoff coordinates.

The proof, is self-contained and occupies the whole book. In the first two chapters the authors treat the spectral problem for the Zakharov–Shabat operator. In particular, the final result of these chapters is the construction a global coordinate map in $H^0_r$ which is canonical and adapted to the the description of the isospectral sets which are important for the dNLS equation.

Then in Chapter III they construct the actions and the angles for the dNLS equation, which are singular when some of the gaps are closed. The construction of the action variables is based on some known formulae which are used in order to study the differentiability properties of the actions themselves. The introduction of the angles is more subtle, since it requires constructing the basis of the holomorphic differentials of the isospectral manifold, namely of the manifold formed by all the potentials having the same spectrum. In general, this is a Riemann manifold of infinite genus, and the construction of the holomorphic differential requires solving an infinite system and studying the analytic properties of its solution.

Finally, in Chapter IV the authors remove the singularity at the origin, they introduce the Birkhoff coordinates, and they show that the so-obtained coordinates are globally analytic.

The presentation is very clear and precise, and all the proofs can be followed without problems even from a reader who is not expert in the field of integrable systems.

The Birkhoff coordinates are a fundamental tool for the study of perturbations of integrable systems, for example for the application of KAM theory, as was done in the previous book by Kappeler and Pöschel on KdV [KP03].

The present book also shows that the technology and the theory of integrable systems is now well developed and applies in a more-or-less standard way to integrable PDEs in one space dimension. The situation is completely different for equations in more than one space dimension, such as the KP equation, for which a construction of action angle/Birkhoff coordinates suitable for the application of functional analytic tools is still completely lacking and is one of the most important open problems in the theory of integrable PDEs.

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