MATHEMATICAL PERSPECTIVES

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 53, Number 3, July 2016, Pages 477–481
<http://dx.doi.org/10.1090/bull/1534>
Article electronically published on March 16, 2016

ABOUT THE COVER: ZETA-FUNCTIONS ASSOCIATED WITH QUADRATIC FORMS IN ADOLF HURWITZ'S ESTATE

NICOLA M. R. OSWALD AND JÖRN STEUDING

The first published proof of key analytic properties of zeta-functions associated with quadratic forms is due to Paul Epstein [2] in 1903 (submitted January 1902). The corresponding name “Epstein zeta-functions” was introduced by Edward Charles Titchmarsh in his article [15] from 1934; soon after, this terminology became common through influential articles of Max Deuring and Carl Ludwig Siegel. While examining Adolf Hurwitz’s mathematical estate at ETH Zurich, however, the authors discovered that Hurwitz could have published these results already during his time in Königsberg (now Kaliningrad) in the late 1880s. Unpublished notes of Hurwitz testify that he was not only aware of the analytic properties of zeta-functions associated with quadratic forms but prepared a fair copy of his notes, probably for submission to a journal. The cover of this issue shows the first page (see Figure 1).

Given a quadratic form $\varphi(x_1, x_2, \dots, x_p) = \sum_{i,k=1}^p a_{ik}x_i x_k$ and real parameters $u_1, u_2, \dots, u_p; v_1, v_2, \dots, v_p$, Hurwitz considers the associated Dirichlet series

$$\mathcal{F}(s|u, v, \varphi) = \sum' \frac{e^{2\pi i(x_1 v_1 + x_2 v_2 + \dots + x_p v_p)}}{[\varphi(x_1 + u_1, x_2 + u_2, \dots, x_p + u_p)]^{\frac{ps}{2}}}$$

as a function of a complex variable s , where the summation is taken over all integers $x_1, x_2, \dots, x_p$ with possible exceptions of the combinations $x_1 = -u_1, x_2 = -u_2, \dots, x_p = -u_p$ (indicated by the dash). Hurwitz succeeds in proving meromorphic continuation to the whole complex plane and a functional equation. His reasoning relies on Riemann’s famous paper on the number of primes below a given magnitude [13] in which a meromorphic continuation of the Riemann zeta-function to the whole complex plane and its functional equation is established (and path-breaking conjectures about the zeta-function with respect to prime number distribution are formulated). Actually, Riemann gave two proofs for the functional equation: one by contour integration (which has been generalized by Hurwitz in

FIGURE 1. CLEAR COPY OF THE COVER. An excerpt from Hurwitz's fair copy showing the first page with the definition of the Dirichlet series associated with a quadratic form.

[3]), and a second one relying on Mellin transformation and Poisson's summation formula. It should be noted that the Riemann zeta-function appears as a special case of a zeta-function associated with the quadratic form of a single variable.

In his notes Hurwitz makes use of a theta transformation formula; for this purpose he follows Riemann's work on Abelian functions [12] where theta-functions of several variables were treated for the first time. A detailed proof of the transformation formula for the theta-function of several variables can be found in Hurwitz's mathematical diary no. 5. Where Hurwitz derives the transformation formula by means of Fourier analysis, Epstein quotes a recent book [5] of Friedrich Prym and Adolf Krazer which had not been published when Hurwitz was doing his investigations. We note that Prym was a pupil of Riemann and a professor, first in Zurich and beginning in 1869 in Würzburg; most of his studies were dedicated to complex analysis and theta-functions in particular. The latter was also the main area of research of his doctoral student Krazer who became professor in Strasbourg in 1889. Actually, Epstein obtained his doctorate in Strasbourg in 1895 under supervision of Elwin Christoffel and remained there as docent until 1919 when he received the call of the recently founded university in his native town of Frankfurt/Main. Therefore, Epstein knew Krazer. Moreover, Epstein probably knew Hermann Minkowski who was a former pupil at Königsberg and later friend of

FIGURE 2. An excerpt from Hurwitz's fair copy showing the Mellin transformation technique applied to his Dirichlet series $\mathcal{F}(s|u, v, \varphi)$ attached to a quadratic form.

Hurwitz. In the 1890s Minkowski was frequently visiting his brother Oskar, the famous physician, in Strasbourg (see [10]). During these visits, Minkowski also kept in touch with mathematicians at Strasbourg University (see [10], pp. 34, 36).

Interestingly, the last sentences of Hurwitz's fair copy, translated into English, are: "Hence, the latter integral is simultaneously treatable termwise with the integral

$$\int_0^\infty \sum' e^{-\pi \lambda (x_1^2 + x_2^2 + \dots + x_p^2)} \lambda^{\frac{\bar{s}}{2}} d\log \lambda.$$

For the latter one this is valid for $\bar{s} > 1$ as follows in a different way. Let " (see Figure 2). The text stops in the middle of the sentence. Here $\bar{s}$ denotes the real part of s (and not its conjugate). In Hurwitz's mathematical diary no. 5, written between February 1886 and the end of March 1888, one can find an undated entry (pp. 17–42) which fills the gap left in the fair copy. A later entry in this diary bears the date 1 May 1887. The detailed fair copy indicates that Hurwitz considered publishing his notes on zeta-functions associated with quadratic forms, but it seems that for some reason he never realized his idea.

Hurwitz's motivation on this topic dates back to the early 1880s when he was studying Dirichlet's proof of the infinitude of primes in arithmetic progression [1] and related generating functions by means of his Hurwitz zeta-functions [3]. First studies about Dirichlet series related to quadratic forms and their meromorphic continuation appear already in Hurwitz's mathematical diary no. 3 from 1883–84, pp. 79–93 (a time when Hurwitz was still in Göttingen); here only the case of binary quadratic forms is treated. As mentioned above, in diary no. 5 the general case has been established. In his paper Epstein mentions Hurwitz's paper [3] on the Hurwitz zeta-function as well as further influential articles by Rudolf Lipschitz [9] and Mathias Lerch [7]. Epstein went slightly beyond what Hurwitz discussed in his

draft by using his results in order to deduce Leopold Kronecker's limit formula for the Laurent expansion of real-analytic Eisenstein series at their singularity [6].

There is another interesting detail. Hurwitz's mathematical diary no. 6 [4] contains a draft of a letter Hurwitz planned to send to Lipschitz bearing the date 17 August 1889 in which Hurwitz announces the main results about zeta-functions associated with quadratic forms (see [4, 11] for details). In his first note on Dirichlet series [8] Lipschitz studied the real and imaginary part of the $\zeta(s)$ defining Dirichlet series and generalizations thereof; in his second article [9] from 1889 Lipschitz succeeded in giving an almost complete account of Dirichlet L -functions within an analysis of a more general Dirichlet series. Hurwitz's letter (if written and sent) seems to be lost; at least, Hurwitz is not mentioned in Scharlau's list [14] of letters Lipschitz had received from his contemporaries. Also this link between Lipschitz and Hurwitz could have been established by Minkowski. In 1887, after studying with Hurwitz in Königsberg, Minkowski became a colleague of Lipschitz at the University of Bonn. Concerning Lipschitz's correspondence, we notice that the mathematical-historical seminar of the University of Bonn was destroyed during World War II. There are also no letters between Lipschitz and Hurwitz in the latter one's estate, neither in Zurich nor in the archive of the Staats- und Universitätsbibliothek Göttingen where most of his correspondence is stored.

Taking all these notes into account, it seems reasonable to believe that there were first the sketches in his mathematical diaries nos. 3 and 5, then the draft of the letter to Lipschitz from 1889 in diary no. 6, and only afterwards his fair copy. In view of the entries in his diaries, it is clear that in 1887 Hurwitz was already aware of how to prove meromorphic continuation and a functional equation for Dirichlet series associated with quadratic forms. Hurwitz always aimed at perfection, and since there are many ways of generalizations and extensions possible, this might have been the reason that he did not publish his results.

ACKNOWLEDGMENTS

The authors would like to thank the ETH Zurich library.

REFERENCES

- [1] G. Lejeune Dirichlet, *Recherches sur diverses applications de l'Analyse infinitésimale à la Théorie des Nombres* (French), J. Reine Angew. Math. **21** (1840), 1–12, DOI 10.1515/crll.1840.21.1. MR1578250
- [2] Paul Epstein, *Zur Theorie allgemeiner Zetafunktionen* (German), Math. Ann. **56** (1903), no. 4, 615–644, DOI 10.1007/BF01444309. MR1511190
- [3] A. Hurwitz, *Einige Eigenschaften der Dirichletschen Funktionen $F(s) = \sum \left(\frac{D}{n}\right) \frac{1}{n^s}$, die bei der Bestimmung der Klassenzahlen binärer quadratischer Formen auftreten*, Zeitschrift f. Math. u. Physik **27** (1882), 86–101
- [4] A. Hurwitz, *Die Mathematischen Tagebücher und der übrige handschriftliche Nachlass von Adolf Hurwitz*, Handschriften und Autographen der ETH-Bibliothek Zürich, Hs 582:5,6; doi.org/10.7891/e-manuscripta-12821, 12823
- [5] A. Krazer, F. Prym, *Neue Grundlagen einer Theorie der allgemeinen Thetafunktionen. Kurz zusammengefasst und herausgegeben von A. Krazer*. Teubner, Leipzig 1892
- [6] L. Kronecker, *Zur Theorie der elliptischen Functionen*, Berl. Sitzungsber. **XX** (1890), 99–120, 123–130, 219–241, 307–318, 1025–1029
- [7] M. Lerch, *Note sur la fonction $\mathfrak{K}(w, x, s) = \sum_{k=0}^{\infty} \frac{e^{2k\pi ix}}{(w+k)^s}$* (French), Acta Math. **11** (1887), no. 1-4, 19–24, DOI 10.1007/BF02418041. MR1554747

- [8] R. Lipschitz, *Untersuchung einer aus vier Elementen gebildeten Reihe* (German), J. Reine Angew. Math. **54** (1857), 313–328, DOI 10.1515/crll.1857.54.313. MR1579049
- [9] R. Lipschitz, *Untersuchung der Eigenschaften einer Gattung von unendlichen Reihen* (German), J. Reine Angew. Math. **105** (1889), 127–156, DOI 10.1515/crll.1889.105.127. MR1580195
- [10] H. Minkowski, *Briefe an David Hilbert*, L. Rüdtenberg, H. Zassenhaus (eds.), Springer 1973
- [11] N.M.R. Oswald, J. Steuding, Aspects of Zeta-Function Theory in the Mathematical Works of Adolf Hurwitz, in: *From Arithmetic to Zeta-Functions. Number Theory in Memory of Wolfgang Schwarz*, J. Sander et al. (eds.), Birkhäuser 2016 (to appear).
- [12] B. Riemann, *Theorie der Abel'schen Functionen* (German), J. Reine Angew. Math. **54** (1857), 115–155, DOI 10.1515/crll.1857.54.115. MR1579035
- [13] B. Riemann, *Über die Anzahl der Primzahlen unterhalb einer gegebenen Grösse*, Monatsber. Preuss. Akad. Wiss. Berlin (1859), 671–680
- [14] Winfried Scharlau, *The mathematical correspondence of Rudolf Lipschitz*, Historia Math. **13** (1986), no. 2, 165–167, DOI 10.1016/0315-0860(86)90029-7. MR851875
- [15] E. C. Titchmarsh, *On Epstein's Zeta-Function*, Proc. London Math. Soc. **S2-36**, no. 1, 485, DOI 10.1112/plms/s2-36.1.485. MR1575971

DEPARTMENT OF MATHEMATICS AND INFORMATICS, UNIVERSITY OF WUPPERTAL, GAUSSSTR. 20, 42 119 WUPPERTAL, GERMANY; AND DEPARTMENT OF MATHEMATICS, WÜRZBURG UNIVERSITY, EMIL-FISCHER-STR. 40, 97 074 WÜRZBURG, GERMANY

E-mail address: `oswald@uni-wuppertal.de`; `nicola.oswald@mathematik.uni-wuerzburg.de`

DEPARTMENT OF MATHEMATICS, WÜRZBURG UNIVERSITY, EMIL-FISCHER-STR. 40, 97 074 WÜRZBURG, GERMANY

E-mail address: `steuding@mathematik.uni-wuerzburg.de`