COMMENTARY ON
 “NUMBERS OF SOLUTIONS
 OF EQUATIONS IN FINITE FIELDS”
 BY ANDRÉ WEIL

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ABSTRACT. Immediately following the commentary below, this previously published article is reprinted in its entirety: André Weil, “Numbers and solutions of equations in finite fields”, *Bull. Amer. Math. Soc.*, **55** (1949), no. 5, 497–508.

The paper “Numbers of solutions of equations in finite fields” of André Weil [W1] ranks among the most influential articles in the mathematics of the last hundred years. On the last page of this paper Weil proposes a collection of precise conjectures which initiated a vast development in mathematics.

It would require a much longer article than these few paragraphs to describe the evolution of the “Weil conjectures”. The Wikipedia article [Wik] is a good starting point, perhaps followed by J. Milne’s comprehensive study [M]. See also [Di, H, O, K].

Weil’s article begins with a laborious computation of the number of solutions over a finite field to a certain equation $\sum a_r x^r = 0$. Initially, Weil offers a somewhat strained motivation for doing this calculation, and (other than in the title of the article) he gives no indication that he is interested in situations more general than this particular equation. However, at the end of the article the real point of the calculation is revealed: it is a nontrivial example which he uses to motivate the conjectures. Weil then states that “this, and many other examples that we cannot discuss here” support the conjectures. He had clearly been thinking about this circle of ideas for some time.

Weil’s conjectures were revolutionary because they proposed a precise and deep relation between two completely different sorts of objects: the topology of a complex algebraic variety versus the number of points (mod p) in that variety. For most mathematicians this probably looked like a message from outer space, and Carlitz in his Mathematical Reviews article does little more than quote, without comment, Weil’s sentence that introduces the conjectures.

In his 1954 ICM lecture [W2], perhaps as a result of daydreaming about these questions for the intervening five years, Weil points out that his conjectures could be conveniently rephrased in terms of the Lefschetz fixed point formula for the self-map that is defined by the Frobenius morphism, thus cementing the relation between topology and number theory that he had proposed in the *Bulletin* article. He does not discuss the obvious problem that a new cohomology theory (valid for

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algebraic varieties defined over finite fields) would be needed in order to make sense of such a Lefschetz formula.

Soon after Weil's talk at the ICM, J. P. Serre found an example proving that such a cohomology theory, *with rational coefficients*, could not exist. Some people suspect that Weil may have resisted the idea of a "Weil cohomology theory" for years. His skepticism may have been reinforced when, in 1960, B. Dwork [Dw] proved the first of the conjectures (rationality of the zeta function) using (non-cohomological) p -adic methods. Nevertheless the whole picture was so compelling that, at some point between 1954 and 1958, mathematicians began to actively seek a cohomology theory in which Weil's conjectures could be formulated and proven. In 1958 Serre published [S] a reasonable but ultimately unsuccessful attempt. In 1958 Grothendieck [G1] made a first outline of a general theory, and by 1972 [A] M. Artin and A. Grothendieck had succeeded in constructing the required *étale cohomology theory*. Together with J. L. Verdier they used it to prove the first three of Weil's conjectures. (The theory was later simplified by Deligne [D3].)

Applications of these results were immediate. For example, even before the last of the Weil conjectures had been established, Deligne showed [D2] that, if true, it could then be used to prove the Ramanujan conjecture (and its generalization, the Ramanujan–Petersson conjecture) from the theory of modular forms.

The remarkable collection of mathematical ideas involved in Deligne's proof [D1] of the fourth conjecture is described in [K, O]. In the current literature, applications of the "Weil conjectures" (that is, the resulting theorems of Artin, Deligne, Grothendieck, and Verdier) have become so common that many authors simply cite "Weil conjectures" without reference, whenever these results are used.

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