BOOK REVIEWS

Prime numbers and the Riemann hypothesis, by Barry Mazur and William Stein, Cambridge University Press, Cambridge, 2016, xi+142 pp., ISBN 978-1-107-49943-0; Honourable Mention, 2017 PROSE Award for Mathematics

Those of you who have found yourselves explaining to a lay audience what the Riemann Hypothesis is and why it is such a central mathematical problem, will know that this is no easy task. I have vivid memories of my failed effort at an MSRI workshop devoted to presenting mathematics to science writers. The book under review succeeds handsomely in making the case for the Riemann Hypothesis to a wide audience.

Mazur and Stein assume that the reader has been exposed to some high school mathematics (for some of the later chapters some calculus is assumed as well) and starting from scratch they proceed to explain the mathematical notions and ideas in Riemann's famous paper "Über die Anzahl der Primzahlen unter einer gegebenen Grösse", where he formulates the Hypothesis.

The short and very readable book comes in three sections. The first and longest section requires little in the way of mathematical background. Beginning with the definition of prime numbers, the authors weave their way through concrete and picturesque presentations of elementary techniques and descriptions of unsolved problems connected with the primes. They provide many insightful footnotes, concrete and illuminating figures, pointers to arXiv pages for added information, and a rich set of endnotes that contain further descriptions and details with varying levels of sophistication. After 23 short sections (a few pages each) they have arrived at a formulation of the Riemann Hypothesis in terms of counting primes up to a given size. By this point in their masterful and compelling presentation, the Hypothesis appears to be completely natural and inevitable.

The next two sections require a little more mathematics, specifically calculus. They introduce some basic Fourier series and distribution theory that they need in order to formulate Riemann's explicit formula connecting the primes to certain fundamental frequencies (or spectrum) via a Fourier duality. These are illustrated with many concrete graphs which clarify the meanings and content of various formulae and even the reader who does not know calculus will, I believe, come away with a pretty good idea of what these formulae convey. The short third section

completes the treatment of Riemann's paper. The reader with some basic knowledge of complex analysis will at this point have been led through Riemann's paper and will appreciate the formulation of the Hypothesis in terms of the location of the zeros of the Riemann Zeta function.

The presentation is very much in the style of Riemann, very concrete and with the scientific bent of developing approximations to the counting of primes (to Riemann and Gauss before him, pure and applied mathematics were one and the same). As such, the book will have a very wide appeal. While Riemann had neither the power of modern computers to calculate many of the zeros (he did compute the first few as was later discovered in his *Nachlass*) nor the ability to see graphs and instances of his formulae and many other related numerical computations, the authors do and they make effective use of such technology and do a marvelous job of integrating all this information into an exposition of the underlying mathematics.

The last pages (pp. 124–126) point to one of the most important features of the Riemann Hypothesis; that it is just the first of a very general conjecture about zeros of zeta functions, some analogues of which have been proven and which constitute major achievements of modern mathematics. I have no doubt that many newcomers to the subject who have read to the end of the book will be eager to learn more and will be drawn into this fertile playground.

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