BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 55, Number 4, October 2018, Pages 553–560
http://dx.doi.org/10.1090/bull/1593
Article electronically published on August 16, 2017

Functions with disconnected spectrum: Sampling, interpolation, translates, by Alexander M. Olevskii and Alexander Ulanovskii, University Lecture Series, Vol. 65, American Mathematical Society, Providence, RI, 2016, x+138 pp., ISBN 978-1-4704-2889-1

One of the basic facts of Fourier analysis is that the sequence of complex exponentials with integer frequencies $\{e^{2\pi inx} : n \in \mathbb{Z}\}$ forms an orthonormal basis for $L^2[0,1]$. This is a consequence of the fact that $2\pi\mathbb{Z}$ and $[0,1]$ (with summation modulo 1) are Abelian groups dual in a very precise sense. It is natural to wonder what would happen with the basis properties of this system if we perturb one or both of these two groups. Clearly, the group structure and the corresponding orthonormality will almost always be lost. Still, one might hope that some of the basis properties of the perturbed exponential system will be preserved. More precisely, we have the following question:

(Q) What will happen if we replace the integers with an arbitrary sequence of real numbers $\Lambda = \{\lambda_n\}$ and $[0,1]$ with a more general measurable set $S \subset \mathbb{R}$? More precisely, can we describe in terms of the frequency sequence Λ the basis properties (if any) of the sequence of nonharmonic complex exponentials $\mathcal{E}(\Lambda) := \{e^{2\pi i\lambda x} : \lambda \in \Lambda\}$ in the space $L^2(S)$?

This is probably one of the most classical questions in Fourier analysis, which, as expected, has received significant attention in the past. It is safe to say that this question and its higher-dimensional analogue represent a central topic of the Olevskii–Ulanovski book, with clear accent on the case when S is disconnected.

The case of connected S, at least in the one-dimensional setting, is much simpler (although by no means simple). This is probably why for these sets the problem was essentially settled much earlier. The reason is as follows. Applying the Fourier transform in the usual way,

$$\hat{f}(\xi) = \int f(x)e^{2\pi i\xi x}dx,$$

this problem can be transferred to an equivalent question in the corresponding Paley–Wiener space $\mathcal{PW}_S$. This space is still a Hilbert space (closed subspace of $L^2(\mathbb{R})$), but its advantage is that when S is bounded, $\mathcal{PW}_S$ consists of entire functions. Moreover, if S is an interval, due to the well-known Paley–Wiener theorem, these entire functions can be characterized by their truly exponential growth depending only on the length of the interval S (exponential type is the more precise notion here). This characterization is not possible for more general sets S, even in the case when S is just a union of two intervals. Therefore, in the case when S is an interval, one can use the full arsenal of techniques from classical complex analysis, which makes the problem significantly more tractable (though not at all easy).

Historically, the first results related to question (Q) were connected to the Riesz basis problem. Since even a slight perturbation of the integer frequencies destroys the orthonormality of $\mathcal{E}(\Lambda)$, it is natural to see whether the next best possible basis property (that of a Riesz basis) remains. In the Hilbert space context, Riesz bases

2010 *Mathematics Subject Classification.* 41-XX, 42-XX, 94A12, 94A20.

are nothing but sequences obtained as images of orthonormal bases under bounded invertible linear operators. The first result in this direction is probably the one of Paley and Wiener who showed that $\mathcal{E}(\Lambda)$ is a Riesz basis for $L^2[0,1]$ whenever $|\lambda_n - n| < 1/\pi^2$ for all $n \in \mathbb{Z}$. The optimal constant $1/4$ was obtained by Kadec [5], and the complete description was obtained by Pavlov [16] much later. In the case when S is not an interval, this problem is significantly more difficult. For a general measurable set S it is not even clear whether there exists any sequence Λ for which $\mathcal{E}(\Lambda)$ is an orthonormal or even just a Riesz basis for $L^2(S)$. The famous Fuglede conjecture is that compact sets S that admit translations that tile $\mathbb{R}^n$ are exactly the sets for which one could find Λ such that $\mathcal{E}(\Lambda)$ is an orthonormal basis for $L^2(S)$. This conjecture was proved for convex planar bodies S by Iosevich, Katz, and Tao [4], but it was disproved for general S in dimension $n \geq 5$ by Tao [19]. There are counterexamples now showing that both directions of the conjecture are false in dimension $n \geq 3$. The Riesz basis counterpart is also still wide open, although important recent progress was made by Kozma and Nitzan [7] who proved that any S that is a finite union of intervals admits a Riesz basis of complex exponentials. Very recently, it was proved by Nitzan, Olevskii, and Ulanovski [12] that any set S of finite measure admits a frame (spanning part of the Riesz basis) of complex exponentials. The proof relies crucially on the recent solution of the Kadison–Singer and Feichtinger conjectures by Marcus, Spielman, and Srivastava [9]. It is presented in Lecture 10 of the book under review.

In the basic question (Q) above, we loosely used the term *basis properties.* Let us explain more precisely what is meant by this. As their names suggest, Riesz and orthonormal bases are both bases, meaning that they both have linear independence and spanning property incorporated in them. In other words, any element in the space can in a unique way (due to independence) be expanded (due to the spanning property) as an appropriate linear combination of the basis elements. Among many ways to generalize linear independence and spanning property in the Hilbert space setting, due to their stability, are that the notions mostly used nowadays are those of a Riesz sequence and a frame. (It is interesting to note that frames were actually introduced first in the context of nonharmonic complex exponentials [2].)

A sequence of vectors $\{e_n\}$ in a Hilbert space $\mathcal{H}$ is said to be a Riesz sequence if there exist $0 < c \leq C < \infty$ such that for any $\{a_n\} \in l^2$,

$$c \sum |a_n|^2 \leq \left\| \sum a_n e_n \right\| \leq C \sum |a_n|^2.$$

So, basically we have linear independence with bounds. A sequence $\{e_n\}$ is called a frame if there exist $0 < c \leq C < \infty$, such that for any $f \in \mathcal{H}$ we have

$$c \|f\|^2 \leq \sum |\langle f, e_n \rangle|^2 \leq C \|f\|^2 .$$

In this case we can find a family $\{e_n^*\}$, a so-called dual frame, such that any $f \in \mathcal{H}$ can be expanded as

$$f = \sum \langle f, e_n^* \rangle e_n = \sum \langle f, e_n \rangle e_n^*,$$

with control on the l^2-norm of the coefficient sequence. It is easy to show that on the Paley–Wiener space these two concepts are equivalent to the notions of interpolation (corresponding to Riesz sequences) and sampling (corresponding to frames). Namely, Λ is sampling/interpolating for $\mathcal{PW}_S$ if and only if $\mathcal{E}(\Lambda)$ is a frame/Riesz sequence for $L^2(S)$.

Since Riesz sequences and frames represent two dual and complementary sides of a Riesz basis, it is natural to ask whether we can characterize these two basic basis properties of $\mathcal{E}(\Lambda)$ in $L^2(S)$ in terms of the frequency sequence Λ only. For intuitive reasons, let us consider the following analogy with the finite-dimensional situation: As we know from linear algebra, the number of elements in any linearly independent set is not greater than the dimension of the corresponding vector space, and similarly the number of elements in any spanning set is not smaller than the dimension. It turns out that there is a very close analogue in the context of complex exponentials. Namely, for a sequence of complex exponentials $\mathcal{E}(\Lambda)$ to be a Riesz sequence for $L^2[0,1]$, it is necessary for the frequency sequence Λ to be asymptotically sparser than the sequence $\mathbb{Z}$ everywhere on $\mathbb{R}$. Similarly, for $\mathcal{E}(\Lambda)$ to be a frame for $L^2[0,1]$, Λ needs to be asymptotically denser than the sequence $\mathbb{Z}$ everywhere on $\mathbb{R}$. It is remarkable that almost a full converse of these statements holds as well. These comparison results are most elegantly expressed with the concept of Beurling densities (other names such as Landau density or uniform density are sometimes attached to them). The lower Beurling density $D^-(\Lambda)$ of a sequence $\Lambda \subset \mathbb{R}$ is defined by

$$D^-(\Lambda) = \lim_{r\to\infty} \inf_{a\in\mathbb{R}} \frac{\#(\Lambda \cap (a-r, a+r))}{2r}.$$

The upper Beurling density $D^+(\Lambda)$ and their higher-dimensional analogues are defined analogously. As mentioned above, Beurling densities can be used to almost characterize Riesz sequences and frames of complex exponentials in $L^2[a,b]$, i.e., to almost answer question (Q) when S is an interval. The answer is provided by the following result of Kahane [6]. For simplicity, we state it in the case when Λ is uniformly discrete (i.e., $\inf |\lambda_i - \lambda_j| > 0$) and $S = [0,1]$.

Theorem 1. *Let $\Lambda \subset \mathbb{R}$ be a uniformly discrete sequence.*

(i) *If $D^-(\Lambda) > 1$, then $\mathcal{E}(\Lambda)$ is a frame for $L^2[0,1]$. If $D^-(\Lambda) < 1$, it is not.*
(ii) *If $D^+(\Lambda) < 1$, then $\mathcal{E}(\Lambda)$ is a Riesz sequence for $L^2[0,1]$. If $D^+(\Lambda) > 1$, it is not.*

In other words, for a system $\mathcal{E}(\Lambda)$ to be a frame in $L^2[0,1]$, it is necessary to have $D^-(\Lambda) \geq 1$, with $D^-(\Lambda) > 1$ being sufficient, and similarly for Riesz sequences. The borderline cases $D^-(\Lambda) = 1$ and $D^+(\Lambda) = 1$ are not covered by the previous result. In these cases both options can occur and additional analysis is required. This remaining open part of the problem was addressed in [15].

It should be noted that earlier Beurling gave a characterization, which is equivalent to the characterization of sampling and interpolation in the Bernstein space (a close relative of the Paley–Wiener space). Since it is not hard to derive the Paley–Wiener result from the one for Bernstein space, many people attribute the Kahane result to Beurling as well.

The first part of the book is devoted to proving Theorem 1. The full proof is given in Lectures 3 (frames) and 4 (Riesz sequences). The proof is based on the Beurling's original approach, but it is fully adapted to the Paley–Wiener (and the Bernstein) space. It should be noted that Beurling's approach was instrumental in Seip's characterization of sampling and interpolating sets in the Bergman and the Bargmann–Fock spaces [17, 18].

Simple counterexamples show that the sufficiency part of Theorem 1 fails (more on this below). Therefore, it is natural to ask (as Beurling himself did) whether the necessity part remains true in higher dimensions. The answer was given by Landau in his influential paper [8]. Using a completely new method, Landau proved the following general result. (Here and elsewhere we use $|S|$ to denote the Lebesgue measure of S.)

Theorem 2. *Let $S \subset \mathbb{R}^n$ be a bounded measurable set, and let $\Lambda \subset \mathbb{R}^n$ be a uniformly discrete sequence.*

(i) *If $\mathcal{E}(\Lambda)$ is a frame for $L^2(S)$, then $D^-(\Lambda) \geq |S|$.*
(ii) *If $\mathcal{E}(\Lambda)$ is a Riesz sequence for $L^2(S)$, then $D^+(\Lambda) \leq |S|$.*

Landau's method is based on spectral analysis of certain (depending on S) concentration operators. The advantage of his method is that it does not rely on complex analysis and is considerably less technical, compared to the Beurling's approach. Very recently, Olevskii jointly with Nitzan [11] found an even simpler proof of this important result. It is not surprising that the authors chose to present this proof (it is presented in the Lecture 5). Their proof also shows that even a weaker independence concept, such as that of uniform minimality, implies the same upper bound on the upper density $D^+(\Lambda)$. Recall that a sequence of vectors $\{e_n\}$ in a Hilbert space is said to be uniformly minimal if $\inf d_n > 0$, where

$$d_n := \operatorname{dist}(e_n, \operatorname{span}\{e_m, m \neq n\}).$$

If we only have $d_n > 0$ for all n (without the uniform bound), then we say that the sequence is minimal. In the intermediate case $d_n \geq Ce^{-n^\gamma}$ for some $C > 0$ and $0 < \gamma < 1$, the authors of the book proved the following slightly weaker upper bound,

$$D^*(\Lambda) := \limsup_{a\to\infty} \frac{\Lambda \cap (-a, a)}{2a} \leq |S|.$$

The upper density $D^*(\Lambda)$ cannot be replaced by the Beurling upper density $D^+(\Lambda)$. In the case when S is an interval, it was proved by the authors of the book that even minimality implies the same upper bound for $D^*(\Lambda)$. It is an open problem whether this continues to hold for more general sets S. These results are presented in Lecture 8.

The above-mentioned result of Kahane and Beurling (Theorem 1) gives an almost complete solution of (Q) in the one-dimensional case, when S is a (bounded) interval. Landau's follow-up result (Theorem 2) provides an extension of the necessity part for general bounded S of finite measure (the boundedness restriction was recently removed by Nitzan and Olevskii) and holds in any dimension. One may hope that the sufficiency part might hold for some S and Λ of special type (e.g., when S is a union of intervals or when Λ is a lattice). Unfortunately, it is not hard to construct examples that show that even in the case when S is a union of two intervals, the density conditions appearing in the above results are not sufficient either for sampling or for interpolation. In the view of these negative results, it is remarkable that there still do exist sequences Λ of very special type which are sampling sets for any $\mathcal{PW}_S$ with compact S of measure $|S| < D^-(\Lambda)$. Similarly, there also exist sets Λ which are interpolating for any $\mathcal{PW}_S$ with open S of measure $|S| < D^+(\Lambda)$. Such sets are called universal sampling/interpolation sets. The existence of such sets is far from obvious, and their existence was first proved by Olevskii and Ulanovski in [14]. Their proof is presented in the first part of Lecture 6. The universal sampling set Λ that they construct is a perturbation of $2\pi\mathbb{Z}$ and represents a universal interpolation set in the same time. Namely, what they do is construct a set Λ such that the corresponding exponential system $\mathcal{E}(\Lambda)$ is a Riesz basis for a certain class of sets Ω. The class of such sets Ω is sufficiently rich so that any compact set S is contained in some Ω, and any open set S contains some Ω of this class. This implies that Λ is both a universal sampling and a universal interpolation set. It is interesting to note that it is still not known whether universal sampling/interpolation sets of optimal density exist.

There is an alternative approach to the universal sampling problem based on Meyer's model sets. The construction of these sets is presented in the second part of Lecture 6. These interesting sets were first defined by Meyer much earlier in connection with other harmonic analysis questions. The fact that they represent universal sampling sets was observed only recently in [10]. We will briefly explain the way these sets are constructed. For simplicity (as done in the book), we restrict to the one-dimensional case. Denote by p_1 and p_2 the coordinate projections of $\mathbb{R}^2$. For a lattice $\Gamma \subset \mathbb{R}^2$ and a "window" Ω which is a finite union of closed intervals in $\mathbb{R}$, the Meyer model set is defined by

$$\Lambda(\Gamma, \Omega) := \{p_1(\gamma) : \gamma \in \Gamma, p_2(\gamma) \in \Omega\}.$$

Denote by Γ^* the dual lattice of Γ. The following duality result is crucial in establishing that some of these sets are universal sampling/interpolating: Let Ω and S be two windows (disjoint unions of closed intervals). If $\Lambda(\Gamma^*, S)$ is an interpolating set for $\mathcal{PW}_\Omega$, then $\Lambda(\Gamma, \Omega)$ is a sampling set for $\mathcal{PW}_S$. Using this duality result, it is not hard to show that for any $a > 0$ and any lattice (in a generic position), the model set $\Lambda(\Gamma, [-a, a])$ is a universal sampling set.

The topological restriction (requiring S to be only compact/open) in the definition of universal sampling/interpolating sets turns out to be essential. Namely, if one removes this restriction, such sets will cease to exist. More precisely, for any uniformly discrete set Λ one can find a bounded set S with arbitrary small measure such that Λ is not a sampling set for $\mathcal{PW}_S$. The proof of this result is presented in Lecture 7. The proof (due to Olevskii and Ulanovski) is a beautiful combination of analytic and combinatorial arguments. It is interesting to note that the combinatorial part depends crucially on the celebrated Szemerédi theorem on arithmetic progressions.

As mentioned above, the sufficiency part in Theorem 1 fails miserably in higher dimensions. It is therefore natural to ask whether there are some stronger conditions that would imply sampling/interpolation in $\mathcal{PW}_S$ in the higher-dimensional case. In the case of interpolation, it is very well known that the following generalization (due to Kahane) of the classical Ingham's theorem gives one such condition.

Theorem 3. *Let B be the unit ball in $\mathbb{R}^n$. If $\Lambda \subset \mathbb{R}^n$ is a sequence with a separation constant greater than* 1*, then Λ is an interpolating sequence for $\mathcal{PW}_B$.*

Can a result like this be obtained for more general sets S? A natural starting point is the case when S is a compact, symmetric convex body in $\mathbb{R}^n$, in which case the full higher-dimensional analog of the Paley–Wiener theorem holds. It is tempting to investigate if there exists some type of symmetry between S and the Ingham-type separation condition. One way to look for such a symmetry is to introduce another compact, symmetric convex body K that will play the role of a gauge (in the usual way) in the Ingham separation condition. Therefore, one could ask if there is a condition on S and K (preferably symmetric) which would imply the following generalization of Theorem 3: if $\Lambda \subset \mathbb{R}^n$ is a sequence such that

$$(\Lambda - \Lambda) \cap K = \{0\},$$

then Λ is an interpolating set for $\mathcal{PW}_S$. It is proved in Lecture 9 (using the Minkowski lattice) that any such pair of sets would have to satisfy $|S||K| > 1$. This condition, of course, brings to mind the following uncertainty principle: if a function $F \in L^2(\mathbb{R}^n)$ is supported on a set S and its Fourier transform is supported on a set K, then $|S||K| \geq 1$ (see, e.g., [3]). So it is natural to hope that some type of weakening of the support condition will give us a sufficient condition on S and K. There is one such condition: the pair of sets (S, K) admits ϵ-concentration if there exist $\epsilon > 0$ and $F \in L^2(\mathbb{R}^n)$ of norm 1 such that

$$\int_K |F(x)|^2 \, dx > 1 - \epsilon, \qquad \int_S |F(\xi)|^2 \, d\xi > 1 - \epsilon.$$

Now the question becomes whether there exists a simple metric condition on S and K that would imply ϵ-concentration for some $\epsilon > 0$. One conjecture is that $|S||K| > 1$ already implies such a concentration. Some results (for a special type of S and K) in favor of this conjecture are presented in the book, but despite the attention of some top analysts, this question (and many other questions surrounding it) still remains open.

The last part of the book is concerned with a slightly different topic. The central problem is the description of completeness spectra, i.e., the sets $\Lambda \subset \mathbb{R}$ for which there exists a weight $w \in L^1(\mathbb{R}), w > 0$ a.e. on $\mathbb{R}$, such that $\mathcal{E}(\Lambda)$ is complete in the weighted space $L^2(\mathbb{R}, w)$. Notice that here the frame property is replaced by the weaker spanning property, that of completeness. This is for a good reason, since it can be shown that no matter how we choose a sequence Λ we cannot find a weight w such that $\mathcal{E}(\Lambda)$ is a frame in $L^2(\mathbb{R}, w)$ (notice here that we require $w > 0$ a.e. on $\mathbb{R}$ which excludes indicator functions). As in many of the results discussed above (connected to question (Q) for sets S which are not intervals), the density of Λ is not the sole factor entering in the description of completeness spectra. To illustrate this, consider the following examples. It can be shown that any Λ that is sufficiently dense (e.g., with infinite upper density) is a completeness spectrum. On the other hand the sequence of integers $\mathbb{Z}$ is not. For a long time it was believed that the reason for this is that $\mathbb{Z}$ is not sufficiently dense. However, the following striking result of Olevskii [13] shows that the reason must be more intricate.

Theorem 4. *Any sequence $\Lambda = \{\lambda_n\}$ of the form $\lambda_n = n + \gamma_n$ with $0 < |\gamma_n| = o(1)$ as $|n| \to \infty$ is a completeness spectrum.*

This theorem shows that besides the density, the arithmetic of the set Λ plays a significant role in the completeness spectrum problem.

The completeness spectrum problem above can be easily reformulated in the following way. Describe those sets $\Lambda \subset \mathbb{R}$ for which there exists a generator $g \in L^2(\mathbb{R})$ such that the corresponding set of translates $\{g(x - \lambda) : \lambda \in \Lambda\}$ is complete in $L^2(\mathbb{R})$. This reformulation makes perfect sense in $L^p(\mathbb{R})$ for $p \neq 2$ as well. The $p \neq 2$ problems are considered in the last Lecture 12. Without going into details, we just point out that the case $p = 1$ has an especially elegant solution due to Bruna, Olevskii, and Ulanovski [1]: A discrete set Λ is a completeness spectrum for $L^1(\mathbb{R})$ if and only if it has infinite upper Beurling–Malliavin density.

We will end with a few general comments about the book. A big portion of the book has a monographic character, mainly containing results obtained by the authors and their collaborators in the last twenty years or so. These results are nicely supplemented with a quick introduction and a detailed discussion of two of the most important results in sampling theory. The book is remarkable in the way it manages to introduce the reader from the very basics to the current state of the art in the field in just 130 pages, doing this in a very readable and lucid way. The material is organized into twelve lectures, many of which can be read independently. Still, the book tells a story which can only be appreciated if the lectures are covered in the right order. The book is essentially self-contained and assumes only very basic knowledge in real, complex, and functional analysis. The text is supplemented by well-chosen exercises (many of which are given with hints) and some open problems. It is suitable for a one semester graduate topics course in Fourier analysis, but can equally well be used for self-study for those interested in a relatively quick introduction into this classical, but still very vibrant, area of mathematical analysis.

References

[1] J. Bruna, A. Olevskii, and A. Ulanovskii, *Completeness in $L^1(\mathbb{R})$ of discrete translates*, Rev. Mat. Iberoam. **22** (2006), no. 1, 1–16, DOI 10.4171/RMI/447. MR2267311

[2] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), 341–366, DOI 10.2307/1990760. MR0047179

[3] G. B. Folland and A. Sitaram, *The uncertainty principle: a mathematical survey*, J. Fourier Anal. Appl. **3** (1997), no. 3, 207–238, DOI 10.1007/BF02649110. MR1448337

[4] A. Iosevich, N. Katz, and T. Tao, *The Fuglede spectral conjecture holds for convex planar domains*, Math. Res. Lett. **10** (2003), no. 5-6, 559–569, DOI 10.4310/MRL.2003.v10.n5.a1. MR2024715

[5] M. Ĭ. Kadec′, *The exact value of the Paley-Wiener constant* (Russian), Dokl. Akad. Nauk SSSR **155** (1964), 1253–1254. MR0162088

[6] J.-P. Kahane, *Sur les fonctions moyenne-périodiques bornées* (French), Ann. Inst. Fourier, Grenoble **7** (1957), 293–314. MR0102702

[7] G. Kozma and S. Nitzan, *Combining Riesz bases*, Invent. Math. **199** (2015), no. 1, 267–285, DOI 10.1007/s00222-014-0522-3. MR3294962

[8] H. J. Landau, *Necessary density conditions for sampling and interpolation of certain entire functions*, Acta Math. **117** (1967), 37–52, DOI 10.1007/BF02395039. MR0222554

[9] A. W. Marcus, D. A. Spielman, and N. Srivastava, *Interlacing families II: Mixed characteristic polynomials and the Kadison-Singer problem*, Ann. of Math. (2) **182** (2015), no. 1, 327–350, DOI 10.4007/annals.2015.182.1.8. MR3374963

[10] B. Matei and Y. Meyer, *Simple quasicrystals are sets of stable sampling*, Complex Var. Elliptic Equ. **55** (2010), no. 8-10, 947–964, DOI 10.1080/17476930903394689. MR2674875

[11] S. Nitzan and A. Olevskii, *Revisiting Landau's density theorems for Paley-Wiener spaces* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **350** (2012), no. 9-10, 509–512, DOI 10.1016/j.crma.2012.05.003. MR2929058

[12] S. Nitzan, A. Olevskii, and A. Ulanovskii, *Exponential frames on unbounded sets*, Proc. Amer. Math. Soc. **144** (2016), no. 1, 109–118, DOI 10.1090/proc/12868. MR3415581

[13] A. Olevskiĭ, *Completeness in $L^2(\mathbf{R})$ of almost integer translates* (English, with English and French summaries), C. R. Acad. Sci. Paris Sér. I Math. **324** (1997), no. 9, 987–991, DOI 10.1016/S0764-4442(97)87873-1. MR1451238

[14] A. Olevskiĭ and A. Ulanovskii, *Universal sampling and interpolation of band-limited signals*, Geom. Funct. Anal. **18** (2008), no. 3, 1029–1052, DOI 10.1007/s00039-008-0674-7. MR2439002

[15] J. Ortega-Cerdà and K. Seip, *Fourier frames*, Ann. of Math. (2) **155** (2002), no. 3, 789–806, DOI 10.2307/3062132. MR1923965

[16] B. S. Pavlov, *The basis property of a system of exponentials and the condition of Muckenhoupt* (Russian), Dokl. Akad. Nauk SSSR **247** (1979), no. 1, 37–40. MR545940

[17] K. Seip, *Density theorems for sampling and interpolation in the Bargmann-Fock space*, Bull. Amer. Math. Soc. (N.S.) **26** (1992), no. 2, 322–328, DOI 10.1090/S0273-0979-1992-00290-2. MR1136138

[18] K. Seip, *Beurling type density theorems in the unit disk*, Invent. Math. **113** (1993), no. 1, 21–39, DOI 10.1007/BF01244300. MR1223222

[19] T. Tao, *Fuglede's conjecture is false in 5 and higher dimensions*, Math. Res. Lett. **11** (2004), no. 2-3, 251–258, DOI 10.4310/MRL.2004.v11.n2.a8. MR2067470

Mishko Mitkovski

Department of Mathematical Sciences, Clemson University, Clemson, South Carolina

E-mail address: mmitkov@clemson.edu