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Teichmüller theory and applications to geometry, topology, and dynamics. Volume 1: Teichmüller theory, by J. H. Hubbard, With contributions by Adrien Douady, William Dunbar, Roland Roeder, Sylvain Bonnot, David Brown, Allen Hatcher, Chris Hruska, and Sudeb Mitra, With forewords by William Thurston and Clifford Earle, Matrix Editions, Ithaca, NY, 2006, xx+459 pp., ISBN 978-0-9715766-2-9

Teichmüller theory and applications to geometry, topology, and dynamics. Volume 2: Surface homeomorphisms and rational functions, by J. H. Hubbard, Matrix Editions, Ithaca, NY, 2016, x+262 pp., ISBN 978-1-943863-00-6

1. Teichmüller theory

I will start by recalling a few basic facts about Teichmüller theory and, in particular, its history.

Originally, Teichmüller theory was the study of moduli of complex structures on surfaces. Its roots lie in the work of Riemann and, more particularly, in his doctoral dissertation, *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse* (Foundations for a general theory of functions of a complex variable) [35, 1851] and his memoir *Theorie der Abel'schen Functionen* (The theory of Abelian functions) [36, 1857]. Riemann introduced there, for the first time, the notion of Riemann surface, together with a variety of methods to study it, from topology to complex analysis and potential theory. He defined an equivalence relation (birational equivalence) between such surfaces, and he stated that for a closed surface of genus g, these equivalence classes have $3g-3$ complex "moduli". A question which remained at the center of an intense activity during nearly 80 years after Riemann's dissertation was to give a precise meaning to this statement. It might be useful to recall in this respect that at the time of Riemann, there was no notion of complex manifold and, in fact, no notion of manifold at all. Topology was at its birth, and the notions of topological space and metric space were still not formulated. Thus, in the form he gave it, Riemann's statement was essentially a parameter count and could hardly be understood as a result on dimension.

With his discovery of Riemann surfaces, Riemann was the first to introduce topology in complex analysis. He associated to every multivalued analytic function a new domain of definition, a Riemann surface, on which this function becomes single valued. He also reduced the study of holomorphic functions of a variable $z = x + iy$ to that of (real) harmonic functions of x and y, introducing potential theory in the study of analytic functions.

Among the multitude of major mathematicians who, during the few decades that followed Riemann's premature death, dedicated a substantial amount of effort to questions raised by his work on Riemann surfaces and their moduli, we mention the names of Weierstrass, Schwarz, Clebsch, Neumann, Fricke, Koebe, Hilbert, Brouwer, Weyl, Torelli, Siegel, and there are many others. But two names have to be singled out, Klein and Poincaré, who introduced for the first time hyperbolic

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geometry in the study of Riemann surfaces, considering spaces of Riemann surfaces in the form of fundamental polygons of isometry groups acting on the hyperbolic plane. They did this in their struggle to prove the uniformization theorem, another question raised by Riemann. Their attempt was based on the so-called "method of continuity" which, because of the lack of the right topological setting, presented several shortcomings. In particular, they were unable to deal with the singular points of the moduli spaces they considered (and they knew that these singular points existed).

The next name that has to be highlighted is that of Oswald Teichmüller, who wrote a series of papers, between 1939 and 1943, which transformed the field. I will mention the results obtained in two of his papers, [38] and [40].[1]

In the paper [38], Teichmüller provided the first solution to Riemann's moduli problem by equipping the moduli space of Riemann surfaces with a topology and a metric, and obtaining a formula for its dimension. This formula coincides with Riemann's count and it gave an explanation to this count as the dimension of some geometric structure on that space.

In the same paper, Teichmüller introduced a notion of marking of Riemann surfaces by elements of the mapping class group (and another one by elements of the fundamental group of the surface), and he considered the space of marked Riemann surfaces, as a space of mappings from a fixed topological surface to a varying Riemann surface. The marking allowed him to overcome the problems caused by the singularities of moduli space. He thus obtained a nonsingular covering of Riemann's moduli space on which the mapping class group acts properly discontinuously. This is the space which carries his name. Teichmüller showed that Riemann's moduli space is a quotient of Teichmüller space by this action. He developed a theory of quasiconformal mappings as a tool for understanding *conformal* mappings and moduli of Riemann surfaces, not only as a theory of *nonconformal* mappings, and he introduced quadratic differentials, their trajectory structure and the singular flat metric they induce on the surface as another essential tool in the study of moduli. He defined the so-called Teichmüller metric, proved that this metric is Finslerian, studied its infinitesimal norm and its geodesics, proved that Teichmüller space is homeomorphic to a ball, and computed its dimension (this result was completed in [39]).

Teichmüller made a thorough investigation of the infinitesimal theory of quasiconformal mappings and the partial differential equations that they satisfy. He translated problems concerning conformal structures on surfaces into problems concerning Riemannian metrics and inaugurated the use of such metrics to solve problems in the theory of conformal mappings. He established an identification of the tangent space to Teichmüller space at every point with a space of equivalence classes of Beltrami differentials on a Riemann surface representing that point. This brought the idea of an almost-complex structure on the tangent space to Teichmüller space which was at the basis of the theory of variation of higher-dimensional complex manifolds that was developed later on by Kodaira and Spencer. Teichmüller also identified the tangent space at every point of Teichmüller space with the topological

[1] These two papers, together with several others, are now largely accessible, since they are translated into English, with notes and commentaries, in several volumes of the *Handbook of Teichmüller theory* [12].

dual of the space of holomorphic quadratic differentials on a Riemann surface representing that point. He introduced the so-called Teichmüller discs (which he called "complex geodesics") as isometric embeddings of the hyperbolic plane into Teichmüller space and obtained inequalities involving the hyperbolic length of closed geodesics and the quasiconformal dilatation of a map between hyperbolic surfaces. Equipped with the right topological notions, he made a correct use of the method of continuity into which Poincaré, Klein, and others had bumped over a long period of time. In particular, he used this method in a correct setting, that is, for maps between objects which he showed were manifolds of the same dimension. He raised the question of whether there is a Hermitian metric on Teichmüller space. (Such a metric, the so-called Weil–Petersson metric, was introduced later on by André Weil.) He studied convexity properties of the Teichmüller metric and addressed the question of studying totally geodesic subspaces of Teichmüller space. He formulated the so-called Nielsen realization problem as a question of finding a fixed point of the action of a finite subgroup of the mapping class group on Teichmüller space, and he hinted to the use of convexity properties of the Teichmüller metric for its solution. He inaugurated the study of *nonreduced Teichmüller spaces*, a theory where each point on the boundary of a surface, or of a union of arcs on this boundary, is considered as a distinguished point. He defined a new metric on a Riemann surface, where the distance between two points is the logarithm of the least quasiconformal constant (or the "dilatation quotient") of a self-map of the surface sending one point to the other.

There are other results in the paper [38] but the list we gave should be sufficient to explain why the theory carries the name Teichmüller theory. We note that in this paper, Teichmüller worked in the most general setting of surfaces of finite type: orientable or not, with or without boundary, and with or without distinguished points in the interior or on the boundary.

Teichmüller space is also a complex manifold. This result originates in another paper by Teichmüller, [40]. In fact, the paper contains an even more interesting result, which we state precisely because the author of the two books under review dwells on it in Volume 2. This is the existence and uniqueness of the so-called universal Teichmüller curve. Teichmüller's statement (using his own wording) is the following:

> There exists an essentially unique globally analytic family of topologically determined Riemann surfaces $\underline{\mathfrak{H}}[\mathfrak{c}]$, where $\mathfrak{c}$ runs over a τ-dimensional complex analytic manifold $\mathfrak{C}$ such that for any topologically determined Riemann surface $\underline{\mathfrak{H}}$ of genus g there is one and only one $\mathfrak{c}$ such that the Riemann surface $\underline{\mathfrak{H}}$ is conformally equivalent to an $\underline{\mathfrak{H}}[\mathfrak{c}]$ and such that the family $\underline{\mathfrak{H}}[\mathfrak{c}]$ satisfies the following universal property: If $\underline{\mathfrak{H}}[\mathfrak{p}]$ is any globally analytic family of Riemann surfaces with base $\mathfrak{B}$, there is a holomorphic map $f : \mathfrak{B} \to \mathfrak{C}$ such that the family $\underline{\mathfrak{H}}[\mathfrak{p}]$ is the pullback by f of the family $\underline{\mathfrak{H}}[\mathfrak{c}]$.

In this statement, $\mathfrak{C}$ is Teichmüller space, $\underline{\mathfrak{H}}[\mathfrak{c}]$ is the Teichmüller universal curve (a fiber bundle over $\mathfrak{C}$, $\underline{\mathfrak{H}}[\mathfrak{c}] \to \mathfrak{C}$, where the fiber above each point in $\mathfrak{C}$ is a marked Riemann surface representing this point), the expression "topologically determined" means "marked", and τ is the complex dimension, that is, Riemann's "number of moduli" ($\tau = 0$ if $g = 0$, 1 is $g = 1$ and $3(g-1)$ if $g > 1$).

Stated in other words, the theorem says that any analytic family $\underline{\mathfrak{H}}[\mathfrak{p}] \to \mathfrak{B}$ of Riemann surfaces is obtained from the universal Teichmüller curve $\underline{\mathfrak{H}}[\mathfrak{c}] \to \mathfrak{C}$ as the pullback by some holomorphic map $f : \mathfrak{B} \to \mathfrak{C}$. Teichmüller also stated the uniqueness of the universal curve up to the action of the mapping class group. This result shows at the same time that there is a complex structure on Teichmüller space and that this structure is canonical (it is defined by a universal property). At the same time, Teichmüller stated a theorem saying that the automorphism group of this universal curve coincides with the action of the mapping class group on it, a result that inaugurated a series of several later rigidity results of the same flavor, the next one being Royden's result stating that the automorphism group of the complex structure of the Teichmüller space of a closed surface coincides with the action of the mapping class group on that space.

As a matter of fact, according to Remmert [34], the first time in the history of mathematics where we find the expression "*komplex analytische Mannigfaltigkeit*" (complex analytic manifold) is in this paper by Teichmüller.

Grothendieck took up this idea of Teichmüller and developed it using a new language of algebraic geometry that he invented. He wrote a series of ten papers [11] whose aim was to prove the following version of Teichmüller's theorem:

> *There exists an analytic space T and a $\mathcal{P}$-algebraic curve V above T which are universal in the following sense: For every $\mathcal{P}$-algebraic curve X above an analytic space S, there exists a unique analytic morphism g from S to T such that X (together with its $\mathcal{P}$-structure) is isomorphic to the pullback of V/T by g.*

Here, the expression "algebraic curve over an analytic space" means a family of algebraic curves which depends analytically on a complex parameter. The analytic space T is Teichmüller space, and the $\mathcal{P}$-algebraic curve V above T is the universal curve. The terms "$\mathcal{P}$-algebraic" refers to a rigidification of the curves using a so-called Teichmüller rigidifying functor.

Besides Grothendieck, and during the first post-Teichmüller period, one has also to mention the names of Ahlfors and Bers, who dedicated a large amount of effort in order to give an analytic proof of the so-called Teichmüller existence and uniqueness result on extremal quasiconformal mappings between Riemann surfaces. This is developed in Teichmüller's papers [38] and [39]. The difficult part is existence, and Bers struggled during almost three decades to find a proof of this property that would satisfy him. As a matter of fact, one of the very last papers that Bers wrote, [4], concerns this problem (the paper is entitled *On Teichmüller's proof of Teichmüller's theorem*). At the same time, Ahlfors and Bers used the quasiconformal theory that Teichmüller developed in [38] (rather than his work in the paper [40] mentioned above) to introduce a complex structure on Teichmüller space. Talking about Bers's contribution to Teichmüller theory, one may mention his work on simultaneous uniformization, the parametrization of the space of quasi-Fuchsian manifolds by a product of Teichmüller spaces, and his embedding of Teichmüller space as a bounded domain in $\mathbb{C}^n$; cf. [2].

As other prominent mathematicians who worked on Teichmüller theory during the same period, together with Grothendieck, Ahlfors and Bers, one may mention the names of Weil, Mumford, and Deligne. But the major figure who transformed the subject after Teichmüller is Thurston. In his hands, Teichmüller space plays an essential role in the theory he developed for the geometrization of 3-manifolds, in particular for what concerns Kleinian groups and their degeneration. Teichmüller space is at the center of the inductive step of Thurston's proof on the existence of hyperbolic structures on atoriodal Haken 3-manifolds, where a hyperbolic structure is constructed by gluing the ends of an open 3-manifold. Indeed, this gluing is performed using quasiconformal deformations, and it is based on the existence of a fixed point of a map acting on Teichmüller space; cf. Thurston's survey [41], and the book [17] and the papers [15] and [18] by McMullen. The latter completed several steps of Thurston's proof.

Let us mention some other contributions of Thurston that are related to Teichmüller theory.

A major result in the classification theory of hyperbolic 3-manifolds is the so-called ending lamination theorem, stating that a hyperbolic 3-manifold with finitely generated fundamental group is determined by its topology together with some end invariants which lie on some surfaces which are boundary components of this manifold. The theorem was conjectured by Thurston in 1982 and it was proved in several steps by Brock, Canary, and Minsky, cf. [6] and [21]. The proof uses in an essential way the large-scale geometry of Teichmüller space. Thurston conjectured a relation (which was proved later by Brock in [5]) between the hyperbolic volume of the convex core of a quasi-Fuchsian manifold and the Weil–Petersson distance in Teichmüller space between the two components of the conformal boundary of this manifold. He also introduced an iteration procedure on Teichmüller space to construct rational maps of the sphere satisfying a given combinatorial pattern. This was also thoroughly studied by McMullen; see his survey [19]. Thurston introduced the idea of the Teichmüller space of a holomorphic dynamical system, which is equipped with a metric, a complex structure, and a properly discontinuous action of a modular group, with a theory that parallels the deformation theory of Riemann surfaces. McMullen proved several conjectures of Thurston on various actions on Teichmüller spaces related to Kleinian groups and holomorphic dynamics. In the paper [15], he studies several contracting mappings between Teichmüller spaces (for the Teichmüller metric) related to this subject. See also the survey [16]. These and many other ideas originating in Thurston's work made Teichmüller space a central object in low-dimensional topology, geometric group theory, complex dynamics, and the study of representation spaces.

In the tradition of Thurston, and besides McMullen whom we already mentioned, the name of Sullivan also comes to the forefront, especially for what concerns Kleinian groups and holomorphic dynamical systems (see, e.g., the survey [20]).

Finally, regarding Thurston, let me also mention that a weak metric on Teichmüller space which he introduced in 1985 (and which is called now Thurston's metric), has come, after several decades, to the forefront of several research programs. In particular, it brings in an essential way metric geometry into Teichmüller theory, and in particular the metric methods of A. D. Alexandrov and Busemann.

Today, the expression "Teichmüller theory" has a very broad meaning, namely, the study of geometric structures on surfaces with applications in several fields, including low-dimensional topology, hyperbolic geometry, Kleinian groups, holomorphic dynamics, algebraic topology, representations of discrete groups in Lie groups, symplectic geometry, cluster algebras, mapping class groups, hyperbolic groups, actions on $\mathbb{R}$-trees, the outer automorphism groups of free groups, Thompson groups, *dessins d'enfants*, string theory, topological field theories, quantum topology, de Sitter and AdS geometry, and new directions are pointing toward biology and holography among others. The interactions between Teichmüller theory and all these fields originate in the fact that Teichmüller space can be approached from several points of view, namely, as a space of equivalence classes of marked hyperbolic metrics, as a space of equivalence classes of complex algebraic curves, as a space of equivalence classes of marked conformal structures, as a space of equivalence classes of representations of the fundamental group of a surface into the Lie group $\mathrm{PSL}(2,\mathbb{R})$, and also in a purely combinatorial way. Besides the Teichmüller, the Thurston, and the Weil–Petersson metrics that we mentioned, Teichmüller space is endowed with various other interesting metrics (Bergman, length-spectrum, Funk–Weil–Petersson, Carathéodory, McMullen, Sun–Liu–Yau, etc.). Besides its natural complex structure, the space carries a symplectic structure, a real analytic structure, an algebraic structure, combinatorial cell-decompositions, and various boundary structures, all of them equipped with natural actions by the mapping class group. There is a decorated Teichmüller theory, a discrete Teichmüller theory and a combinatorial theory, a universal Teichmüller space, relations with the theory of harmonic maps, and higher-dimensional generalizations.

There are also interesting geodesic and horocyclic flows on the quotient Riemann moduli space, a quantization theory of its Poisson structure, and the list goes on and on. Over the recent years, the subject of Teichmüller theory has continued to grow at an exceptional rate, and new ideas and new connections between all these domains have emerged.

The relations with dynamics, besides the important one with holomorphic dynamics that we mentioned, include the study of geodesic flows on various bundles over Teichmüller space, e.g., the one defined through the action of the subgroup of $\mathrm{SL}(2,\mathbb{R})$ consisting of matrices of the form $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ on the bundle $\mathcal{Q}$ of quadratic differentials over Teichmüller space, which as we already mentioned, can be seen as its cotangent bundle. This flow leaves invariant a natural stratification of $\mathcal{Q}$ defined by fixing the number and type of singular points of the quadratic differentials. It preserves a natural volume form on each stratum, is invariant by the action of the mapping class group, and it induces a flow on the quotient moduli spaces of quadratic differentials. There is also a horocyclic flow. The study of these flows has important consequences for the distribution of closed geodesics on the surface, and it involves arithmetical problems. In this respect, one should mention the name of Maryam Mirzakhani, who dedicated a large part of her work to this subject, and more generally to the dynamics and the ergodic theory of the Teichmüller flows. See in particular [23], [24], [25], and [1]. Connections with number theory were also discovered through the number fields of dilatations of pseudo-Anosov homeomorphisms, which are related to the length spectrum of the Teichmüller geodesic flow. Again, the name of McMullen should be mentioned.

Regarding the dynamical aspects, one should also mention the name of Sullivan, and at least his eight years of continuous effort (1982–90) on investigating the Teichmüller space of compact Riemann surface laminations, the theory he developed to prove the rigidity of the Feingenbaum Cantor set. This Teichmüller space is a separable Banach manifold. (One should note here that all the other known Teichmüller spaces are either finite-dimensional or not separable.) See [37], where Sullivan introduces the hyperbolic solenoid as the inverse limit of the directed system of finite-sheeted pointed covers of an arbitrary closed oriented surface of negative Euler characteristic. We mention in this respect that Sullivan made the relation between a property of the modular group of the solenoid acting on its Teichmüller space and the Ehrenpreis conjecture (which was not solved at that time).

Penner exploited the combinatorial structure of Teichmüller space and applied it far and wide in mathematics, physics, and biology (see the recent paper [33]).

All this shows that Teichmüller theory is a living and growing subject, involving many fundamental aspects of mathematics. The multivolume *Handbook of Teichmüller theory* [12] contains survey articles on all these connections. This project has involved up to now more than 100 authors, but there is still a need for other good surveys and comprehensive monographs. This brings us to Hubbard's two volumes.

2. On the content of the volumes under review

Volume I is concerned with one aspect of Teichmüller theory, namely, the classical theory, in part from Thurston's point of view. There are roughly three parts in the volume, in the following order: geometric, analytical, and algebraic. Including these three points of view on Teichmüller theory is useful for the reader who will get a feeling of the broadness of the subject. The specific topics include uniformization (Chapter 1); plane hyperbolic geometry (Chapter 2); hyperbolic structures on Riemann surfaces, including pair of pants decompositions and the collar lemma, and Fuchsian groups with their fundamental domains and limit sets (Chapter 3); the quasiconformal theory (Chapter 4); the Douady–Earle extension and holomorphic motions (Chapter 5); the analytic structure of Teichmüller space (Chapter 6); and a last chapter called "The geometry of finite-dimensional Teichmüller spaces", which contains more advanced material, including Teichmüller's result stating the Teichmüller space is contractible, the Mumford compactness theorem, Royden's theorem on the automorphisms of the complex structure, the Teichmüller universal curve, Fenchel–Nielsen coordinates, the Weil–Petersson metric, and Wolpert's theorem expressing the symplectic structure of Teichmüller space in terms of the Fenchel–Nielsen coordinates. The volume ends with appendices on various topics and of different degrees of difficulty, ranging from a discussion of partitions of unity, Dehn twists, and the Riemann–Hurwitz formula to the Dolbeault–Grothendieck lemma, the Cartan–Serre theorem in sheaf cohomology, Serre duality, and the Hirzebruch–Riemann–Roch theorem. Of course, for the advanced material, only sketches of proofs are given. The exposition becomes more and more algebraic, and in the last part of the book it ends up being far away from Thurston's point of view, with the advantage of adding a new perspective to the latter.

The geometric part of the Volume 1 (Chapters 1–3) constitutes a comprehensive introduction to the geometry of surfaces, and it may be used as the basis of a course on the subject. Except for what concerns uniformization, it may be considered as a detailed exposition of material contained in Thurston's book [41]. The second part, the analytic theory, consists of a lucid and useful exposition of works done in the tradition of Ahlfors and Bers.

Volume II is centered around two theorems of Thurston. The first one is his classification theorem of the elements of mapping class groups of surfaces of finite type. There are many good expositions of this theorem, the best one (from the reviewer's point of view) being still the one in [8], which follows Thurston's outline and which was published later [42]. This theorem, in its original form, says the following:

Any homeomorphism f_0 of a surface S of finite type is isotopic to a homeomorphism f which satisfies one of the following properties:

(1) *f fixes an element of Teichmüller space, and has finite order;*
(2) *f is reducible, that is, it fixes the homotopy class of multicurve (a system of disjoint simple closed curves in distinct homotopy classes);*
(3) *f is pseudo-Anosov, that is, there exists $K > 1$ and a pair of measured foliations F^s and F^u such that $f(F^s) = (1/K)F^s$ and $f(F^u) = KF^u$.*

In his exposition of this theorem, Hubbard follows Bers's proof in [3], and he uses a definition of a pseudo-Anosov mapping f that is equivalent to, but more analytically oriented, than Thurston's. With this definition, case (3) above is replaced by the following statement: There exists an element $\phi : S \to X$ of the Teichmüller space of S, a holomorphic quadratic differential Q on X and $K > 1$ such that $\phi \circ f \circ \phi^{-1}$ is a Teichmüller mapping $(X, q) \to (X, q/K)$.

The second result of Thurston presented in Volume 2 is his characterization of rational maps among branched coverings of the Riemann sphere which have finite post-critical set (forward image of the critical points). Thurston made a major contribution to the study of these maps. The condition on the post-critical set allowed him to associate to such a map an action on the Teichmüller space of an orbifold whose underlying space is the sphere and whose orbifold points are defined using this post-critical set. There is also an action on the space of simple closed curves on that orbifold, a notion of stable multicurve, and the induced action on each stable multicurve is linear. The theorem states that (provided the orbifold is hyperbolic) the branched covering map is equivalent to a rational function if and only if the eigenvalues of the action on stable multicurves are all < 1. The equivalence relation between maps is a weak topological equivalence relative to the orbifold points: two post-critically finite branched coverings f and g are equivalent if there exist homemorphisms ϕ and ϕ' of the sphere which coincide on the post-critical set and which are isotopic relative to this set, such that $g \circ \phi' = \phi \circ f$.

With such a result, there is a kind of return to Riemann, who addressed explicitly the question of when a topological ramified covering of the sphere is equivalent to a meromorphic map. His results on this problem are part of the so-called Riemann existence theorem.

There are strong relations between the two theorems of Thurston: each of them provides canonical representatives of maps between surfaces, they both involve actions on sets of simple closed curves, and their proofs are based on actions on Teichmüller space. In particular, the proof of the theorem on rational functions is obtained, in the tradition of several of Thurston's results, as a fixed point theorem of a self-map of Teichmüller space. The proof that Hubbard presents is Thurston's proof that the latter gave in various versions of his lecture notes [43] and in several lectures on the theorem.[2] The one written by Douady and Hubbard in [7] also follows Thurston's. Part of Thurston's notes on the dynamics of iterations of rational maps is edited in [42].

Besides these two theorems, Volume 2 contains an introduction to the dynamics of polynomial maps as well as several appendices, some of which are on the

[2] At some point, Thurston's notes were announced as to appear in the CBMS Regional Conference Series of the AMS.

topology of surfaces, complementing the chapter on Thurston's classification of homeomorphisms, and others on dynamics, complementing Thurston's result on the characterization of rational maps.

Hubbard's two volumes constitute a valuable addition to our mathematical literature. They contain detailed proofs of results in this important field and they will certainly contribute to the training of the new generation of geometers.

3. On attributing results, and on bibliographical references

I would like to point out a few omissions in the two volumes under review, essentially for what concerns references. It is good, in a book, to give at least a minimum of original references for the important results. This is essentially a way of giving credit to those who deserve it, a tradition to which mathematicians are attached, and the fact that this is taken care of in a book is usually reflected in the list of bibliographical references. These references are also a useful tool for the readers who may want to check the original ideas or get more information.

The first omission concerns Teichmüller. A reference to some of his papers would have been desirable, and the two theorems that are attributed to him, the contractibility of Teichmüller space and the existence of uniqueness of quasiconformal extremal maps, are not his only contributions to the material discussed in these books. We already talked about his paper [40] concerning the complex structure, which also contains his result on the universal property of the Teichmüller universal curve. On p. 262 of Volume 1 of the books under review, we find a historical remark on the complex analytic structure of Teichmüller space, where the author essentially says that Ahlfors, Rauch, Grothendieck, Earle, Eells, and Kuranishi contributed to this topic. He also writes, on p. 162, that "the complex analytic structure of Teichmüller space is a highly nontrivial result, with a rich and contentious history, involving Ahlfors, Rauch, Grothendieck, Bers, and many others." This is correct, except that the name of the initiator of the theory is missing. Incidentally, let me mention that the exposition made in Chapter 6 of this volume concerning Grothendieck's work on the universal curve could have been more detailed. For instance (I refer to the notation used in the result on this curve that I stated above), there is no statement or hint to the fact that an arbitrary analytic family of Riemann surfaces is isomorphic to a pullback of the family $\underline{\mathfrak{H}}[\mathfrak{c}]$, which is an important element of the theory, and which makes the family "universal". A universal property is stated (p. 279) in terms of an equivalence between two functors, with a map in each direction, but it is not proved that these two maps are inverses to each other.

Another author whose work could have been better quoted is Jakob Nielsen. We read, in a note on p. 1 of Volume 2:

> Apparently, Jakob Nielsen has some claim to having proved the result long before Thurston. However, I have spoken with the people who know Nielsen's work best, and they say he never made any definition similar to "pseudo-Anosov." Without it, no classification theorem seems possible.

In fact, several authors showed that Thurston's classification can be easily deduced from Nielsen's papers [26], [27], [28], and [30]. These authors include J. Gilman in her two articles [9] and [10], A. Miller in his article [22], and Thurston himself, in his article [13] with Handel. In particular, Miller showed that the measured foliations that are invariant by the pseudo-Anosov maps correspond exactly to the boundaries of the so-called Nielsen principal regions. Another remark that should be corrected is made on p. 26 of the same volume: "We have proved along the way the following rather surprising conjecture, known as the *Nielsen conjecture*: if $f^{\circ k}$ is homotopic to the identity for some k, then f is homotopic to a map of order k." In fact, this is not Nielsen's conjecture, but a theorem proved by Nielsen in his paper [29] (another proof was given later on, in a more general context, by H. Zieschang). There was a question, known as "Nielsen's realization problem", which was later referred to as a conjecture. The question was whether an arbitrary finite subgroup of the mapping class group can be realized as a finite group of homeomorphisms of the surface. It was answered positively by S. Kerckhoff. It concerned arbitrary finite subgroups, and not cyclic elements, of the mapping class group.

Anosov's 1967 paper on the geodesic flows on Riemannian manifold of negative curvature is mentioned in the bibliography for the classification of the mapping classes of the torus. Despite the name "Anosov" given to hyperbolic toral automorphisms, Anosov did not study these mappings in his monograph. The study of the mapping class group of the torus had been carried out long ago by Dehn and Nielsen. (In fact, Nielsen studied these maps in the last chapter of his doctoral dissertation.)

The names of Dehn and Nielsen are also missing in Appendix C4 which concerns the fact that the mapping class group of a closed orientable surface is naturally isomorphic to the outer automorphism group of its fundamental group. Another missing reference regarding homeomorphisms of surfaces concerns the theorem proved in Appendix C3, stating that for surface homeomorphisms homotopy implies isotopy, which should be attributed R. Baer.

Regarding the modern literature, and whereas several papers with a somehow limited interest are quoted, the references to Thurston and to McMullen's papers are completely absent from the bibliography of Volume 2. Levy cycles are mentioned, with no hint at all to their origin. A mention of the thesis of Silvio Levy [14] where they were introduced would have been welcome. As a matter of fact, Levy's thesis is probably the first place where Thurston's theorem on the characterization of rational maps was used. To give another example, Penner's name is completely absent from the bibliography of this volume, whereas it should be there, at least for his lucid exposition, with Harer, of train tracks [32], for his general construction of pseudo-Anosov diffeomorphisms [31], his algorithms to find them, and for his work on the dilatation factors of these maps, a topic which is discussed in this book.

But this is not the right note to end on. Hubbard has written a beautiful set of books. The main advantage of his approach over all the existing monographs on Teichmüller theory is that it makes a synthesis between geometry, analysis, algebra, and dynamics. We all look forward to the sequel.

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