BOOK REVIEWS

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Tensor categories, by Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik, Mathematical Surveys and Monographs, Vol. 205, American Mathematical Society, Providence, RI, 2015, xvi+343 pp., ISBN 978-1-4704-2024-6, US$65.00

> MAIN PRINCIPLE OF
> CATEGORY THEORY:
> In any category, it is unnatural and undesirable to speak about equality of two objects.
>
> Kapranov and Voevodsky [11]

Once affectionately disparaged as "general abstract nonsense" (see [13]), category theory has become a highly active and broadly applied field of study. A major impetus for this development is the ubiquity of tensor categories in the distinct yet deeply connected fields of representation theory, von Neumann algebras, quantum topology, mathematical physics, and condensed matter physics. We clearly see the thread of tensor categories woven through the work of Drinfeld, Jones, and Witten in the work cited for their 1990 Fields Medals [1]. Indeed, quantum groups, link invariants, and topological quantum field theories (TQFTs) occupy three sides of the same coin. This book is based on the lecture notes (by the same authors) for a course taught at MIT in 2009, and it gives a cohesive and rigorous axiomatic development of the theory of tensor categories.[1]

Before providing some details on the role of tensor categories in various fields, let me motivate the categorical concepts with two related perspectives. The first, more pedestrian, approach is to consider a particular kind of category whose structures and properties you admire and axiomatize them. The second idea is to take mathematical concepts built on sets and define categorical versions of them, i.e., categorification.

Fix a group G and a field $\mathbb{F}$, and denote by $\mathrm{Rep}(G)$ the category of finite-dimensional $\mathbb{F}$-representations of G. Here the objects are pairs (ρ, V), where $\rho : G \to \mathrm{GL}(V)$ is a homomorphism and the morphisms sets $\mathrm{Hom}_G(V_1, V_2)$ are linear maps $T \in \mathrm{Hom}(V_1, V_2)$ such that $T\rho_1(g) = \rho_2(g)T$ for all $g \in G$. A careful review

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[1]The term "tensor category" does not have a consistent definition in the literature; we will use the definition found in the book under review.

of representation theory reveals the following facts (we suppress the corresponding homomorphisms, as is customary) for any $V, W \in \mathrm{Rep}(G)$:

(1) $\mathrm{Hom}_G(V, W)$ is a finite-dimensional $\mathbb{F}$-vector space, and
(2) $V \oplus W \in \mathrm{Rep}(G)$.
(3) $V \otimes W \in \mathrm{Rep}(G)$, and
(4) $\mathbb{F} \cong \mathbf{1} \in \mathrm{Rep}(G)$, where each g acts by 1, and $V \otimes \mathbf{1} \cong \mathbf{1} \otimes V \cong V$.
(5) $V^* = \mathrm{Hom}(V, \mathbb{F}) \in \mathrm{Rep}(G)$, and the maps $ev : f \otimes v \mapsto f(v)$ and $coev : a \mapsto \sum_i a(v_i \otimes f_i)$ where $\{v_i\}, \{f_i\}$ are dual bases are in $\mathrm{Hom}_G(V^* \otimes V, \mathbf{1})$ and $\mathrm{Hom}_G(\mathbf{1}, V \otimes V^*)$, respectively.

These structures satisfy various compatibility and coherence constraints. For example, $\otimes$ satisfies an associativity constraint: for any $V, W, U \in \mathrm{Rep}(G)$ the following linear map is an isomorphism:

$$\alpha_{V,W,U} : (V \otimes W) \otimes U \cong V \otimes (W \otimes U), \quad \alpha_{V,W,U}((v \otimes w) \otimes u) = v \otimes (w \otimes u).$$

Notice that it is not true that these two representations are *equal* (even as vector spaces). However, there is a coherence condition in terms of an equality of maps: applying α to three factors at a time, we find two maps from $((V \otimes W) \otimes U) \otimes Y)$ to $V \otimes (W \otimes (U \otimes Y))$ which one may verify coincide. The corresponding commutative diagram has the form of a pentagon.

Each of the items above are parts of honest axioms of tensor categories: (3) and (4) are required of a *monoidal category*, (5) is part of the *rigidity* axiom while (1) and (2) axiomatize $\mathbb{F}$-*linear abelian categories*. For the record, the following definition is found in Chapter 4 of the book under review: a *tensor category* is a locally finite $\mathbb{F}$-linear abelian rigid monoidal category with simple unit such that $\otimes$ is bilinear on morphisms. After unpacking this definition, one can verify that $\mathrm{Rep}(G)$ is a tensor category for any group G, as is the category $\mathrm{Vec}_{\mathbb{F}}$ of finite-dimensional $\mathbb{F}$-vector spaces.

A more systematic point of view is that these categorical axioms are abstractions of familiar algebraic structures and properties. A key point of this perspective is that the objects are not assumed to have any internal structure; their structure and properties are shifted to the morphisms. Take the example of a *monoid*: a set with an associative binary operator $\times$ with a distinguished identity element 1. The categorical structure one obtains as an abstraction is a *monoidal category*: a category $\mathcal{M}$ endowed with a bifunctor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, a distinguished unit object $\mathbf{1}$ and certain natural families of isomorphisms abstracting the identity and associativity axioms. From this perspective one can view tensor categories as a categorification of the notion of a ring (with some additional properties).

Properties may also be abstracted. For example, the obvious fact that the one-dimensional trivial representation $\mathbf{1}$ of a finite group G is irreducible becomes the requirement that in a tensor category $\mathbf{1}$ should be a simple object (i.e., having no nontrivial proper subobjects). This axiom is not superfluous: the monoidal category A-bimod of A-bimodules over a multimatrix algebra $A = \bigoplus_{i=1}^{k} M_{n_i}(\mathbb{F})$ with bifunctor $\otimes_A$ has unit object $\mathbf{1} = A$, which is not simple if $k \geq 2$. In fact, the category A-bimod satisfies all axioms of a fusion category *except* possibly simplicity of $\mathbf{1}$.

When G is a finite group and $\mathbb{F}$ has characteristic 0, then $\mathrm{Rep}(G)$ has additional properties: first, there are finitely many simple (irreducible) objects, and second, every object is isomorphic to a finite direct sum of simple objects. This makes

Rep(G) a *fusion category*, i.e., a finite, semisimple tensor category. The number of isomorphism classes of simple objects for a fusion cateogry is called the *rank*, which is an important invariant. An essential part of the data of a fusion category $\mathcal{C}$ is the Grothendieck ring $K_0(\mathcal{C})$, generated by the isomorphism classes of simple objects with operations coming from $\oplus$ and $\otimes$. The structure constants of $K_0(\mathcal{C})$ are $N_{i,j}^k = \dim \mathrm{Hom}(X_i \otimes X_j, X_k)$, known as the fusion rules.

Another property enjoyed by Rep(G) is that the $\otimes$ operation is commutative in the following sense: $\sigma_{V,W} : V \otimes W \cong W \otimes V$ via $\sigma_{V,W}(v \otimes w) = w \otimes v$. The abstraction of this idea is a structure called a *braiding* [10] on a monoidal category $\mathcal{C}$: a natural family of isomorphisms $c_{X,Y} \in \mathrm{Hom}(X \otimes Y, Y \otimes X)$ satisfying certain compatibility conditions (essentially a categorical version of the Yang–Baxter equation). In particular, a braided tensor category yields representations of the braid group $\mathcal{B}_n$: the generators σ_i that exchange the i and $i+1$st strands act by

$$\mathrm{Id}_X^{\otimes i-1} \otimes c_{X,X} \otimes \mathrm{Id}_X^{\otimes n-i-1} \in \mathrm{Aut}(X^{\otimes n}).$$

Notice there is a slight subtlety between properties and structures: the *existence* of a braiding is a property of $\mathcal{C}$ whereas a *choice* of a braiding is a structure.

Modular categories, topological quantum field theories, and link invariants. Around the same time as Witten's interpretation [22] of the Jones polynomial in terms of TQFTs, Moore and Seiberg [14] developed categorical connections to the topologically invariant part of rational conformal field theories.[2] They observed that one obtains representations of mapping class groups (which they call modular groups) of compact genus g surfaces with punctures using the cutting and gluing axioms. Turaev [20] axiomatized these categorical connections as a *modular category*, which is a spherical braided fusion category with a further nondegeneracy condition. Reshetikhin and Turaev [18] showed that one may construct a $(2+1)$-dimensional TQFT from a modular category. Some of the most interesting $(2+1)$-TQFTs come from quantum group $U_q\mathfrak{g}$ at roots of unity; see [3] for a concise treatment. The reverse construction[3] is also known for sufficiently nice (i.e., "extended") $(2+1)$-TQFTs. A philosophically similar idea to reconstruct a modular category (and hence a $(2+1)$-TQFT) from a sufficiently nice link-invariant is found in [19].

To complete this triangle of related notions, we observe that a modular category[4] $\mathcal{C}$ gives rise to link invariants in essentially the same way that the Jones polynomial [8] arises from categories obtained from $U_q\mathfrak{sl}_2$. First, Alexander's theorem tells us that any link L can be expressed as the closure $\hat{\beta}$ of a braid $\beta \in \mathcal{B}_n$. Then the $\mathcal{B}_n$ representation provided by the braiding in $\mathcal{C}$ gives us an operator $\rho(\beta) \in \mathrm{End}(X^{\otimes n})$ for some $X \in \mathcal{C}$. Now the spherical structure gives us a linear functional $\mathrm{Tr}_{\mathcal{C}} : \mathrm{End}(X^{\otimes n}) \to \mathbb{F}$ and $\mathrm{Tr}_{\mathcal{C}}(\rho(\beta))$ is a link invariant. The invariance (via Markov moves) follows from the functorial properties of the maps involved.

Categories for their own sake. A traditional motivation for category theory was to understand how structures and results in different areas of mathematics are instances of general phenomena, often redefining concepts in terms of universal

[2]Perhaps reluctantly, as they say, "We will need to use some very simple notions of category theory, an esoteric subject noted for its difficulty and irrelevance."

[3]A proof is promised in a series of four papers, the second and fourth of which have been posted on `arXiv`; see [4].

[4]Actually one only needs a *ribbon* tensor category, but modularity is essential for constructing a TQFT.

properties. In the modern theory of tensor categories we are often interested in studying problems, such as classifying and characterizing categories with given properties, understanding the relationships between different properties, developing new constructions, and generalizing old ones. Another perspective contributing to the recent popularity of tensor categories is that of categorification and higher category theory; see [2]. For example, one speaks of an *algebra object* A in a tensor category as an object equipped with morphisms $m : A \otimes A \to A$ and $u : \mathbf{1} \to A$ satisfying appropriate commutative diagrams. Then an algebra object in the tensor category $\mathrm{Vec}_{\mathbb{F}}$ of vector spaces is just a usual finite-dimensional algebra. The development of tensor categories as the categorification of rings has reached a very mature stage: nearly any algebraic concept has a categorical analogue. Module categories over a monoidal category $\mathcal{C}$ are of special importance, these are categories $\mathcal{M}$ equipped with a functor $\mathcal{C} \times \mathcal{M} \to \mathcal{M}$ satisfying the relevant constraints.[5] A higher algebra version can be given if A is an algebra object in $\mathcal{C}$: one may similarly define A-modules as objects $M \in \mathcal{C}$, appropriately equipped with functors that play nicely with the m, u morphisms. Taking it a step further, one can consider the category $\mathrm{Mod}_{\mathcal{C}}(A)$ of A-modules in $\mathcal{C}$, which is itself a $\mathcal{C}$-module category! Under certain restrictions *every* $\mathcal{C}$-module category is of this form. The authors carefully develop these notions, so I will just mention a few more concrete applications.

Fix an integer r and a ring R that is a free $\mathbb{Z}$-module of rank r, with structure coefficients $\{N_{i,j}^k : 1 \leq i, j, k \leq r\} \subset \mathbb{Z}_+$. Is there a *categorification* of R, i.e., a fusion category $\mathcal{C}$ with $K_0(\mathcal{C}) \cong R$? If so, can we classify or characterize all inequivalent categorifications? Do any of them admit other structures (e.g., a braiding)? What properties can be inferred from R? Notice that a specific *given* ring may or may not have a categorification, whereas the axiomatic *concept* of a ring does have a categorification

We know that there are (at most) finitely many categorifications of a given R, due to a result known as Ocneanu rigidity; see [7]. More generally, it is an open question whether there are finitely many inequivalent fusion categories of a given rank r, which has been shown for braided fusion categories; see [5] for the modular case.

In analogy with $\mathrm{Rep}(G)$, we can define categorical dimensions as $\dim(X) = \mathrm{Tr}_{\mathcal{C}}(\mathrm{Id}_X)$, provided $\mathcal{C}$ admits a (spherical) trace. From $K_0(\mathcal{C})$ we can extract another real-valued dimension function FPdim on simple objects X_i as the largest (Frobenius–Perron) eigenvalue of the matrix $(N_i)_{k,j} = N_{i,j}^k$ encoding fusion with X_i. The FP-dimension of $\mathcal{C}$ is then $\mathrm{FPdim}(\mathcal{C}) = \sum_i \mathrm{FPdim}(X_i)^2$, and $\mathcal{C}$ is called pseudo-unitary if $\mathrm{FPdim}(\mathcal{C}) = \dim(\mathcal{C})$. Many properties of $\mathcal{C}$ can be inferred from the FP-dimensions. A common theme is to categorify notions from group theory and prove generalized versions of classical theorems using dimension in place of group order. A categorical Cauchy's theorem equating the prime ideals dividing $\dim(\mathcal{C})$ and a generalized notion of exponent for spherical fusion categories is found in [5]. Group theoretical results generalize more easily to so-called *weakly integral* categories for which $\mathrm{FPdim}(\mathcal{C}) \in \mathbb{Z}$. For example, if $\mathrm{FPdim}(\mathcal{C}) = p^m q^n$ for primes p, q, then $\mathcal{C}$ is a solvable category, generalizing Burnside's theorem. Many classification results for weakly integral categories are known.

[5]The concept of a module category goes back at least to the 1970s [17]. As category theory languished in the 1970s and 1980s, this and many other concepts have been rediscovered.

Quasi-Hopf algebras and von Neumann algebras. The interplay between Hopf algebras and tensor categories is a very rich subject; see Chapter 5 of the text. One reason is that there is a reconstruction theory relating tensor categories to quasi-Hopf algebras. Indeed, if A is a finite-dimensional semisimple (quasi-)Hopf algebra, then $\operatorname{Rep}(A)$ is an *integral* fusion category, i.e., the dimensions of the simple objects are ordinary integers. Conversely, if $\mathcal{C}$ is an integral fusion category, then there is a (nonunique) semisimple quasi-Hopf algebra A such that $\mathcal{C} \cong \operatorname{Rep}(A)$ as fusion categories. Dropping conditions on $\mathcal{C}$ (e.g., semisimplicity, simplicity of the unit, rigidity) leads to similar reconstruction results. The categorical perspective often yields powerful results. An example is found in [6] in which they prove Kaplansky's conjecture for quasi-triangular Hopf algebras H (that is, $\dim(V) \mid \dim(H)$ for any simple H-module V). General (possibly nonintegral) fusion categories are reconstructed as $\operatorname{Rep}(H)$ for some *weak Hopf algebra* H, but these are somewhat unwieldy.

There is also a useful connection between type II_1 subfactors $N \subset M$ of finite index $[M:N]$ and C^* (i.e., unitary) tensor categories. Let ${}_NM_N$ be M, viewed as an $N-N$ bimodule. Then one constructs the C^*-tensor subcategory $\mathcal{C} \subset N$-bimod generated by simple subobjects of $({}_NM_N)^{\otimes n}$. If $N \subset M$ has finite depth, then $\mathcal{C}$ is a (unitary) fusion category. This construction has led to a plethora of interesting new fusion categories and brings categorical techniques to bear on the classification of (planar algebra) subfactors of low index; see [9]. Notice that M is naturally an algebra object in $\mathcal{C}$—indeed, algebra objects categorify Ocneanu's "quantum subgroups" in subfactor theory [16]; see [12]. Now let me sketch how subfactors can be constructed from a unitary fusion category $\mathcal{C}$. Choose a simple object $X \in \mathcal{C}$ with $\dim(X) > 1$. We have obvious inclusions of finite-dimensional C^*-algebras: $A_n := \operatorname{Id}_X \otimes \operatorname{End}(X^{\otimes n-1}) \subset B_n := \operatorname{End}(X^{\otimes n})$ and $A_n \subset A_{n+1}, \subset B_n \subset B_{n+1}$ via the identification $f \leftrightarrow f \otimes \operatorname{Id}_X$. By taking the inductive limit of these inclusions, we obtain finite depth subfactors $N \subset M$ with $[M:N] = \dim(X)^2$; see [21] for details. A general, precise correspondence includes an object in a (right) $\mathcal{C}$-module category $Y \in \mathcal{M}$ replacing B_n above by $\operatorname{End}(Y \otimes X^{\otimes n})$.

Condensed matter. It may come as a surprise that tensor categories play any significant role in condensed matter physics, but they are tremendously useful in modeling topological phases of matter. These are exotic states of matter for which the observable properties are topologically invariant, at least to exponentially close approximation. In [15] they take the definition of a system in topological phase to be one whose low-energy effective field theory is a topological quantum field theory. From the above discussion we understand that modular categories are the algebraic models for these phases of matter. *Anyons* are quasi-particles that appear in topological phases of matter which can have nontrivial exchange (braiding) statistics. Extrapolating a bit, we can define anyons mathematically to be the objects in a modular category. This lines up very well with the Main Principle in the epigraph: it makes no sense to say that two anyons are equal! However, we should view them as equivalent if their responses to quantum processes and measurements do not distinguish them. Morphism spaces play the role of state spaces, and quantum processes are also morphisms that act on the state spaces by composition. For example $\operatorname{Hom}(\mathbf{1}, X^{\otimes n})$ describes a disk with n anyons of type X at fixed positions, and interchanging the first and second anyons is represented by the braiding $c_{X,X} \otimes \operatorname{Id}_X^{\otimes n}$.

The correspondence between topological phases and categories extends to nearly every topic in this book. A recent important example is *de-equivariantization*, which corresponds to a topological phase transition known as *boson condensation*. Connections of this kind have made category theory the language of choice for many condensed matter physicists!

Summary. Structured as a text for a one- or two-semester course or for self-study, this book assumes familiarity with the basic notions of category theory and builds the axioms of tensor categories from there, with numerous examples and exercises provided. Many of the concepts discussed have appeared in other sources (published and unpublished), but never in the present generality in a single reference. The first seven chapters include the general theory of abelian, monoidal, and tensor categories and their representations along with two more specialized chapters on $\mathbb{Z}_+$-rings (focusing on the fusion rules) and the theory of Hopf algebras (a major source of both categories and category theorists!). Readers interested in applications may find Chapters 8 and 9 on braided and fusion categories most valuable. As the text develops the categorical abstraction of essentially all of classical algebra, some of the treatment is somewhat economical. However, at the end of each chapter some references on the concepts discussed are given, as well as sketches of results that are beyond the scope of the text. This is a book that was long in coming and will endure as a key resource for years to come.

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