BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 56, Number 4, October 2019, Pages 691–703
https://doi.org/10.1090/bull/1677
Article electronically published on July 30, 2019

SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
JÁNOS KOLLÁR

MR0000169 (1,28f) 14.0X

Cartan, Elie

Sur des familles remarquables d'hypersurfaces isoparamétriques dans les espaces sphériques.

Mathematische Zeitschrift **45** (1939), 335–367.

An investigation of the existence of families of isoparametric hypersurfaces in a spherical space (space of constant positive curvature) of an arbitrary number of dimensions. It is shown that there exist families of isoparametric hypersurfaces having three distinct principal curvatures in spherical spaces of 4, 7, 13 and 25 dimensions and that such families exist only in spherical spaces of these dimensions. The following general theorem is proved: If in a spherical space there exists a family of isoparametric hypersurfaces having p distinct curvatures of the same degree of multiplicity, the general equation of these hypersurfaces is of the form $P(x_1, x_2, \cdots, x_{n+1}) = \cos pt$ where P is a harmonic polynomial of degree p satisfying the condition

$$\sum_i \left(\frac{\partial P}{\partial x_i}\right)^2 = p^2({x_1}^2 + {x_2}^2 + \cdots + {x_{n+1}}^2)^{p-1}.$$

Finally it is pointed out that the investigation of families of isoparametric hypersurfaces for which all the principal curvatures have the same degree of multiplicity is reducible to an algebraic problem.

T. Y. Thomas

From MathSciNet, July 2019

MR0009471 (5,154f) 14.0X

Segre, B.

A note on arithmetical properties of cubic surfaces.

Journal of the London Mathematical Society. Second Series **18** (1943), 24–31.

The first of these two papers concerns non-singular cubic surfaces F which are rational in the sense that they are definable over the field of rational numbers. The questions treated pertain to the existence of rational points on F and, more generally, of rational lines and of non-trivial rational curves on F. The results are stated without proofs; these will be published elsewhere. Concerning the existence of rational points it is stated that F either carries no rational points or infinitely many. The existence of non-trivial rational curves (that is, curves which are not obtained as the complete intersection of F with another rational surface) is correlated to the existence of a rational line, a rational doublet, triplet or sextuplet of such lines and to the existence of a parametric rational solution of the equation of F. The paper also contains a discussion of the equation $a_1{x_1}^3 + a_2{x_2}^3 + a_3{x_3}^3 + a_4{x_4}^3 = 0$ and of special cases of this equation, such as the Reley equation ${x_1}^3 + {x_2}^3 + {x_3}^3 + p{x_4}^3 = 0$,

p not the cube of a rational number. In the second paper the author proves by geometric considerations that a cubic rational surface F with more than one rational point necessarily carries infinitely many rational points. The essential part of the proof is that in which it is shown that, if F carries three collinear rational Eckhardt points, then it carries an infinity of rational points (compare with the next review [MR0009472] of the paper by Mordell).

O. Zariski

From MathSciNet, July 2019

MR0568901 (58 #27964) 14B05

Teissier, Bernard

The hunting of invariants in the geometry of discriminants.

Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), 565–678, *Sijthoff and Noordhoff, Alphen aan den Rijn*, 1977.

This long, interesting paper is another in a series of self-contained, semi-expository articles by the author [*Algebraic geometry* (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974), pp. 593–632, Amer. Math. Soc., Providence, R.I., 1975; MR0422256]. The general theme is equisingularity for hypersurface germs $f = 0$; the moral is: "primitive invariants of the discriminant D {of the versal deformation $V \to T$ of $f = 0$}$\cdots$ yield rather subtle invariants {of the original hypersurface}." For instance, the multiplicity of D equals the Milnor number μ of $f = 0$. The main notion discussed here is the Newton polygon of the plane curve $D \cap H$, where H is a generic "vertical" 2-plane section of T. This is an invariant of equisingularity type, this (for this paper) is by definition equivalently: μ^*-constancy; Whitney conditions; or condition (c). M. Merle has proved [Invent. Math. **41** (1977), no. 2, 103–111, MR0460336] that this Newton polygon of D completely recovers the topological type of an irreducible plane curve $f = 0$. Here are some highlights of the paper.

Fitting ideals are used in §1 to define a scheme structure on the image of a finite map (this is not the obvious definition); the goal is a definition of discriminant compatible with base change, but Bezout's theorem happens to pop out as a corollary.

§2, on the module of differentials, has 2 main aspects. First, if X is a reduced d-dimensional space, the development (or Nash blow-up) is the proper modification $g\colon X_1 \to X$, defined via the closure of the graph of a section of $\mathrm{Grass}_d(\Omega^1_X) \to X$ over the nonsingular points of X. X_1 is the space of limits of tangent vectors (a theme picked up recently by Lê, Henry, and the author). It is proved that g is an isomorphism if and only if X is nonsingular. Later, it is shown that the map $C \to D$ of critical to discriminant locus is not only the normalization, but the development as well. Second, the "idealistic Bertini's theorem" is a beautiful algebraic version of Sard's theorem. It says that for $f\colon X \to Y$ flat, X reduced, and Y nonsingular, generically the singular subscheme of X has the same integral closure as the singular subscheme of f (Sard's theorem says the radicals of the corresponding ideals are the same). This condition (on integral closures) is then used to define equisingularity (= condition (c)) for a family of hypersurfaces.

A key result in §3 says that for a 1-parameter deformation of a (reduced) curve, the following are equivalent: δ-constancy, simultaneous resolution (by normalizing the total space), and simultaneous parametrization. For a plane curve, δ is shown to equal δ_0, the maximum number of singular points in a fibre of $V \to T$; from the point of view of Morse theory, δ_0 =maximum number of critical points with the same critical value (for a perturbation of f). (This is contrasted with μ, the maximum number of critical points with distinct critical values.) For a general hypersurface, Iomdin has shown δ_0 is related by inequalities to some $\mu^{(i)}$.

§4 performs the useful function of differentiating between deformation theory and the Thom-Mather theory of unfoldings (in particular, the role of R, A and K-equivalence).

In §5, polar curves are introduced and related to vertical plane sections $D \cap H$ (most of this material is in another article by the author [ibid. **40** (1977), no. 3, 267–292; MR0470246]). In particular, examples show the Newton polygon is not an invariant of μ-constancy, and the number of components of $D \cap H$ is not even an invariant of μ^*-constancy.

{Reviewer's remark: the author has pointed out that p. 649, 1.17′ to 1.3′, is incorrect. His attempt to keep this out of the final version was thwarted by the publisher.}

{For the entire collection see MR0457430.}

Jonathan M. Wahl
From MathSciNet, July 2019

MR0877010 (88d:14004) 14B05; 13C15
Knörrer, Horst
Cohen–Macaulay modules on hypersurface singularities. I.
Inventiones Mathematicae **88** (1987), *no.* 1, 153–164.

Let P be a regular analytic k-algebra, where k is an algebraically closed field of characteristic different from two, and let f be a nonzero element in the maximal ideal of P. The main result of this paper is the following theorem: If $R = P/(f)$ is the local ring of a simple hypersurface singularity in the sense of V. I. Arnol′d [*Proceedings of the International Congress of Mathematicians, Vol. 1* (Vancouver, B.C., 1974), 19–39, Canad. Math. Congr., Montreal, Que., 1975; MR0431217] and K. Kiyek and G. Steinke [Arch. Math. (Basel) **45** (1985), no. 6, 565–573; MR0818299], then there are only finitely many isomorphism classes of indecomposable maximal Cohen–Macaulay modules over R (i.e. R-modules M with $\operatorname{depth} M = \dim R$ and such that M is not a nontrivial direct sum).

To prove this result the author studies the relations between maximal Cohen–Macaulay modules over the ring of a singularity and the ring obtained by adding sums of squares of the variables to the equation of the singularity. Thus, the author proves that there are only finitely many isomorphism classes of indecomposable maximal Cohen–Macaulay modules over $R = R/(f)$ if and only if the same is true for $R_1 = P_1/(f + y^2)$, where $P_1 = P\langle y\rangle$. Then, the main result is obtained by iterated application of the first result because every n-dimensional simple hypersurface singularity is isomorphic to a singularity with an equation $g(x,y) + z_1^2 + z_2^2 + \cdots + z_{n-1}^2 = 0$, where $g(x,y) = 0$ defines a simple plane curve singularity and, on the other hand, for the case $n = 2$, the result of M. Artin and J.-L. Verdier [Math. Ann. **270** (1985), no. 1, 79–82; MR0769609] proves that there are only finitely many isomorphism classes of indecomposable maximal

Cohen–Macaulay modules. The results of this paper allow us to determine the Auslander–Reiten quivers for simple hypersurface singularities.

For part II see the following review [MR0877011].

Tomás Sánchez-Giralda
From MathSciNet, July 2019

MR0877011 (88d:14005) 14B05; 13C15

Buchweitz, R.-O.; Greuel, G.-M.; Schreyer, F.-O.

Cohen–Macaulay modules on hypersurface singularities. II.

Inventiones Mathematicae **88** (1987), *no.* 1, 165–182.

In the paper under review the main purpose of the authors is to prove that, if there are only finitely many isomorphism classes of indecomposable maximal Cohen–Macaulay modules over $R = P/(f)$ (P is a regular analytic k-algebra, where k is an algebraically closed field of characteristic different from two, and f is a nonzero element in the maximal ideal of P), then R is the local ring of a simple hypersurface singularity. This result and its converse, proven by H. Knörrer [part I, same journal **88** (1987), no. 1, 153–164; see the preceding review; MR0877010] provides a characterization of the simple hypersurface singularities in terms of the indecomposable maximal Cohen–Macaulay modules over the corresponding hypersurface ring R.

This theorem of characterization of simple hypersurface singularities for the 1-dimensional case $\mathrm{char}(k) = 0$ was proven by Greuel and Knörrer [Math. Ann. **270** (1985), no. 3, 417–425; MR0774367] and later by K. Kiyek and G. Steinke [Arch. Math. (Basel) **45** (1985), no. 6, 565–573; MR0818299]. In the 2-dimensional case the result is due to M. Artin and J.-L. Verdier [Math. Ann. **270** (1985), no. 1, 79–82; MR0769609] and later M. Auslander [Trans. Amer. Math. Soc. **293** (1986), no. 2, 511–531; MR0816307] and H. Esnault [J. Reine Angew. Math. **352** (1985), 63–71; MR0809966]. The authors prove that if R is the local ring of a nonsimple hypersurface singularity then there are infinitely many different ideals $I \subseteq P$ such that $f \in I^2$ and from this they conclude that there are infinitely many isomorphism classes of indecomposable maximal Cohen–Macaulay R-modules. Moreover, if k is the complex field the authors study the nonisolated singularities A_∞ and D_∞ of the equations $f = z_1^2 + z_2^2 + \cdots + z_n^2 = 0$ and $f = z_0 z_1^2 + z_2^2 + \cdots + z_n^2 = 0$ respectively, where f belongs to $P = \mathbf{C}\{z_0, z_1, \cdots, z_n\}$, and they prove that these are the only hypersurface singularities which are of countable Cohen–Macaulay representation type (i.e. there are only countably many isomorphism classes of indecomposable maximal Cohen–Macaulay modules over R). Furthermore, a complete classification of such modules over A_∞ and D_∞ is given. This paper also includes some results and applications to vector bundles on projective hypersurfaces with "no cohomology in the middle", deformation theory and other topics.

Tomás Sánchez-Giralda
From MathSciNet, July 2019

MR1092845 (92g:14014) 14F17; 14J99, 14N05

Bertram, Aaron; Ein, Lawrence; Lazarsfeld, Robert

Vanishing theorems, a theorem of Severi, and the equations defining projective varieties.

Journal of the American Mathematical Society **4** (1991), *no.* 3, 587–602.

The key point of the paper is to prove a variant of a statement of Severi [Rend. Circ. Math. Palermo **17** (1903); per revr.] by means of elementary arguments using the Kodaira-Kawamata-Viehweg vanishing theorem. More precisely, apart from a generalization of Severi's statement, the argument proves the following theorem: Let M be a smooth projective complex variety, A an ample line bundle and L a globally generated line bundle on M. Let X be a smooth codimension e subvariety of M with ideal sheaf $\mathcal{I}_X$ which is defined scheme-theoretically in M by the vanishing of m sections $s_i \in H^0(M, L^{\otimes d_i})$ with $d_1 \geq d_2 \geq \cdots \geq d_m$. Then it follows that $H^i(M, \mathcal{I}_X^a \otimes K_M \otimes L^{\otimes k} \otimes A) = 0$ for $i \geq 1$, provided $k \geq ad_1 + d_2 + \cdots + d_e$. This result has a surprising number of applications to questions involving the equations defining projective varieties. In particular, the authors show the following: Let X be a smooth variety of dimension n and codimension e in $\mathbf{P}^r$. Corollary. If X is cut out scheme-theoretically by hypersurfaces of degree $d_1 \geq d_2 \geq \cdots \geq d_m$ and $\sum_{i=1}^e d_i \leq r+1$ then X is projectively normal. If $\sum_{i=1}^e d_i \leq r$ then X is projectively Cohen-Macaulay. Corollary. If X has degree d and $d \leq r/2e$ then X is a complete intersection. Corollary. If X is defined by hypersurfaces of degree $d_1 \geq d_2 \geq \cdots \geq d_m$, then X is $(\sum_{i=1}^e d_i - e + 1)$-regular (in the sense of Castelnuovo-Mumford) and it fails to be $(\sum_{i=1}^e d_i - e)$-regular if and only if X is the complete intersection of hypersurfaces of degrees $d_1, \cdots, d_e$. Corollary. If X is connected and defined scheme-theoretically by hypersurfaces of degrees $d_1 \geq d_2 \geq \cdots \geq d_m$, then the Hodge type of X is $\geq [(r + 1 - \sum_{i=1}^e d_i)/d_1]$. (This result gives a partial answer, under strong hypotheses, to a conjecture of Deligne-Dimca.)

Finally, the theorem applies to extend statements on the projective normality and defining equations of algebraic curves to higher dimension projective varieties, following an observation of Mukai. In particular the following theorem is proved. Let A be a very ample line bundle on a smooth n-fold X; then $K_X \otimes A^{\otimes n+1}$ is projectively normal except in the case of $(\mathbf{P}^n, \mathcal{O}(1))$. This last result was obtained independently by the reviewer and A. J. Sommese [Comment. Math. Helv. **66** (1991), no. 3, 362–367] and the reviewer, E. Ballico and Sommese ["Appendix", ibid., to appear]. Analogous assertions for defining equations and higher order syzygies were established by Ein and Lazarsfeld ["A theorem on the syzygies of smooth projective varieties of arbitrary dimension", to appear].

Marco Andreatta

From MathSciNet, July 2019

MR1491444 (99c:14018)

Cutkosky, Steven Dale

Local factorization of birational maps.

Advances in Mathematics **132** (1997), *no.* 2, 167–315.

Since Castelnuovo, Abhyankar and Zariski, we have known that an inclusion $R \subset S$ of regular local rings with a common quotient field can be factored by a unique finite product of blow-ups with smooth centers if $\dim(R) = \dim(S) = 2$ (in that case, centers are closed points).

If $\dim(R) = \dim(S) \geq 3$, this is not true, even in dimension 3; counterexamples exist [see J. Sally, Trans. Amer. Math. Soc. **171** (1972), 291–300; MR0309929; erratum; MR0382258; D. L. Shannon, Amer. J. Math. **95** (1973), 294–320; MR0330154] where S and R are essentially of finite type over a field.

In his paper, Cutkosky proves the following theorem, which gives a positive answer to Abhyankar's conjecture [S. Abhyankar, *Ramification theoretic methods in algebraic geometry*, Ann. of Math. Stud., 43, Princeton Univ. Press, Princeton, N.J., 1959; MR0105416 (p. 237)]: Theorem A: Suppose that R, S are excellent local regular rings of dimension 3, containing a field k of characteristic 0, with a common quotient field K, such that S dominates R. Let V be a valuation ring of K which dominates S. Then there exists a regular local ring T, with quotient field K, such that T dominates S, V dominates T, and the inclusions $R \to T$ and $S \to T$ can be factored by sequences of monoidal transformations with regular centers.

If you only use Hironaka's theorem of desingularization, you can find T such that either $R \to T$ or $S \to T$ can be factored by sequences of monoidal transformations with regular centers, but not both. Theorem A is a very difficult result.

Define $Z(X)$ to be the set of all valuation rings V of K, in case K is the quotient field of a proper k-variety, $Z(X)$ is a naturally ringed space obtained as the inverse limit of the systems of all projective varieties with K as function field. $Z(X)$ is called the Zariski-Riemann manifold of K. Theorem A is a local result for the topology of $Z(X)$. Cutkosky gives an answer to the problem of globalization. Theorem C: Let k be a field of characteristic 0, $\Phi\colon X \to Y$ a birational morphism of integral nonsingular proper excellent k-schemes of dimension 3. Then there exist a nonsingular complete k-scheme Z and birational complete morphisms $\alpha\colon Z \to X$ and $\beta\colon Z \to Y$ commuting with Φ such that α and β are locally products of monoidal transformations.

You can glue local triangles given in Theorem A and, as $Z(X)$ is quasi-compact, only a finite number of triangles are necessary to get Theorem C. An important point: α and β may be nonseparated morphisms. The global problem: "given a birational map between two complete smooth varieties of the same dimension, is it possible to decompose it in a sequence of blow-ups and blow-downs along smooth centers?" is still open (it has been solved in the case of toric varieties [see D. Abramovich, K. Matsuki and S. Rashid, "A note on the factorization theorem of toric birational maps after Morelli and its toroidal extension", Preprint, http://xxx.lanl.gov/abs/math/9803126; R. Morelli, J. Algebraic Geom. **5** (1996), no. 4, 751–782; MR1486987; J. Włodarczyk, Trans. Amer. Math. Soc. **349** (1997), no. 1, 373–411; MR1370654]). The gap between this last problem and Theorem A is comparable to the gap between local uniformization and desingularization. These last two results together with Theorem A lead to great hope that the answer to the global problem is yes: in [D. Abramovich, K. Matsuki and S. Rashid, op. cit.], a research program is proposed to solve it.

As noted above, the difficult result in Cutkosky's paper is Theorem A. Let us say a few words about the proof, which is 146 pages long. It is carried out case by case. The valuations $\nu \in Z(X)$ dominating R are classified by the triples $\mathrm{rank}(\nu)$, $\mathrm{rrank}(\nu)$, $\dim_R(\nu)$, which satisfy Abhyankar's inequality:

$$\mathrm{rank}(\nu) + \dim_R(\nu) \leq \mathrm{rrank}(\nu) + \dim_R(\nu) \leq \dim(R),$$

where $\dim_R(\nu)$ is the transcendence degree of $R/\mathcal{M}\to\mathcal{O}_\nu/\mathcal{M}_\nu$, $\mathrm{rank}(\nu)=\dim(\mathcal{O}_\nu)$, $\mathrm{rrank}(\nu)=\dim(\Gamma_\nu)\otimes\mathbf{Q}$, Γ_ν being the group of ν. In the theory of birational maps, there is a general fact that is always verified: the greater $\dim(R)-(\mathrm{rrank}(\nu)+\dim_R(\nu))$, the worse the case. The (not easy) case where ν has maximal rank 3 has already been proved by C. Christensen [J. Indian Math. Soc. (N.S.) **45** (1981), no. 1-4, 21–47 (1984); MR0828858]. Here the worst case is $\dim_R(\nu)=0$, with Γ_ν a nondiscrete subgroup of $\mathbf{Q}$. Let us concentrate on this case. First, we suppose k algebraically closed. Lemma 2.1: Let (x,y,z) be coordinates in S, let $\nu(y)/\nu(x)=\alpha/\beta$, with $(\alpha,\beta)=1$, α',β' nonnegative integers such that $\alpha\alpha'-\beta\beta'=1$; then there exists a unique constant $c\in k$ such that $\nu(y^\beta/x^\alpha-c)>0$. Define $x_1=x^{\alpha'}/y^{\beta'}$, $y_1=y^\beta/x^\alpha-c$, $z_1=z$, S_1 the localization of $S[x_1,y_1]$ at (x_1,y_1,z_1). Then $S\to S_1$ is birational and can be factored by a sequence of monoidal transformations, and ν dominates S_1.

Let us note that the monoidal transformation $S\to S_1$ is quite natural: it was used in a paper by O. Zariski [Ann. of Math. **40** (1939), 639–689; MR0000159] to make the uniformization of a surface singularity along a valuation with nondiscrete group.

Let us start at the end: many computations are done to reach the last case, where there exist regular systems of parameters (r.s.p.) (u,v,w) of R, (x,y,z) of S such that $u=\gamma x^a$, $v=y$, $w=x^c\Lambda(x,y,z)$ with γ a unit, $\Lambda(x,y,0)\neq 0=\Lambda(0,0,0)$. By Theorem 2.13, $(x,y,\Lambda(x,y,z))$ is an r.s.p. of S and $a=1$. Theorem A is then trivial. Theorem 2.12: If we have $u=\omega x^{\lambda_0}$, $v=y$, with ω a unit, then there exist $R\to R'$ and $S\to S'$ compositions of monoidal transformations of the type in Lemma 2.1 such that we have the last case for $R'\subset S'$ and ν dominates S'.

The main difficulty in the proof is to desingularize $w\in S$ without losing the assumption $v=y$. The author has to work with pairs of monoidal transformations $S\to S_1$, $R\to R_1$ such that $R_1\subset S_1$ and (with clear notations) $v_1=y_1$.

Let us note that the assumption $u=\omega x^{\lambda_0}$ in Theorem 2.12 is immediate: it is a consequence of Hironaka's desingularization and is stable under any reasonable monoidal transformation of S. The second assumption is obtained in a classical way: you do an expansion of $v\in k[\![x,y,z]\!]=\widehat{S}$:

$$v=x^r(\alpha_0+\alpha_1x+\cdots+\alpha_nx^n+\beta\Lambda(y,z)x^n+x^{n+1}F(x,y,z)),$$

where the Greek letters represent elements of k. By Lemma 2.8, you can suppose $v=x^r(\alpha+\beta\Lambda(y,z)+xF)$. By Theorem 2.10, you can suppose $\mathrm{mult}(\Lambda)=1$; then Lemma 2.9 asserts that you can get the hypotheses of Theorem 2.12.

To get some flavour of the difficulties, note that many different subcases appear: the elements of the expansion of v are formal, so you have to control an extension $\widehat{\nu}$ of ν to $\widehat{S}$ [see J. Sally, op. cit.] in order to control the restriction of $\widehat{\nu}$ to $k[\![x,y]\!]$. For example, it may happen that $\mathrm{rrank}(\widehat{\nu})>\mathrm{rrank}(\nu)$; then the valuations of some elements of $\widehat{S}$ are infinitely greater than all elements of Γ_ν (see Lemmas 2.5, 2.6).

In the case when k is not algebraically closed, the author has to re-prove the results above. Base changes do not help much: you can get k algebraically closed in K; then there is an extension $\overline{\nu}$ of ν to $\overline{K}=K\otimes_k\overline{k}$, and $\overline{T}$ is the localization of $T\otimes_k\overline{k}$ at the center of $\overline{\nu}$ for any local regular ring T with fraction field K. Unfortunately, the assumption k algebraically closed in K is not stable if you have to deal with nonrational points.

Vincent Cossart

From MathSciNet, July 2019

MR1978567 (2004d:14009) 14E15; 32S45

Hauser, Herwig

The Hironaka theorem on resolution of singularities (or: A proof we always wanted to understand).

Bulletin of the American Mathematical Society. (New Series) **40** (2003), *no.* 3, 323–403.

This paper is expository in nature. It seduces the reader with the charming flavour and lightness of a bedtime story told to children: it gets the audience gradually involved in and obsessed with the theme (in the case of children, until they fall asleep; in the case of mathematicians, until they pick up paper and pencil to solve the proposed riddles on their own). In its size and organization the paper has the status of a monograph. It explains how to prove the existence of resolutions of singularities of algebraic varieties over a field of characteristic zero. Many mathematicians have been fascinated by this problem; see the introduction of [*Resolution of singularities (Obergurgl, 1997)*, Progr. Math., 181, Birkhäuser, Basel, 2000; MR1748614]. In [Ann. of Math. (2) 79 (1964), 109–203; ibid. (2) **79** (1964), 205–326; MR0199184] H. Hironaka proved the existence of resolution of singularities in characteristic zero; this was the first result for varieties of any dimension, but the theorem is non-constructive. At the end of the last century the constructiveness of Hironaka's theorem was proved [see, e.g., O. E. Villamayor U., Ann. Sci. École Norm. Sup. (4) **22** (1989), no. 1, 1–32; MR0985852; E. Bierstone and P. D. Milman, Invent. Math. **128** (1997), no. 2, 207–302; MR1440306], and people also got interested in the method for resolving singularities, not only in the existence.

Given an algebraic variety X over a field of characteristic zero and embedded in a regular variety W, an embedded resolution of singularities of $X \subset W$ is a proper and birational morphism $W' \to W$ such that W' is regular, the strict transform X' of X has no singular points and the total exceptional divisor of the morphism has only normal crossings with X'. Let us recall that the strict transform X' may be defined as the closure of the inverse image of regular points of X, and that a scheme E is said to have only normal crossings at a point if there is a regular system of parameters (say a system of coordinates) at the point $x_1, \ldots, x_n$ such that E may be expressed as the zero set of monomials on the x_i's. It is very natural to require in addition the property that the morphism $W' \to W$ is an isomorphism outside the singular points of X and other properties like equivariance under group actions; also, usually, the morphism $W' \to W$ is required to be a sequence of blowups at regular centers. So the problem is to define the several centers to be blown up.

All constructive proofs define an upper semicontinuous (u.s.c.) function on W such that the points where the function is maximum form the first center. One obtains the first blowup $W_1 \to W$; then a u.s.c. function is constructed on W_1 defining $W_2 \to W_1$, and the procedure continues until the resolution is achieved. The proof of termination follows by the improvement of the function, namely the function decreases at each stage. The goal of the paper is to construct those functions which define the sequence of centers of the blowups. But, unlike in the usual mathematical research papers, the author proceeds in a different way: after describing the problem he develops a naive strategy to try to solve it. This strategy immediately hits obstructions. Studying these, the author (as well as the reader) is led to modify and improve the strategy stepwise, exploring and thus discovering the (rather complicated) structure of the final proof. As the author says, the reader develops his own proof (under the auspices of the guide).

In the paper all concepts and ideas are introduced for non-specialists (at least at the beginning). There are three chapters: Chapter 0 for busy readers, Chapter 1 for moderately interested readers and Chapter 2 for highly interested readers. Thus, everyone can read only up to the chapter which best fits his interest. But it could happen that after Chapter 0 the reader may already be so fascinated that he is tempted to continue with Chapter 1 and also with the more technical Chapter 2. As the author says, "The question is: How can I understand in one hour the main aspects of a proof which originally covered two hundred pages?" The objective of the article under review is to reveal to the reader the beauty of the problem and to explain to him the main points of the proof. At the end of the introduction we find the sentence: "The article has accomplished its goal if the reader starts to suspect—after having gone through the complex and beautiful building Hironaka proposes—that he himself could have proven the result, if only he had known that he was capable of it." And the author may have succeeded in this goal.

In the paper there are several pictures illustrating abstract concepts, and also lots of examples are found everywhere, helping to motivate the necessity of defining new objects

Chapter 0 is an overview of the problem of resolution of singularities, written for non-specialists. First it introduces one of the main concepts, the blowup; then there is a dictionary which translates concepts from algebraic geometry (high-tech) to intuitive concepts (low-tech). Followed by an explanation of the result, and a brief exposition of the inductive nature of the proof, several examples illustrate concepts like: the strict and weak transform of an ideal, the coefficient ideal of an ideal and the construction of the centers of the blowups which make the singularities improve and finally define a resolution of singularities.

Chapter 1 starts with the main ideas of Hironaka's proof, which may be summarized as follows: how to choose the center at each stage of the resolution process in order to improve the singularities of X. Here the idea of the coefficient ideal is developed in more detail. At every point, how the coefficient ideal should be defined and how it transforms under blowup is justified with examples. The author concludes by the definition of an invariant which will work: it is well defined (does not depend on any choices), and it has the properties required at the beginning. But immediately some obstructions are explained, in order to have an invariant which bring us to the end. In particular one of the obstructions is the normal crossings condition on the center with respect to the exceptional divisor. Thus the invariant must be refined or transformed. At the end of Chapter 1 the author introduces us to the positive characteristic problem and explains the first obstacles in this case.

Chapter 2 is devoted to giving the precise definitions and proofs to the construction of resolution of singularities in characteristic zero. The use of mobiles has to be mentioned. Mobiles are objects attached to the singularity of X which were introduced in [S. Encinas and H. Hauser, Comment. Math. Helv. **77** (2002), no. 4, 821–845 MR1949115]. In the literature one finds several objects encoding the data for resolving singularities: Hironaka's idealistic exponents; Abhyankar's trios, quartets and quintets; Villamayor's basic objects; and Bierstone-Milman's infinitesimal presentations. Mobiles seem to be the final concept for how to encode the required resolution data. In contrast to the earlier concepts, they are intrinsic and they collect the precise information which one wishes to deal with at each stage of the resolution process. Using mobiles eliminates the need to consider the history of prior blowups and makes equivalence relations superfluous.

At the end of Chapter 2 the author deals with the problems in positive characteristic. Two examples are developed illustrating how the characteristic-zero techniques fail in several aspects, all motivated by the non-existence of maximal contact hypersurfaces in positive characteristic.

The paper terminates with five appendices: Appendices A, B and C explain technical details of the proof of the theorem. Appendices D and E are a resumé of definitions and a table of notations useful to follow the proofs, especially in Chapter 2.

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From MathSciNet, July 2019

MR2079993 (2005i:14055) 14J70; 14M25, 32Q55
Mikhalkin, Grigory
Decomposition into pairs-of-pants for complex algebraic hypersurfaces.
Topology. An International Journal of Mathematics **43** (2004), *no.* 5, 1035–1065.

The aim of this paper is to describe, or find tools for describing, a smooth hypersurface $V \subset \mathbb{CP}^{n+1}$ of arbitrary degree d as a smooth manifold. The results also apply to smooth hypersurfaces in toric varieties, and give some information about V as a symplectic manifold too. The author first explains the cases $n = 1$, $n = 2$ and $d = n + 2$: this is standard material apart from the author's incautious statement that the mountain K2 is in Pakistan, which is not the official opinion in India. (André Weil, whom the author cites, in fact refers more diplomatically to "la belle montagne K2 au Cachemir"). He then gives an outline of his results (some of which I repeat below in abbreviated form).

For any n and d the hypersurface V admits a singular fibration λ over an n-dimensional polyhedral complex Π. A generic fibre of λ is diffeomorphic to a smooth torus T^n. The base Π is homotopy equivalent to the bouquet of p_g copies of S^n. The local topological structure of $\Pi \subset \mathbb{R}^{n+1}$ is known in differential topology as the local structure of special spines. In particular there is a natural stratification of Π and regular neighbourhoods of the vertices essentially exhaust the complex Π. This stratification determines a decomposition of V into d^{n+1} copies of $\mathcal{P}_n$, where $\mathcal{P}_n$ is diffeomorphic to the complement of $n+2$ hypersurfaces of $\mathbb{CP}^n$ in general position, an analogue of the pair-of-pants decomposition of Riemann surfaces.

The hypersurface V can be reconstructed as a smooth manifold from Π, even though Π has real dimension n and V has real dimension $2n$. This is because Π and its piecewise-linear embedding in $\mathbb{R}^{n+1}$ encode the combinatorics of gluing together the d^{n+1} copies of $\mathcal{P}_n$ to obtain V.

The fibration λ produces a number of Lagrangian submanifolds in V. Different fibres of λ are not necessarily homologous, and p_g disjoint embedded Lagrangian tori, linearly independent in $H_n(V)$, arise in this way. Then there are p_g linearly independent Lagrangian spheres arising as (partial) sections of λ.

The complex Π is in a suitable sense dual to a lattice polyhedron: in the basic case of hypersurfaces of degree d in $\mathbb{CP}^{n+1}$ the polyhedron is the simplex Δ_d with vertices at $(d, 0, \ldots, 0), (0, d, \ldots, 0), \ldots, (0, 0, \ldots, d)$ and $(0, \ldots, 0)$. More precisely, it is determined by a function $v\colon A = \Delta_d \cap \mathbb{Z}^{n+1} \to \mathbb{R}$, by $\Pi = \{y \in \mathbb{R}^{n+1} \mid \max_{x \in A}(xy - v(y))$ is not a smooth function at $y\}$. Conversely, given an n-dimensional complex in $\mathbb{R}^{n+1}$ satisfying certain compatibility conditions, A (and hence Δ) and v can be recovered, up to unimportant ambiguities. In general the

Δ that arises is not a simplex, and it is this that leads to the more general consideration of hypersurfaces in toric varieties. Section 2 explains this construction in detail. Section 3 gives precise statements of the results, and Section 4 describes some important examples.

Reconstruction of the hypersurface from Π is done in Section 5. In Section 6 the other main results are proved, using as a main technical tool the machinery of non-Archimedean amoebas, due to M. M. Kapranov ["Amoebas over non-Archimedean fields", preprint, 2000; per bibl.].

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From MathSciNet, July 2019

MR3359052 14M20; 14E08, 14F45, 14J30, 14K30
Voisin, Claire
Unirational threefolds with no universal codimension 2 cycle.
Inventiones Mathematicae **201** (2015), *no.* 1, 207–237.

Let us begin with a consequence (formulated as part of Theorem 1.1; see details below) of general ideas and constructions of the work: quartic double solids satisfying some conditions of generality and having a few nodes are not stably rational (but unirational). Stable rationality of Y means that a product $Y \times \mathbf{P}^r$ (for some $r \geq 0$) is rational. The mentioned solids are three-dimensional projective varieties $X \subset \mathbf{P}(1,1,1,1,2)$ whose weighted homogeneous equations are $u^2 = f(z_0, \dots, z_3)$, where $f \in \mathbf{C}[z_0, \dots, z_3]_4$, $\deg(u) = 2$, $\deg(z_i) = 1$ $(i = 0, \dots, 3)$. Let k denote the number of nodes (i.e. double points with non-degenerate tangent cones) of the quartic branch locus $S\colon = \{z | \, z \in \mathbf{P}^3, f(z) = 0\}$. Assume that S does not have singular points besides these k nodes. Every node of S produces a node in the covering double solid X. The main and essential assumption is that $0 \leq k \leq 7$, but following the extreme cases $k = 0$ and $k = 7$ is interesting enough. As we have seen recently for the case $k = 0$ [A. Beauville, Algebr. Geom. **2** (2015), no. 4, 508–513; MR3403239], there is an extension of the above result to some higher dimensions: if $n = 4$ or $n = 5$, then the double cover of $\mathbf{P}^n$ branched along a very general quartic hypersurface is not stably rational (but unirational). From the historical point of view, a possible reason for the assumption $k \leq 7$ is that a quartic surface S cannot possess more than 7 nodes which are arbitrarily situated, because 8 nodes in a general position must be located on a unique elliptic quartic curve and the sum of these points minus the doubled plane section of the curve is an element of order two in the group of divisor classes of the curve, hence the position of 8 nodes is not general. In fact, this observation with respect to 8 general nodes is due to A. Cayley (1870); see [*The collected mathematical papers of Arthur Cayley. Vol. VII*, Cambridge Univ. Press, Cambridge, 1889 (p. 141)], where he writes (using the same notation 'k' as in the work under review!): "The greatest value of k is thus $k = 7$."

As for other values of k, a specialization to an excluded case $k = 10$ with a special position of nodes is essentially used in the proofs and constructions. These ten points in the quartic $S_0 \subset \mathbf{P}^3$ (the so-called symmetroid) correspond to ten quadrics of rank 2 (i.e. quadrics breaking up into pairs of planes) in a general linear quadric web $z_0F_0 + \cdots + z_3F_3$. By the way, this set of ten points was also discovered in 1870 by G. Salmon [*Analytic geometry of three dimensions. Vol. II*, Art. 572; per revr.], Cayley [op. cit. (p. 139)] and G. Darboux; moreover, in [Bull. Sci. Math.

Astron. **1** (1870), 348–358 (p. 354)], Darboux wrote about “les dix droits qui sont arrêtes des couples de plans du système”.

The Artin-Mumford quartic double solid V_0 branched along S_0 is not stably rational because its desingularization V has nonzero 2-torsion in $H^3(V, \mathbf{Z})$. Artin and Mumford's proof was given in the 1970s; the use of 2-torsion looks historically parallel to Cayley's above-mentioned use of 2-torsion elements in elliptic quartics 100 years before the proof. For a comparison of approaches to the Artin-Mumford example and double solids X with $k \leq 7$, it is necessary to note that for the desingularization of such X, the cohomology torsion is trivial. Therefore, it was necessary to invent new stable birational invariants and to develop a new techniques of proofs. It is a bit paradoxical that the basis of the new techniques is taken from the lowest level by a gradual descent from the cases of enormous infinite-dimensional Chow groups $\mathrm{CH}_0(\cdot)$ of 0-cycles (D. Mumford, 1969; A. A. Roitman, 1971) to the cases of finite-dimensional (A. A. Roitman, 1972) or cyclic groups (S. Bloch, V. Srinivas, J. Murre, 1983). The last case means that $\mathrm{CH}_0(\cdot) = \mathbf{Z}$, i.e. $\mathrm{CH}_0(\cdot)_0 = 0$. Like in mountaineering, a descent needs more attention and produces many more impressions than an ascent. The most applicable to the problems of stable (ir)rationality is the lower cyclic case, more precisely, an improvement and perfection of zero triviality (it is possible indeed!); it is the case of triviality of $\mathrm{CH}_0(\cdot)_0$ by every field extension that is the so-called universal triviality. For the latter condition, the triviality by the extension of the ground complex field to the field of rational functions on the variety is sufficient. The universal triviality is a necessary condition for rationality, but is not sufficient. For example, Enriques surfaces (which appeared in 1896) have universally trivial $\mathrm{CH}_0(\cdot)_0$. According to Theorem 3.10 in the article of Bloch and Srinivas [Amer. J. Math. **105** (1983), no. 5, 1235–1253; MR0714776], for a smooth projective variety Y such that $\mathrm{CH}_0(Y \otimes_{\mathbf{C}} \mathbf{C}(Y))_0 = 0$, the following equality in $\mathrm{CH}^d(Y \times Y)$ $(d = Y)$ takes place: $N\Delta_Y = Z_1 + Z_2$, where $\Delta_Y \subset Y \times Y$ is diagonal, N is a positive integer, Z_1, Z_2 are codimension d cycles in $Y \times Y$ with $\operatorname{Supp} Z_2 \subset Y \times (\text{point})$, $\operatorname{Supp} Z_1 \subset D \times Y$, and D is a proper closed subset of Y. For a given Y, the minimal possible N in the equality is an invariant of Y. So these universal nothings are parametrized by natural numbers! The most useful N for the work under review is $N = 1$. If the above presentation of the diagonal holds true with $N = 1$, then the author writes about an *integral Chow theoretic decomposition of the diagonal*. If an analogous equality with $N = 1$ exists in $H^{2d}(Y \times Y, \mathbf{Z})$, then she says that Y admits an *integral cohomological decomposition of the diagonal*. Certainly, the cohomological decomposition is weaker than the Chow theoretic one, but more convenient in use. Both the properties are invariant with respect to stable birational equivalence $(Y_1 \times \mathbf{P}^r \dashleftarrow\dashrightarrow Y_2 \times \mathbf{P}^s)$.

A complete formulation of Theorem 1.1 mentioned at the beginning of this review is as follows. Let X be the desingularization of a very general quartic double solid with at most seven nodes. Then X does not admit an integral Chow theoretic decomposition of the diagonal, hence it is not stably rational. Moreover, the first part of Theorem 1.9 asserts that these quartic solids do not admit an integral cohomological decomposition of the diagonal. It is necessary to note that the integral cohomological decomposition of the diagonal has strong consequences. For any smooth projective threefold X with such a decomposition, they are enumerated in Theorem 1.7:

(1) $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$,
(2) $\mathrm{Torsion}(H^*(X, \mathbf{Z})) = 0$,
(3) $H^{\mathrm{even}}(X, \mathbf{Z})$ consists of classes of algebraic cycles,
(4) there exists a universal codimension 2 cycle $Z \in \mathrm{CH}^2(J^3(X) \times X)$, where $J^3(X)$ is the intermediate Jacobian of X.

The cycle Z is an analog (for codimension 2) of the Poincaré divisor. In some of her previous talks and publications (e.g., pages 141–142 of [*Chow rings, decomposition of the diagonal, and the topology of families*, Ann. of Math. Stud., 187, Princeton Univ. Press, Princeton, NJ, 2014; MR3186044]), Voisin proposed two questions with respect to the Abel-Jacobi map AJ_Y.

Question 1. Let Y be a smooth projective variety such that this map from the codimension 2 Chow classes $\mathrm{CH}^2(Y)_{\mathrm{hom}}$ homologous to zero to the intermediate Jacobian $J^3(Y)$ is an isomorphism. Is there a codimension 2 cycle Z on $J^3(Y) \times Y$ with $Z_a = Z|_{a \times Y} \in \mathrm{CH}^2(Y)_{\mathrm{hom}}$ for any $a \in J^3(Y)$ such that the morphism $\Phi_Z \colon J^3(Y) \to J^3(Y)$ $(\Phi_Z(a) \colon = \mathrm{AJ}_Y(Z_a))$ is the identity?

Theorem 1.7 gives a sufficient condition for a positive answer. Properties (1)–(4) taken together with property (5) are sufficient for the existence of an integral cohomological decomposition of the diagonal. The additional condition is the following.

(5) There is a 1-cycle $z \in \mathrm{CH}^{g-1}(J^3(X))$ (where $J^3(X)$ is principally polarized with the help of $\Theta \in H^2(J^3(X), \mathbf{Z})$, $g = \dim J^3(X)$) whose cohomology class $[z] \in H^{2g-2}(J^3(X), \mathbf{Z})$ coincides with $\Theta^{g-1}/((g-1)!)$. The last condition resembles T. Matsusaka's characterization (1958) of a Jacobian variety. The second part of the above-mentioned Theorem 1.9 asserts that if X is the desingularization of a very general quartic double solid with exactly seven nodes, then X does not admit a universal codimension 2 cycle z. The author proves a strengthening of the assertion. It is connected with the second question (an extension of Question 1) from the above-mentioned book by Voisin.

Question 2. Let us consider the following property of a smooth threefold Y. There exist a smooth projective variety B and a codimension 2 cycle Z on $B \times Y$ with $Z_b \in \mathrm{CH}^2(Y)_{\mathrm{hom}}$ for any $b \in B$, such that that morphism $\Phi_Z \colon B \to J(Y)$ induced by the Abel-Jacobi map AJ_Y (i.e. $\Phi_Z(b) = \mathrm{AJ}_Y(Z_b)$) is surjective with rationally connected general fiber. For which smooth threefolds Y is this property satisfied?

Theorem 2.10 asserts that the double solid with $k = 7$ does not possess the property. The last (3rd) section of the work is devoted to the connections of preceding objects with the so-called unramified cohomology groups. The introduction of these new stable birational invariants was initiated by Bloch and A. Ogus in 1974, then their treatment was essentially transformed and developed by J.-L. Colliot-Thélène and M. Ojanguren beginning in 1988. We mention a part of Theorem 1.10 (cf. Theorem 3.1): The degree 3 unramified cohomology with torsion coefficients of smooth complex projective variety X (of arbitrary dimension) with $\mathrm{CH}_0(X)_0 = 0$ is universally trivial if and only if there is a universal codimension 2 cycle $Z \in \mathrm{CH}^2(J^3(X) \times X)$. Hence Corollary 1.11 (cf. Corollary 3.4): Such a cohomology group for the natural desingularization of a very general quartic double solid with 7 nodes is not universally trivial.

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From MathSciNet, July 2019