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ABOUT THE COVER: *QUASTE*

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The image on the cover represents the real points of the algebraic surface *quaste* (Figure 1) defined in $\mathbb{R}^3$ (up to some scaling) by the unremarkable equation

$$Q : (x^3-y^2-z^2)^2[(x+1)^3-y^2-z^2]+z^2(3x+1)(x^3-y^2-z^2)+z^2[x^3-y^2(3x+1)^3] = 0.$$

More instructive is the remark that the polynomial on the left-hand side is a generator of the ideal

$$I = \langle y^2 - (x-w)^3, z^2 - w^2 - w^3 \rangle \cap \mathbb{R}[x, y, z].$$

This shows that quaste is the image of the surface P in $\mathbb{R}^4$ defined by the two equations $C : y^2 = x^3$ and $N : z^2 = w^2+w^3$ under the linear projection $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ sending (x, y, z, w) to $(x + w, y, z)$. Note that P equals the product $C \times N$ of the *cusp* and *node* defined in $\mathbb{R}^2$ by the first (resp., second) equation. We can therefore interpret quaste as a rather faithful representation in $\mathbb{R}^3$ of the Cartesian product surface P in $\mathbb{R}^4$.

If we pass to parametrizations, things become clearer. The cusp is parametrized by $s \to (s^2, s^3)$ and the node by $t \to (t^3-t, t^2-1)$; hence P has the parametrization

$$(s, t) \to (s^2, s^3, t^3 - t, t^2 - 1).$$

Composing with π, we obtain the parametrization of quaste,

$$(s, t) \to (s^2 + t^2 - 1, s^3, t^3 - t).$$

The polynomial relation satisfied by the three components produces the implicit equation for Q.

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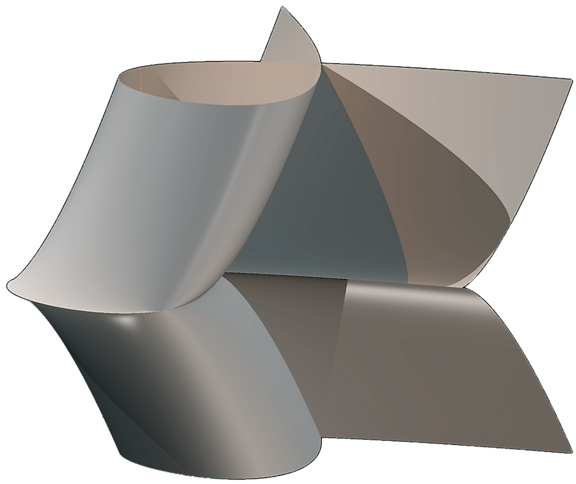

FIGURE 1. CLEAR COPY OF THE COVER. The surface *quaste*. (The German word “quaste” means “tassel” (), a fabric decoration used for pillows, blankets, and other textile purposes.)

The two projections $P \to C$ and $P \to N$ on the first and second factor of $P = C \times N$ produce a fibration of P by a family of (identical) nodes (resp., cusps). We can see these fibrations also on Q:

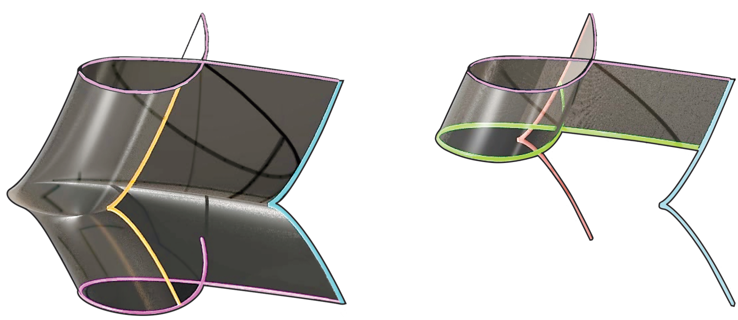

FIGURE 2. Vertical and horizontal fibration of quaste

Similarly to the torus T in $\mathbb{R}^3$, where a small circle travels along a (perpendicular) larger circle while rotating (or vice versa), we can produce quaste by moving a vertical cusp along a horizontal node (without rotation), or, conversely, by moving a horizontal node along a vertical cusp (Figure 2). But here, contrary to the isomorphic surfaces $S = S^1 \times S^1 \subset \mathbb{R}^4$ and $T \subset \mathbb{R}^3$, the surfaces P and Q are not isomorphic as affine algebraic varieties (their local embedding dimensions at 0 are different, namely, two for P and one for Q). Their singular loci are, however, the same: they are in both cases the union of a node and a cusp meeting at the origin.

Cartesian products abound in various branches of mathematics. In algebraic geometry, for instance, they give rise to the famous cancellation problem: Assume that $X \times Z$ is isomorphic to $Y \times Z$. Is then X already isomorphic to Y? Here,

X, Y, Z can be algebraic or analytic varieties or germs thereof, and isomorphic may mean biregular, birational, biholomorphic, or just formal isomorphism. The answer depends very much on the context, with scattered results in the biregular case (small dimensions, smooth Z [1], [2]), counterexamples and various subtle theorems in the birational setting [3], [4], [5], [6], and rather complete positive results for local analytic varieties [7].

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