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MINIMAL SURFACES AND FREE BOUNDARIES: RECENT DEVELOPMENTS

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Abstract. Free boundaries occur in a lot of physical phenomena and are of major interest both mathematically and physically. The aim of this contribution is to describe new ideas and results developed in the last 20 years or so that deal with some nonlocal (sometimes called anomalous) free boundary problems. Actually, such free boundary problems have been known for several decades, one of the main instances being the thin obstacle problem, the so-called (scalar) Signorini free boundary problem. We will describe in this survey some new techniques that allow to deal with long-range interactions. We will not try to be exhaustive since the literature on this type of problem has been flourishing substantially, but rather we give an overview of the main current directions of research. In particular, we want to emphasize the link, very much well-known in the community, between minimal surfaces, their "approximation" by the Allen–Cahn equation and free boundary problems.

Contents

1. Introduction

Free boundaries occur when there is a drastic (nonsmooth) change between two "phases". Typical examples of free boundary problems are incompressible flow through porous media, heat optimization, or the pricing of an American option to name just a few. Mathematically, one can translate those problems by saying the PDE governing the phenomenon is set in a domain whose boundary is also

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unknown. It is of major interest both mathematically and in applications to develop a general program to understand free boundary problems. The aim of the present contribution is to shed light on several new techniques developed in the last 15 years to handle phenomena exhibiting long-range interactions and/or anomalous diffusion. We do not pretend to be exhaustive; instead we want to emphasize several aspects of the theory and provide several open problems. See also the survey paper [CS15].

Free boundary problems are one of the possible mathematical translations of phase transitions. For this reason, our starting point is the canonical model for phase transitions, the so-called Allen–Cahn equation (see [AC79])

$$-\Delta u = u - u^3 = W'(u) \text{ in } \mathbb{R}^n, \quad n \geqslant 1. \tag{1.1}$$

Since our goal is to introduce new techniques for free boundary problems, we now explain the link between the Allen–Cahn equation and such problems. The intermediate step, well-known by experts in the field, is minimal surface theory. Then the Allen–Cahn energy in Ω associated to the double well potential W is defined by

$$\int_\Omega \varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\,\mathrm{d}\,x\,, \qquad \varepsilon \in (0,1)\,, \tag{1.2}$$

where $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$. Critical points of (1.2) satisfy the so-called elliptic Allen–Cahn (or scalar Ginzburg–Landau) equation

$$-\Delta u_\varepsilon - \frac{1}{\varepsilon^2}W'(u_\varepsilon) = 0 \quad \text{in } \Omega\,. \tag{1.3}$$

When ε is small, a control on the potential implies that $u_\varepsilon \simeq \pm 1$ away from a region whose volume is of order ε. Formally, the transition layer from the phase -1 to the phase $+1$ has a characteristic width of order ε. It should take place along a hypersurface which is expected to be a critical point of the area functional, i.e., a minimal surface. More precisely, the region $\{u_\varepsilon \simeq 1\}$, which is essentially delimited by this hypersurface and the container Ω, should be a stationary set in Ω of the perimeter functional, at least as $\varepsilon \to 0$. To make this heuristics mathematically precise, we need the following notion of Γ-convergence.

Definition 1.1 (Γ-convergence)**.** Let X be a topological space, and let J_n be a sequence of functional on X taking values in $[0, \infty)$. Let J be another functional on X taking values in $[0, \infty)$. Then we say J_n Γ-converges to J if the following are true.

(i) For any sequence $x_n \in X$ satisfying $x_n \to x \in X$ as $n \to \infty$, we have

$$J(x) \leqslant \liminf_{n\to\infty} J_n(x_n).$$

(ii) For any $x \in X$ there exists a sequence $x_n \in X$ converging to x such that

$$J(x) \geqslant \limsup_{n\to\infty} J_n(x_n).$$

A very important property of Γ-convergence is that the minimizers converge to the minimizers. More precisely, if J_n Γ-converges to J and x_n is a minimizer of J_n, then every limit point of $\{x_n\}$ is a minimizer of J.

The connection between energy minimizing solutions (under their own boundary conditions) of (1.3) and minimal surfaces was first found in [MM77] through one of the first examples of Γ-convergence. The result shows that if the energy is equibounded, then $u_\varepsilon \to u_*$ in $L^1(\Omega)$ as $\varepsilon \to 0$ for some function $u_* \in BV(\Omega;\{\pm 1\})$ (up to subsequences). The set $\{u_* = 1\}$ minimizes (locally) its perimeter in Ω, and, up to a multiplicative constant, the energy converges (in the sense of Γ-convergence) to the relative perimeter of $\{u_* = 1\}$ in Ω.

In order to state mathematically the convergence result, one needs to define the De Giorgi perimeter of a set E into Ω, denoted $P(E,\Omega)$. The reader is referred to Section 2 for a formal definition and is reminded here that this is a suitable generalization of the formula

$$P(E,\Omega) = \mathcal{H}^{N-1}(\partial E \cap \Omega),$$

which holds in the smooth setting.

The previous discussion makes a clear link between minimizers of (1.1) and minimizers of the perimeter functional (i.e., area-minimizing minimal surfaces), since Γ-convergence is designed exactly for this property to hold. This led to the famous conjecture by De Giorgi (see Appendix A for late accounts). In particular it gives a *level-set approach* to the theory of minimal surfaces. In this model, the internal energy in each phase completely disappears under the scaling. If one wants to keep such a term, one must make a different scaling. We introduce the new functional

$$\int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon^2}\mathcal{H}^n(\{|u| < \varepsilon\}).$$

Now, the transition between $u_\varepsilon \simeq \pm\varepsilon$ holds with a characteristic width of order ε^2. Formally, in the limit $\varepsilon \to 0$ one gets the functional

$$\int_\Omega |\nabla u|^2 + \mathrm{Area}(u = 0).$$

The previous discussion gives rise to another model, with increased complexity, of phase transition. This latter model has been investigated thoroughly in [ACKS01].

The previous models were actually known among the free boundary community, however, and were motivated by applications in several areas of physics, biology, or population dynamics, to name a few. Chen and Fife introduced several models of phase transitions taking into account long-range interactions [CF00]. The purpose of our survey is precisely to report on recent progress in [CF00].

In the last decade, a large number of works have been focused on understanding properties of nonlocal equations. The simplest operator one could consider is the so-called *fractional laplacian*, a Fourier multiplier of symbol $|\xi|^{2s}$, $s > 0$, on $\mathbb{R}^n$. Whenever $s \in (0,1)$, it admits the integral representation

$$(-\Delta)^s u(x) = p.v. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy,$$

provided u is (say) $C^{1,1}$ locally and with suitable decay at infinity. Fractional laplacians appear in several areas such as harmonic analysis, probability [Ber96], crystal dislocation [FIM09], fluid dynamics [MT96], quantum mechanics [Las02], and more generally mathematical physics to mention a few. These are related to anomalous diffusion (Levy processes) where the space and time variables do not scale like the ones of the Brownian motion. Due to the huge number of articles and surveys focusing on the fractional laplacian and its variants in the last 20 years, we will not pretend to exhaust the literature and instead refer the reader to several well-known papers in the field. Let us just mention that the power 1/2 of the laplacian appears actually in an old problem, the so-called (scalar) Signorini problem or thin-obstacle problem. The mathematical formulation is as follows. Consider

$$\min_{v\in\mathcal{X}} \int_\Omega |\nabla v|^2,\ \Omega \text{ open } \subseteq \mathbb{R}^n, n \geqslant 2,$$

where v ranges in the closed set

$$\mathcal{X} = \left\{v \in H^1(\Omega) : v = g \text{ on } \partial\Omega \setminus \mathcal{M}, v \geqslant \phi \text{ on } \mathcal{M}\right\}.$$

Here $\mathcal{M} \subseteq \partial\Omega$ is a codimension 1 manifold, g is the boundary datum, and $\phi : \mathcal{M} \to \mathbb{R}$ is the "thin" obstacle. This problem is actually an obstacle problem for the operator $(-\Delta)^{1/2}$ since $(-\Delta)^{1/2}$ is the Dirichlet-to-Neumann operator of the harmonic extension of a function v.

2. A quick review of the theory of minimal surfaces in Euclidean space

The ideas developed in the theory of sets of finite and minimal perimeter played a fundamental role in the theory of local and nonlocal phase transition problems. Consider the problem of finding a surface with minimal area enclosed by a given curve. Such a surface is called a minimal surface. The problem of finding a minimal surface is called Plateau's problem, named after the famous physicist Joseph Plateau who first demonstrated that a minimal surface can be obtained by immersing a wire frame, representing the boundaries, into soapy water. We will follow the approach of De Giorgi (see the monograph [Giu84]). In this approach we will see the minimal surface as boundaries of special sets.

2.1. **Sets of finite perimeter, and existence of minimal surfaces.** We have the following definition.

Definition 2.1. Let E be any measurable set in $\mathbb{R}^n$. Then the perimeter of E in an open set Ω is defined to be the total variation of $\nabla\chi_E$ in Ω, i.e.,

$$P(E,\Omega) = \sup\left\{\int_E \operatorname{div}(g)\,dx : g \in C_0^1(\Omega;\mathbb{R}^n), \|g\|_\infty \leqslant 1\right\}.$$

Also, $P(E)$ stands for the perimeter of E in $\mathbb{R}^n$.

A set E is said to be a Caccioppoli set if it has locally finite perimeter. Plateau's problem can be formulated by using the notion of perimeter,

$$\text{minimize } P(E) \text{ among all sets } E \text{ satisfying } E \setminus \Omega = L \setminus \Omega,$$

for a given Caccioppoli set L. The boundary of any minimizer of the above problem is called a minimal surface in Ω with prescribed boundary $\partial L \cap \partial\Omega$. If Ω is bounded open set in $\mathbb{R}^n$, then the existence of the minimal surface easily follows from compactness of BV functions in L^1.

Remark 2.2. A more geometric characterization of sets of finite perimeter is the following. An open set $\Omega \subseteq \mathbb{R}^n$ is of finite perimeter if there exists a sequence $\{\Omega_k\}$ of polyhedra such that

- $\sup_k P(\Omega_k) < \infty$, where here $P(\Omega_k)$ is the area of the polyhedron;
- the symmetric difference satisfies $|\Omega\Delta\Omega_k| \to 0$ as $k \to \infty$.

We then define

$$P(\Omega) = \liminf_{\substack{\Omega_k \to \Omega \\ \Omega_k \text{ a polyhedra}}} P(\Omega_k).$$

A set Ω, in a given class, is of minimal perimeter if $P(\Omega) \leqslant P(\Omega^*)$ for all sets Ω^* in that class.

2.2. **The regularity of minimal surfaces.** The regularity of minimal surfaces asserts that minimizers obtained by the above-mentioned theory are more than just sets of finite perimeter; i.e., their boundary enjoys some regularity properties. The final result is the following:

Theorem 2.3. *Let $E \subseteq \mathbb{R}^n$ be a minimal set. Then one has*

- *if $n \leqslant 7$, then ∂E is analytic;*
- *if $n \geqslant 8$, then there exists $\mathcal{S} \subseteq \partial E$ such that $\mathcal{S}$ is closed, $\partial E \backslash \mathcal{S}$ is analytic and the δ-Hausdorff measure of $\mathcal{S}$, $\mathcal{H}^\delta(\mathcal{S}) = 0$ for any $\delta > n - 8$.*

This remarkable result is a combination of several deep results of De Giorgi, Giusti, Bombieri, Simons, and Federer. One can find the full proof in [Giu84]. We also refer to the very nice survey by Cozzi and Figalli [CF17], providing a proof (of an ε-regularity result) based on ideas of Savin [Sav07].

Let us summarize the main ingredients of the proof. Assume that E is a minimal set, i.e., ∂E is a minimal surface, and consider $x_0 \in \partial E$.

(1) *Positive densities of the phases.* There exists $c(n)$ such that

$$\frac{|E \cap B_r(x_0)|}{r^n} \text{ (and) } \frac{|E^c \cap B_r(x_0)|}{r^n} \geqslant c(n) > 0.$$

This property asserts that L^1-convergence of a sequence of minimal sets E_k ($|E_k \Delta E| \longrightarrow 0$) implies uniform convergence, and then

$$E_k \subseteq \mathcal{N}_{\varepsilon_k}(E),$$

where $\mathcal{N}_h(E)$ is an h-neighborhood of E.

(2) *Blowups.* The dilation of E (i.e., $x \in E_\lambda$ if $\lambda x \in E$ (and translation)) of a set of minimal perimeter is again a set of minimal perimeter. Therefore, to see if ∂E has a tangent plane at x_0, we can perform a sequence of dilations (E_{ϵ_k} with $\epsilon_k \searrow 0$) that locally converges (uniformly) to a set E_0, a global minimal surface, and then attempt to classify the possible limits E_0. We go back afterward to E at x_0 (shrink down), provided one has kept information along the blow-up procedure. This procedure is achieved via a *monotonicity formula*; i.e., one can prove that

$$\frac{\mathcal{H}^{n-1}(\partial E \cap B_r(x_0))}{r^{n-1}}$$

is monotone nondecreasing in r.

(3) *Global minimal cones.* It then follows that ∂E_0 is a global minimal cone, and one has to classify all of them in $\mathbb{R}^n$. Note that in $\mathbb{R}^8$, the Simons' cone

$$\mathcal{S} = \left\{x \in \mathbb{R}^8,\ x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\right\}$$

is minimal. In lower dimensions, by results of De Giorgi, Almgren, and Simons, all minimal cones are trivial (i.e., half-spaces). If E_0 is a half-space, then we want to prove that at x_0, E had a tangent plane, together with some "universal" convergence to the plane (i.e., in $B_r, \partial E$ is trapped in a cylinder of width $r^{1+\alpha}$). This implies that at "regular points" ∂E is a $C^{1,\alpha}$ surface in a neighborhood of the point. This is achieved by an *improvement of flatness*.

3. Taking into account long-range effects: nonlocal minimal surfaces

Motivated by nonlocal phase transitions [CF00] and motions by nonlocal curvature [CS10], Caffarelli, Roquejoffre, and Savin in [CRS10a] introduced a notion of "area" encoding long-range interactions, the so-called nonlocal minimal surfaces. The purpose of this section is to introduce their basic theory and their link with a version of the Allen–Cahn equation.

3.1. Definitions and first properties.

Definition 3.1. The fractional s-perimeter of E into $\Omega \subseteq \mathbb{R}^n$ is defined for $s \in (0, 1/2)$ by

$$P_s(E,\Omega) := \int_{E\cap\Omega}\int_{E^c\cap\Omega} \frac{\mathrm{d}\,x\,\mathrm{d}\,y}{|x-y|^{n+2s}} + \int_{E\cap\Omega}\int_{E^c\setminus\Omega} \frac{\mathrm{d}\,x\,\mathrm{d}\,y}{|x-y|^{n+2s}} + \int_{E\setminus\Omega}\int_{E^c\cap\Omega} \frac{\mathrm{d}\,x\,\mathrm{d}\,y}{|x-y|^{n+2s}}\,.$$

Analogously to Section 2, one has the following

Definition 3.2. The set $E_* \subseteq \mathbb{R}^n$ is minimizing its s-perimeter in Ω if

$$P_s(E_*,\Omega) \leqslant P_s(F,\Omega) \qquad \forall F \subseteq \mathbb{R}^n\,,\ F\setminus\Omega = E_*\setminus\Omega\,. \tag{3.1}$$

Sets satisfying the minimality condition (3.1) have been introduced in [CRS10a], where they proved an existence result. Their boundary $\partial E_* \cap \Omega$ is referred to as (minimizing) *nonlocal* (s-)*minimal surfaces* in Ω. By the minimality condition (3.1), the first inner variation of the s-perimeter vanishes at E_*, i.e.,

$$\delta P_s(E_*,\Omega)[X] := \left[\frac{\mathrm{d}}{\mathrm{d}\,t} P_s\big(\phi_t(E_*),\Omega\big)\right]_{t=0} = 0 \tag{3.2}$$

for any vector field $X \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ compactly supported in Ω, where $\{\phi_t\}_{t\in\mathbb{R}}$ denotes the flow generated by X. If the boundary $\partial E \cap \Omega$ of a set $E \subseteq \mathbb{R}^n$ is smooth enough (e.g., a C^2-hypersurface), the first variation of the s-perimeter at E can be computed explicitly, and it gives

$$\delta P_s(E,\Omega)[X] = \int_{\partial E\cap\Omega} \mathrm{H}^{(s)}_{\partial E}(x)\, X\cdot\nu_E \,\mathrm{d}\,\mathcal{H}^{n-1}\,, \tag{3.3}$$

where ν_E denotes the unit exterior normal field on ∂E, and $\mathrm{H}^{(s)}_{\partial E}$ is the so-called *nonlocal* (or fractional) (s-)*mean curvature* of ∂E, defined by

$$\mathrm{H}^{(s)}_{\partial E}(x) := p.v.\left(\int_{\mathbb{R}^n} \frac{\chi_{\mathbb{R}^n\setminus E}(y) - \chi_E(y)}{|x-y|^{n+2s}}\,\mathrm{d}\,y\right)\,, \quad x\in\partial E\,.$$

Therefore, a set E_* whose boundary is a minimizing nonlocal s-minimal surface in Ω (i.e., such that (3.1) holds) satisfies in a weak sense the Euler–Lagrange equation

$$\mathrm{H}^{(s)}_{\partial E_*} = 0 \quad \text{on } \partial E_*\cap\Omega\,. \tag{3.4}$$

The weak sense here is precisely the relation (3.2). It has been proved in [CRS10a] that minimizing nonlocal minimal surfaces also satisfies (3.4) in a suitable viscosity sense. This is one of the key ingredients in the regularity theory of [CRS10a]. There they prove that a minimizing nonlocal minimal surface is a $C^{1,\alpha}$-hypersurface away from a (relatively) closed subset of Hausdorff dimension less than $(n-2)$. Later the size of the singular set has been reduced to $(n-3)$ in [SV13]. Whether or not the singular set can be further reduced in full generality remains a largely open question (see [CV13] for a nonquantitative result when s is close enough to $1/2$).

3.2. **Fractional Allen–Cahn and nonlocal minimal surfaces.** In this section, we describe several results in connection with a fractional version of the Allen–Cahn equation and nonlocal minimal surfaces. More particularly, we will be considering a semilinear equation of the type

$$(-\Delta)^s u = f(u) \text{ in } \mathbb{R}^n, \tag{3.5}$$

where f is a suitable nonlinearity. Then it is easy to see that this equation admits a "formal" variational structure. In terms of distributions, the action of $(-\Delta)^s v$ on a test function $\varphi \in \mathcal{D}(\Omega)$ is defined by

$$\big\langle (-\Delta)^s v, \varphi \big\rangle_\Omega := \frac{\gamma_{n,s}}{2} \iint\limits_{\Omega\times\Omega} \frac{\big(v(x)-v(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{n+2s}}\, dxdy + \gamma_{n,s} \iint\limits_{\Omega\times(\mathbb{R}^n\setminus\Omega)} \frac{\big(v(x)-v(y)\big)\varphi(x)}{|x-y|^{n+2s}}\, dxdy\,. \tag{3.6}$$

Indeed, this formula defines a distribution on Ω whenever $v \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$. It allows us to define a natural energy

$$\mathcal{E}(v,\Omega) := \frac{\gamma_{n,s}}{4} \iint\limits_{\Omega\times\Omega} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\, dxdy + \frac{\gamma_{n,s}}{2} \iint\limits_{\Omega\times(\mathbf{R}^n\setminus\Omega)} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\, dxdy < \infty \tag{3.7}$$

and

$$\mathcal{E}_\varepsilon(v,\Omega) := \mathcal{E}(v,\Omega) + \frac{1}{\varepsilon^{2s}} \int_\Omega W(v)\, dx\,.$$

Denote

$$\widetilde{\mathcal{E}}_\varepsilon(\cdot,\Omega) := \begin{cases} \varepsilon^{2s-1}\mathcal{E}_\varepsilon(\cdot,\Omega) & \text{if } s \in (1/2,1)\,, \\ |\ln\varepsilon|^{-1}\mathcal{E}_\varepsilon(\cdot,\Omega) & \text{if } s = 1/2\,. \end{cases}$$

In this case the analogue of the Modica–Mortola Γ-convergence result has been proved in [SV12]; namely, the functional $\widetilde{\mathcal{E}}_\varepsilon$ Γ-converges in L^1 as $\varepsilon \to 0$ to the functional $\widetilde{\mathcal{E}}_0(\cdot,\Omega)$ defined on $BV(\Omega;\{\pm1\})$ by

$$\widetilde{\mathcal{E}}_0(v,\Omega) := \sigma\, \mathrm{P}\big(\{v=1\},\Omega\big)\,,$$

where $\sigma = \sigma(W,n,s)$ is a positive constant and $\mathrm{P}(E,\Omega)$ denotes the standard perimeter of the set E in Ω.

For $s \in (0,1/2)$, the variational convergence of $\mathcal{E}_\varepsilon(\cdot,\Omega)$ appears to be much simpler since H^s-regularity does not exclude (all) characteristic functions. In particular, there is no need in this case to normalize $\mathcal{E}_\varepsilon(\cdot,\Omega)$. The functionals $\mathcal{E}_\varepsilon(\cdot,\Omega)$ converge as $\varepsilon \to 0$ both in the variational and pointwise sense to

$$\mathcal{E}_0(v,\Omega) := \begin{cases} \mathcal{E}(v,\Omega) & \text{if } v \in H^s(\Omega;\{\pm 1\})\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

Now it is worth noting that

$$\mathcal{E}(v,\Omega) = 2\gamma_{n,s} P_s\big(\{v=1\},\Omega\big) \quad \forall v \in H^s(\Omega;\{\pm 1\})\,. \tag{3.8}$$

The previously described asymptotic result provides an interesting link between long-range interfaces and fractional Allen–Cahn equations, where the genuinely nonlocal regime is for $s < 1/2$. On the other hand, one observes that for $s \geqslant \frac{1}{2}$, despite the nonlocality of the Allen–Cahn equation, the asymptotic limit gives rise to a standard area-minimizing minimal surface.

4. Free boundary problems consisting of a bulk and a surface term

The seminal works of the first author shed a completely new light on free boundary problems by making mathematically rigorous their analogy with the theory of minimal surfaces. This aspect is particularly emphasized in the survey [CS15] and the book of Caffarelli and Salsa [CS05], where an extremely detailed account of the theory of several free boundary problems is described.

4.1. **A general program for studying free boundaries.**

Let us start with a simple example: one of the simplest configurations consists in finding a function u_0 in a domain D of $\mathbb{R}^n$ with positive Dirichlet data h along ∂D, such that one of the following formulations holds:

(1) *Variational version.* Minimize the energy integral

$$E(u) = \int_D \frac{|\nabla u|^2}{2} + u dx$$

among all nonnegative functions u;

(2) *Nonvariational version.* This is the least supersolution u_0 of $\Delta u \leqslant 1$ (reminiscent of the Perron's method).

The resulting configuration of the solution looks like a balloon pressed against a table, for instance. Note that condition (2) forces u_0 to be $C^{1,1}$; i.e., it lifts tangential to horizontal plane along the "free boundary". The program is then:

(1) get the optimal regularity of u_0;
(2) since the problem is invariant under dilations, one can blow up and obtain global solutions;
(3) classify all global solutions;
(4) then go back to obtain the regularity of the free boundary.

In the problem described above, it is rather straightforward to observe that u_0 is $C^{1,1}$ with $\|D^2 u_0\| \leqslant C(n)$ and grows quadratically away form the free boundary. Because of this, the right scaling is, if u is a solution, then $\frac{1}{\lambda^2}u(\lambda x)$ is also a solution for any $\lambda > 0$. From the $C^{1,1}$ regularity, a sequence of dilations has a subsequence converging to a global solution u^0. The solution u^0 is convex and

$$u^0(x) = C(x_1^+)^2$$

for some constant $C > 0$, and this blowup plays the role of the plane solutions for minimal surfaces. Going back under blowdown, at a neighborhood of a point x_0, the free boundary is a $C^{1,\alpha}$ hypersurface, or the zero set (i.e., the free boundary) forms a cusp. Using a monotonicity formula, one can show that the set of singular points is contained in a countable union of C^1 manifolds (i.e., is stratified).

As can be seen from the discussion above, the strategy to attack free boundary problems is parallel to the one to prove regularity of minimal surfaces (of codimension 1). This analogy is of crucial importance. In the following subsections, we describe several free boundary problems of particular interest in connection with nonlocal phase transitions.

4.2. **Phase transition models consisting of bulk and classical perimeter terms.** In order to shed light on the nonlocal phase transitions and their new features, let us start with a problem investigated in [ACKS01]. Consider the functional

$$\mathcal{E}(u) = \int_\Omega |\nabla u|^2 dx + \text{Area}\{u = 0\}.$$

In [ACKS01], the following results have been proved.

(1) The minimizing solution u is Lipschitz, and this is optimal.
(2) Along the interface Σ, one has the Euler–Lagrange equation

$$|\nabla u^+|^2 - |\nabla u^-|^2 = \mathcal{H}_\Sigma,$$

where $\mathcal{H}_\Sigma$ is the mean curvature of the interface.
(3) Σ is smooth except for a closed set of dimension at most $n - 8$

How do we obtain those results, and why is this problem very difficult? Formally, by making a domain variation, the interface condition reads

$$\left(u_\nu^+\right)^2 - \left(u_\nu^-\right)^2 = \mathcal{H}_\Sigma.$$

However, the natural scaling for u is the one giving Hölder regularity of exponent $\frac{1}{2}$. This indicates that the leading term in the free boundary regularity is the surface term. Indeed, since "generically" a harmonic function is Lipschitz along a boundary surface where it vanishes, the term $\left(u_\nu^+\right)^2 - \left(u_\nu^-\right)^2$ is of lower order with respect to the curvature. Therefore, if we manage to prove that u is Lipschitz across Σ, then it falls in a class of geometric problems known as *almost-minimal surfaces*, and the theory follows from Tamanini (and Almgren) results [Tam84]. The strategy goes as follows. First one gets low Hölder continuity of u and positive density of the sets $\{u > 0\}$ and $\{u < 0\}$ by creating a De Giorgi type iteration in dyadic rings. To reach the Lipschitz optimal regularity, one has to be more careful. One notices that

$$\left(u_\nu^+\right)^2 - \left(u_\nu^-\right)^2 = \mathcal{H}_\Sigma$$

is a good free boundary condition since positive mean curvature implies superharmonicity of the distance function, and this competes with the interface condition. Now, using an Alt–Caffarelli–Friedman monotonicity formula [ACF84], one gets that if u_ν^+ is large, then u_ν^- is small. But then from the curvature condition, Σ

bends toward the positivity set, and in turn that should control u_ν^+, using the distance function as a barrier for u^+. This is the heuristics toward showing that both u^+ and u^- are Lipschitz, hence implying regularity of the interface.

4.3. **Nonlocal phase transitions: a local bulk term and a long-range perimeter term.** We now turn our attention to phase transitions of nonlocal type as described by Chen and Fife [CF00]. More concretely, consider the functional

$$\mathcal{E}(u) = \int |\nabla u|^2 \, dx + \mathrm{P}_s(\{u > 0\}),$$

where P_s denotes the fractional perimeter. This is a very natural model and has been investigated in [CSV15]. The existence and regularity theory initially closely follows the ideas in the previous section, since the replacement method affects the "nonlocal area" proportionally in the same way that the "standard area" was affected before (via De Giorgi iteration). This gives Hölder continuity of u and positive density along the free boundary. As before, a domain variation gives analogously the free boundary condition

$$\left(u_\nu^+\right)^2 - \left(u_\nu^-\right)^2 = \mathcal{H}_\Sigma^{(s)},$$

where this time it is the fractional mean curvature introduced before. A quick inspection of the Euler–Lagrange equation of nonlocal minimal surfaces shows that the use of the distance function is not a suitable candidate. Furthermore, it is a major open problem in the field to derive an Alt–Caffarelli–Friedman monotonicity formula.

According to the previous discussion, several new ideas are in order. The idea is to bypass these difficulties by introducing a "Weiss type" monotonicity formula that combines both energies, namely

$$\Phi_u(r) = r^{2s-n} \Bigg(\int_{B_r} |\nabla u|^2 \, dx + C_{n,s} \int_{B_r^+} z^{1-2s} |\nabla U|^2 \, dx \\ - \left(1 - s r^{s-(n+1)}\right) \int_{\partial B_r} u^2 d\mathcal{H}^{n-1} \Bigg).$$

In the previous expression U stands for the fractional extension in one more dimension of u (see [CS07]). One can then prove

Lemma 4.1. *The quantity*

$$\Phi_u(r) = r^{2s-n} \Bigg(\int_{B_r} |\nabla u|^2 \, dx + C_{n,s} \int_{B_r^+} z^{1-2s} |\nabla U|^2 \, dx \\ - \left(1 - s r^{s-(n+1)}\right) \int_{\partial B_r} u^2 d\mathcal{H}^{n-1} \Bigg)$$

is nondecreasing in r.

Then by using an improvement of flatness, Federer dimension reduction, and the absence of minimal cones in $\mathbb{R}^2$, one can show that the free boundary is $C^{1,\alpha}$ except on a closed set of Hausdorff dimension $n - 3$.

Remark 4.2. Several steps of this strategy have been applied in [DSV15] where the local Dirichlet form in the energy is replaced by a Gagliardo one.

5. Plain free boundary problems

We now turn our attention on the so-called Alt–Caffarelli type problems, by analogy with the seminal paper of Alt and the first author [AC81]. The classical one is given by

$$
\begin{cases} \Delta u = 0, & \text{in } \Omega \cap \{u > 0\}\,, \\ |\nabla u| = 1, & \text{on } \Omega \cap \partial\{u > 0\}\,, \end{cases} \tag{5.1}
$$

with Ω a domain in $\mathbb{R}^n$. A pioneering investigation of this problem was that of Alt and Caffarelli in [AC81] (variational context) and then the first author [Caf87, Caf89, Caf88] (viscosity solutions context). In the book by Duvaut and Lions [DL76], one can recast some cavitation models for semi-permeable membranes by a free boundary problem of the previous type but for the fractional laplacian, namely

$$
\begin{cases} (-\Delta)^s u = 0, & \text{in } \Omega \cap \{u > 0\}\,, \\ \lim_{t\to 0^+} \dfrac{u(x_0 + t\nu(x_0))}{t^\alpha} = const., & \text{on } \Omega \cap \partial\{u > 0\}, \end{cases} \tag{5.2}
$$

with u defined on the whole $\mathbb{R}^n$ with prescribed values outside of Ω. This problem was investigated for the first time by the two authors and Roquejoffre in [CRS10b]. By a result in [CS07], one can recast it as a *local* free boundary problem in one more dimension; i.e., given $s \in (0,1)$ and a function $u \in H^s(\mathbb{R}^n)$, minimize for v the functional

$$
\min\left\{ \int_{\mathbb{R}^{n+1}_+} z^{1-2s}\,|\nabla v|^2\; dxdz \;:\; v|_{\partial\mathbb{R}^{n+1}_+} = u \right\}. \tag{5.3}
$$

Due to the variational structure of the extension problem, one can consider the following functional, associated to (5.2),

$$
J(u, B_1) = \int_{B_1} |z|^{1-2s} |\nabla u|^2 dxdz + \mathcal{H}^n(\{u > 0\} \cap \mathbb{R}^n \cap B_1),
$$

where $B_1 \subseteq \mathbb{R}^{n+1}$ and u is even through the hyperplane. The minimizers of J have been investigated in [CRS10b], where general properties (optimal regularity, nondegeneracy, classification of global solutions), corresponding to those proved in [AC81] for the classical Bernoulli problem, have been obtained. In [CRS10b], only a partial result concerning the regularity of the free boundary is obtained. The question of the regularity of the free boundary in the case $s = 1/2$ was subsequently settled in a series of papers by De Silva, and Roquejoffre [DSR12] and De Silva and Savin [DSS12, DSS15]. In [DSSS14], De Silva, Savin, and the second author considered the case of any power $s \in (0,1)$, more particularly the improvement of flatness argument. In the case $n = 1$, a particular two-dimensional solution U to the free boundary problem is given by

$$
U = \left(r^{1/2} \cos\frac{\theta}{2} \right)^{2s}, \tag{5.4}
$$

with r, θ the polar coordinates in the plane. This function is simply the extension of the model entire solution to the upper half-plane, reflected evenly across $z = 0$. The typical "flatness implies regularity" that one proves is

Theorem 5.1. *There exists a small constant $\bar{\varepsilon} > 0$ depending on n and s, such that if u is a viscosity solution satisfying*

$$\{x \in B_1 : x_n \leqslant -\bar{\varepsilon}\} \subseteq \{x \in B_1 : u(x,0) = 0\} \subseteq \{x \in B_1 : x_n \leqslant \bar{\varepsilon}\}, \tag{5.5}$$

then the free boundary is $C^{1,\gamma}$ in $B_{1/2}$, with $\gamma > 0$ depending on n and s.

Theorem 5.1 extends the results in [DSR12] to any power $0 < s < 1$. It follows the strategy developed in [DSR12], i.e., a compactness argument and a Harnack principle. This family of free boundary problems are called *thin one-phase* problems since the free boundary lies on a codimension 1 surface.

Remark 5.2. We would like to mention that a very important result on the regularity of the singular set for the Alt–Caffarelli free boundary has been recently obtained by Edelen and Engelstein [EE19].

Remark 5.3. In all of the previously described free boundary problems, monotonicity of a suitable quantity plays a crucial role in obtaining regularity. Such a type of monotonicity is known only for pure powers of second-order elliptic operators (in particular for the fractional laplacian). It is a major open problem in the field to prove monotonicity for a larger class of operators, not admitting in particular extensions. Nevertheless, in the case of the obstacle problem for general integro-differential operators, lack of monotonicity can be overcome by a careful analysis of blowups/blowdowns (see [CROS17]).

Appendix A. Late progress on several versions of a conjecture by De Giorgi

In this section we give an account on the state of the art for an analogue of De Giorgi conjecture for nonlocal phase transitions. For the classical De Giorgi conjecture, we refer the reader to the survey [FV09] for a more thorough review. We first state the original De Giorgi conjecture.

Conjecture A.1 (De Giorgi). *Let u be a bounded, entire, smooth solution of*

$$-\Delta u = u - u^3 \quad \text{in } \mathbb{R}^n$$
$$\partial_{x_N} u > 0.$$

Then u is one-dimensional at least up to $n \leqslant 8$.

Motivated by the result of Savin and Valdinoci [SV12] on nonlocal phase transitions, one can formulate the following conjecture.

Conjecture A.2 (Fractional De Giorgi). *Let u be a bounded, smooth solution of*

$$(-\Delta)^s u = u - u^3 \quad \text{in } \mathbb{R}^n$$
$$\partial_{x_N} u > 0,$$

Then u is one-dimensional at least up to $n \leqslant 8$ and for $s \geqslant \frac{1}{2}$.

We now describe the state of the art concerning Conjecture A.2: the conjecture holds for

(1) $n = 2$: see [SV09] and [CS11] with different methods.
(2) $n = 3$: see [CC10, CC14].
(3) $4 \leqslant n \leqslant 8$ with the additional flatness at infinity in [Sav18, Sav19].

Actually, for $n = 2$, the conjecture holds for any $s \in (0, 1)$ and not only $s \geqslant 1/2$ (see [SV09, CS11]). In dimension $n = 3$ the argument of Cabré and Cinti [CC14] based on a variational approach and optimal energy estimates cannot be improved for $s < 1/2$. Indeed the authors proved the following:

$$\mathcal{E}_s(u, B_R) \leqslant CR^{n-1}, s > \frac{1}{2},$$
$$\mathcal{E}_s(u, B_R) \leqslant CR^{n-1} \log R, s = \frac{1}{2},$$
$$\mathcal{E}_s(u, B_R) \leqslant CR^{n-2s}, s < \frac{1}{2}.$$

They are optimal since they are satisfied by the one-dimensional solution. However, the growth of the energy for $s < 1/2$ is not slow enough to apply the usual Liouville theorem. The case of $s \leqslant 1/2$ is much less clear. Indeed, the result by Savin and Valdinoci on the Γ-convergence for nonlocal Allen–Cahn suggests that one needs to better understand the Bernstein problem for nonlocal minimal surfaces. The result in [FV17] suggests that one should get flatness of level sets of fractional Allen–Cahn for $s < 1/2$ in dimension 3. This has been indeed proved recently by introducing a very clever, intrinsically nonlocal improvement of flatness in [DPSV19] and supplemented by an argument in [DFV18]. An important open problem in the field is the flatness (or nonflatness) for $s < 1/2$ and $n \geqslant 4$. Of course, this is related to the Bernstein problem for nonlocal minimal surfaces and, more broadly speaking, their regularity, which is largely open, as already mentioned. Also, a counterexample for dimension $n \geqslant 9$ needs to be constructed. In the local case, this counterexample is due to del Pino, Kowalczyk, and Wei [dPKW11]. The latter is not of a variational nature but is based on a Lyapunov–Schmidt reduction. Such a reduction is quite challenging in the nonlocal setting due to the slow decay at infinity introduced by the power tails of the generators of Levy processes.

A last question one could ask about is the existence of the one-dimensional solution. For nonlocal problems one should emphasize that even in one dimension (or radial solutions), the PDE never boils down to an ODE. This makes the problem more challenging mathematically. However, Cabré and one of the authors managed to prove it in [CS11]

Theorem A.3. *There exists a unique, up to translations, heteroclinic solution of* $(-\partial_{xx})^s u = f(u)$ *in* $\mathbb{R}$ *connecting* -1 *to* 1 *if and only if* $F' = -f$ *is a double-well potential.*

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