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THE MATHEMATICS OF ANDREI SUSLIN

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Contents

0. Introduction

Andrei Suslin (1950–2018) was both friend and mentor to us. This article discusses some of his many mathematical achievements, focusing on the role he played in shaping aspects of algebra and algebraic geometry. We mention some of the many important results Andrei proved in his career, proceeding more or less chronologically beginning with Serre's Conjecture proved by Andrei in 1976 (and simultaneously by Daniel Quillen). As the reader will quickly ascertain, this article does not do justice to the many mathematicians who contributed to algebraic K-theory and related subjects in recent decades. In particular, work of Hyman Bass, Alexander Beĭlinson, Spencer Bloch, Alexander Grothendieck, Daniel Quillen, Jean-Pierre Serre, and Christophe Soulè strongly influenced Andrei's mathematics and the mathematical developments we discuss. Many important aspects of algebraic K-theory (e.g., the study of manifolds using surgery and the study of operator algebras) are not mentioned here; such topics are well addressed in various books on algebraic K-theory such as that of Charles Weibel [83].

In discussing Andrei's mathematics, we hope the reader will get some sense of the sweep and evolution of algebraic K-theory in the past 50 years. Andrei was deeply

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involved in both the formulation and the solution of many of the most important questions in algebraic K-theory. His own evolution from a "pure algebraist" led to a partnership with Vladimir Voevodsky in building the edifice of motivic cohomology. The close relationship of arithmetic algebraic geometry to algebraic K-theory, seen frequently in Andrei's work, has contributed much to the development of algebraic K-theory which is situated at the interface of algebra, algebraic geometry, number theory, homotopy theory, and K-theory.

Towards the end of his career Andrei made important contributions to the modular representation theory of finite group schemes. This represented something of a return by Andrei to more algebraic questions, though still reflecting his engagement in K-theory and algebraic geometry.

Andrei was primarily a problem solver, a mathematician confident that clearly formulated questions could be answered by "direct, imaginative attack". Time and again, Andrei introduced new techniques and structures in order to solve challenging problems. Although he did not incline to "theory building", he has left us considerable theory with which to continue his efforts. For many years, Andrei's clear, precise, careful approach to fundamental questions placed him as the "final judge" of many current efforts at the interface of algebraic geometry and K-theory.

Andrei freely shared his ideas, gave brilliantly clear lectures, and encouraged the work of others. Many of us felt that while stepping to the edge of this new mathematics, we needed Andrei's guidance and confirmation of validity. His death is a great loss to us personally as well as to our mathematical world.

1. Projective modules

The early period of K-theory was led by Richard Swan and Hyman Bass, beginning in the late 1950s. Much of this early work was very algebraic, though motivation from topology and functional analysis played an important role. There was experimentation of definitions of higher K-theory, questions of stability for such groups as $E_n(R) \subset GL_n(R)$, and much discussion about projective modules.

We recall that if R is an associative, unital ring and M is a (left) R-module, then a (finitely generated) projective R-module P is a summand of a free module $R^{\oplus N}$; in other words, there is some R-module Q and an isomorphism of R-modules $R^{\oplus N} \simeq P \oplus Q$.

Andrei began his independent research as an undergraduate, solving special cases of Serre's problem which asks whether every finite generated projective module P over a polynomial ring $F[x_1, \dots, x_n]$ over a field F is free [57], following Serre's earlier proof that any such projective module is stably free (see [3]). Andrei, working in relative isolation, first verified (in the affirmative) cases known to experts and then, while still an undergraduate, verified new cases. In 1976, Andrei and Daniel Quillen independently and essentially simultaneously proved the following theorem, at the time the most famous problem of commutative algebra.

Theorem 1.1 ([56], [59]). *Let S be a finitely generated polynomial algebra over a field F, so that $S = F[x_1, \dots, x_d]$ for some d. Then every finitely generated projective S-module P (i.e., any direct summand P of some free S-module M) is a free S-module.*

Andrei's proof of Theorem 1.1 proceeds by induction on d. Thanks to the stable freeness of projective S-modules proved earlier by Serre, it suffices to prove the cancellation property for a finitely generated projective S-module: given an isomorphism $P \oplus S \simeq S^{\oplus n}$ for some $n > 0$, then P is isomorphic to $S^{\oplus n-1}$. Andrei observes that this is equivalent to proving that the group $GL_n(S)$ acts transitively on the set of unimodular rows $\mathrm{Um}_n(S)$ of length n, where a row is unimodular if its entries generate S as an S-module.

The key step in Andrei's proof is the following elementary algebraic fact designed for his goal: Let R be a commutative ring, and set $f = (f_1, f_2) \in R[t]^{\oplus 2}$ (not necessarily unimodular). Let $c \in R \cap (f_1R[t] + f_2R[t])$. Then for any commutative R-algebra A and $a, a' \in A$ such that $a \equiv a'$ modulo cA, we have that $f(a)$ and $f(a')$ are conjugate by an element of $GL_2(A)$. This is a remarkable piece of ingenuity!

As seen especially in Quillen's proof, Serre's problem can be usefully considered as a question of *extendability*: under what conditions can one say that a projective module for $R[x]$ arises by extension from a projective module for R? Quillen's proof relied on earlier work of Geoffrey Horrocks [33] and was somewhat more geometric in nature than Andrei's. Andrei preferred Quillen's approach, and this might have led him into more geometric considerations of algebraic problems.

During the 1970s, Andrei refined and extended the above theorem with many other cancellation results (see, for example, [60]). These studies appeared to have led Andrei to consider other aspects of algebraic groups, in particular related questions of algebraic K-theory. Other algebraic constructions occupied Andrei in this period, including considerations of division algebras, which appeared repeatedly in his work in the 1980s. We mention one beautiful result of Andrei's concerning unimodular rows whose proof involved elaborate algebraic arguments.

Theorem 1.2 ([61]). *Let $m_1, m_2, \dots, m_n$ be positive integers. Given a unimodular row $(x_1, x_2, \dots, x_n)$ over a commutative ring R, the unimodular row $(x_1^{m_1}, x_2^{m_2}, \dots, x_n^{m_n})$ can always be completed to an invertible $n \times n$ matrix over R if and only if the product $m_1 m_2 \cdots m_n$ is divisible by $(n-1)!$.*

2. K_2 of fields and the Brauer group

In this section, we discuss Andrei's role in revealing how important Quillen's foundations for algebraic K-theory [55] proved to be in the study of the Galois cohomology of fields. Andrei developed much of this mathematics in the 1980s. The remarkable Merkurjev–Suslin theorem (Theorem 2.2) has had a tremendous influence on the study of division algebras; it also paved the way for Vladimir Voevodsky's proof of Theorem 5.3, usually referred to as the Bloch–Kato conjecture.

We begin with the definition of the Grothendieck group $K_0(R)$ of an associative, unital ring R. This group is the free abelian group generated by isomorphism classes $[P]$ of finitely generated projective left R-modules P, which we denote by $\mathbb{Z}\{[P]\}$, modulo the relations $\{[P_1 \oplus P_2] = [P_1] + [P_2]\}$. This construction (extended to locally free, coherent sheaves on a scheme) was introduced by Alexander Grothendieck in formulating his Riemann–Roch theorem [9]. Grothendieck's introduction of $K_0(R)$ was soon followed by work of Michael Atiyah and Fritz Hirzeburch [1], who adapted Grothendieck's construction to the context of algebraic topology. Ever since, algebraic geometers have tried to establish fundamental properties for algebraic K-theory analogous to those established by Atiyah and Hirzebruch for topological K-theory.

Various important questions in algebraic number theory and the structure of algebraic groups have formulations in terms of *low degree algebraic K-theory*. As mentioned above, Grothendieck launched the subject with his definition of K_0; Bass [3] extensively considered K_1 of a ring, in some sense viewing $K_1(R)$ as units $R^\times$ of R modulo *algebraic homotopies.* In his remarkable book [46], John Milnor defined $K_2(R)$ and proved certain exact sequences giving us the sense that a general theory with good properties lurked amonst these groups.

There is a natural product map $R^\times \otimes R^\times \to K_2(R)$ for a commutative ring R which can be viewed as the *universal symbol*, one of the many connections between algebraic number theory and algebraic geometry. Hideya Matsumoto gave a presentation of $K_2(F)$ as a quotient of $F^\times \otimes F^\times$ [39], which led Milnor to define the *Milnor K-theory* of a field: $K_*^M(F)$ is defined to be the tensor algebra $T^*(F^\times)$ modulo the relations $\{a \otimes (1-a) : 0, 1 \neq a \in F\}$,

$$K_*^M(F) \ := \ T^*(F^\times)/\{a \otimes (1-a) : 0, 1 \neq a \in F\}. \tag{1}$$

The class of the tensor $a_1 \otimes \cdots \otimes a_n$ in $K_n^M(F)$ is denoted by $\{a_1, \ldots, a_n\}$.

These developments encouraged various mathematicians to formulate $K_i(R)$ for all $i \geq 0$. Quillen gave a definition of $K_*(R)$ in terms of the Quillen plus construction applied to $GL_\infty(R)$ [54]. Perhaps to the despair of algebraists, Quillen's K-groups were defined as the homotopy groups of a topological space; homotopy groups are notoriously difficult to compute. Quillen followed his plus construction definition with a more general, more widely applicable definition using what we now call *Quillen's Q-construction* [55]. Very quickly, Andrei recognized the power and potential of the theoretical tools and many new results that Quillen introduced.

Galois cohomology groups provide a powerful tool in the study of fields, especially in class field theory, which can be viewed as an early forerunner of algebraic K-theory. Let Γ_F be the absolute Galois group of a field F, i.e., $\Gamma_F = \mathrm{Gal}(F_{\mathrm{sep}}/F)$, where F_{sep} is a separable closure of F. Then Γ_F acts on the multiplicative group $F_{\mathrm{sep}}^\times$ of nonzero elements of F_{sep}, typically denoted $\mathbb{G}_m$. Thus the cohomology groups

$$H^d(F, \mathbb{G}_m) \ := \ H^d(\Gamma_F, F_{\mathrm{sep}}^\times)$$

are defined for every $d \geq 0$, and are viewed (following Grothendieck) as the étale cohomology of $\operatorname{Spec} F$ with coefficients in the associated sheaf (also denoted $\mathbb{G}_m$), $H^d(F, \mathbb{G}_m) \ \simeq \ H^d_{\mathrm{et}}(\operatorname{Spec} F, \mathbb{G}_m)$.

Theorem 2.1. *Let F be a field.*

- $H^0(F, \mathbb{G}_m) \ = \ F^\times$.
- $H^1(F, \mathbb{G}_m) \ = 0$, *Hilbert's Theorem 90.*
- $H^2(F, \mathbb{G}_m) \ = \mathrm{Br}(F)$, *the Brauer group of F.*

The vanishing of $H^1(F, \mathbb{G}_m)$ is "essentially equivalent" to the exactness of the sequence

$$K_1(L) \xrightarrow{1-\sigma} K_1(L) \xrightarrow{\mathrm{Norm}_{L/F}} K_1(F)$$

for a cyclic field extension L/F with σ denoting a generator of $\mathrm{Gal}(L/F)$ (frequently referred to as *Hilbert's Theorem 90*). This is an early appearance of norm maps in algebraic K-theory.

The only finite groups isomorphic to Γ_F for some field F are the trivial group and $\mathbb{Z}/2$. Further restrictions on Γ_F are implicit in some of the results we proceed to discuss. Let M be a Galois module over F (i.e., M is a discrete Γ_F-module with a continuous action) and consider the graded cohomology ring

$$H^*(F, M^{\otimes *}) \ := \ \bigoplus_{i \geq 0} H^i(F, M^{\otimes i}).$$

If m is an integer prime to $\mathrm{char}(F)$ and $M = \mu_m$ is the Γ_F-module of mth roots of unity in F_{sep}, then the cohomology ring $H^*(F, \mu_m^{\otimes *})$ is related to the Milnor ring $K_*^M(F)$ of F via the norm residue homomorphism defined as follows. The Kummer exact sequence of Galois modules

$$1 \to \mu_m \to F_{\mathrm{sep}}^\times \xrightarrow{m} F_{\mathrm{sep}}^\times \to 1$$

yields a connecting homomorphism $l : F^\times = H^0(F, F_{\mathrm{sep}}^\times) \to H^1(F, \mu_m)$. The cup-product in Galois cohomology yields a homomorphism $(F^\times)^{\otimes n} \to H^n(F, \mu_m^{\otimes n})$ for every $n \geq 0$ sending the tensor $a_1 \otimes \cdots \otimes a_n$ to $l(a_1) \cup \cdots \cup l(a_n)$. One shows that this determines the *norm residue homomorphism* (so named, presumably, because of the Hilbert symbol taking values in the Brauer group)

$$h_n : K_n^M(F)/mK_n^M(F) \to H^n(F, \mu_m^{\otimes n}). \tag{2}$$

If F contains a primitive mth root of unity, we have $\mu_m = \mathbb{Z}/m\mathbb{Z}$, so that in this case $H^*(F, \mu_m^{\otimes *}) \ = \ H^*(F, \mathbb{Z}/m)$.

Theorem 2.1 suggests that one investigates $\mathrm{Br}(F) := H^2(F, \mathbb{G}_m)$. Miraculously, $\mathrm{Br}(F)$ is naturally isomorphic to the group of Brauer equivalence classes of simple F-algebras with center F with the group operation given by the tensor product over F. Thus, the Brauer group can be studied by means of the theory of noncommutative associative algebras and the study of the algebraic geometry of Severi–Brauer varieties. (The Severi–Brauer variety of a central simple algebra A of dimension n^2 is the variety of right ideals in A of dimension n. It is a twisted form of the projective space $\mathbb{P}^{n-1}$.) Conversely, simple algebras can be studied with the help of Galois cohomology.

Let F be a field. A *central simple F-algebra* (c.s.a.) is an (associative) finite-dimensional F-algebra with center F and no nontrivial (two-sided) ideals. By Weddenburn's theorem, every c.s.a. A over F is isomorphic to the matrix algebra $M_r(D)$ over a unique (up to isomorphism) central division F-algebra D (called the *underlying division algebra of* A). Two c.s.a.s A and B over F are *Brauer equivalent* if $M_s(A) \simeq M_t(B)$ for some s and t, or, which is the same, the underlying division algebra of A and B are isomorphic. The tensor product over F endows the set $\mathrm{Br}(F)$ of equivalence classes of all c.s.a.s over F a group structure called the Brauer group of F. The Brauer group $\mathrm{Br}(F)$ is an abelian torsion group. Weddenburn's theorem establishes a bijection between $\mathrm{Br}(F)$ and the set of isomorphism classes of central finite-dimensional division F-algebras. Moreover, two c.s.a.s over F are isomorphic if and only if the classes of A and B in $\mathrm{Br}(F)$ are equal and $\dim(A) = \dim(B)$.

The first computations one typically encounters include $\mathrm{Br}(\mathbb{F}_q) = 0$ for any finite field $\mathbb{F}_q$ and $\mathrm{Br}(\mathbb{R}) \simeq \mathbb{Z}/2$, generated by the class of the classical quaternion algebra. Work of Adrian Albert, Richard Brauer, Helmut Hasse, and Emmy Noether determined $\mathrm{Br}(F)$ for any number field F, showing that every c.s.a. over F is *cyclic*, namely constructed as follows.

Let L/F be a cyclic field extension of degree m and σ a generator of the Galois group and $b \in F^\times$. We introduce an F-algebra structure on the m-dimensional vector space $C(L/F, \sigma, b)$ over L with basis $1, u, u^2, \ldots, u^{m-1}$ by $u^m = b$ and $(xu^i)(yu^j) = x\sigma^i(y)u^{i+j}$ with $x, y \in L$. Then $C(L/F, \sigma, b)$ is a c.s.a. over F of dimension m^2. An F-algebra isomorphic to $C(L/F,\sigma,b)$ for some L/F, σ and b is called a *cyclic algebra*. If F contains a primitive mth root of unity ξ, then $L = F(\alpha)$ where $\alpha = a^{1/m}$ and $\sigma(\alpha) = \xi\alpha$ for some $a \in F^\times$. We write $C(a,b)_\xi$ for the algebra $C(L/F,\sigma,b)$. This algebra is generated over F by two elements v and u subject to the relations $v^m = a$, $u^m = b$ and $vu = \xi uv$.

Possibly the most well known and most influential work of Andrei's concerns this norm residue homomorphism. First, Merkurjev proved that h_2 is an *isomorphism* if $m = 2$, answering a long-standing question of Adrian Albert. This work was much influenced by Andrei and it utilized Andrei's paper [62].

Here is the famous Merkurjev–Suslin theorem.

Theorem 2.2 ([42]). *Let F be a field, let m be an integer prime to* $\mathrm{char}(F)$*, and let $\xi \in F$ be a primitive mth root of unity. Then the norm residue homomorphism*

$$h_2 = h_{F,2} : K_2^M(F)/mK_2^M(F) \ \to \ H^2(F,\mu_m), \quad \{a,b\} \mapsto C(a,b)_\xi, \tag{3}$$

is an isomorphism, where $H^2(F,\mu_m) = \mathrm{Br}(F)[m] \subset Br(F)$ consists of all elements of whose exponent divides m. In particular, the subgroup $\mathrm{Br}(F)[m]$ of the Brauer group is generated by the classes of cyclic algebras $C(a,b)_\xi$ for $a,b \in F^\times$.

We remark that a tensor product of two cyclic algebras is not necessarily cyclic. There are c.s.a.s that are not tensor products of cyclic algebras (see [75]).

The idea of the proof of Theorem 2.2 is as follows. Injectivity of h_2 is proved by induction on the number of symbols in the presentation of an element in K_2 as a sum of symbols, using the passage to the function field of a Severi–Brauer variety splitting a cyclic algebra $C(a,b)_\xi$ and hence the symbol $\{a,b\}$ modulo m. To prove surjectivity of h_2, it suffices to construct a field extension F'/F such that

(1) $K_2(F') = mK_2(F')$ and $\mathrm{Br}(F')[m] = 0$, so $h_{F',m}$ is an isomorphism trivially.
(2) The natural homomorphism $\mathrm{Coker}(h_{F,m}) \to \mathrm{Coker}(h_{F',m})$ is injective.

In fact, we may assume that m is prime and $\xi \in F$. The property (1) implies that every symbol $u = \{a,b\}$ with $a,b \in F^\times$ is contained in $mK_2(F')$. There is a generic way to make u divisible by m over a field extension: as in the proof of injectivity, one passes to the function field $F(X)$ of the *Severi–Brauer* variety X of the algebra $C(a,b)_\xi$. The dimension of X equals $m-1$; since m is assumed to be prime, $(m-1)!$ is relatively prime to m which is useful in applying the Riemann–Roch theorem. Quillen's computation of higher K-theory of X, the Brown–Gersten spectral sequence [10], and Grothendieck's Riemann–Roch theorem yield the surjectivity of $\mathrm{Coker}(h_{F,m}) \to \mathrm{Coker}(h_{F(X),m})$.

Iterating this passage to the function fields of Severi–Brauer varieties for various cyclic algebras, we find a field extension F'/F satisfying (2) and such that $K_2(F')/mK_2(F') = 0$. Finally, one shows that $\mathrm{Br}(F') = 0$ and hence (1).

Theorem 2.2 quickly led to various new algebro-geometric results, typically guided by Andrei. One such development was the following theorem of Andrei's (occurring as Theorem 24.8 in the somewhat difficult to access paper [66]). This theorem partially answered a question of Serre, who asked whether $H^1(F,G) = 0$ for every simply connected semi-simple algebraic group G over a field F of cohomological dimension ≤ 2. In other words, such G have no nontrivial principal homogeneous spaces over F. This is known informally as Serre's Conjecture II.

Theorem 2.3 ([66]). *Let F be a field of cohomological dimension ≤ 2. Then the reduced norm homomorphism $Nrd : A^\times \to F^\times$ is surjective for every central simple algebra A over F of degree n. This implies that Serre's Conjecture II is valid for simply connected groups of inner type A_{n-1}.*

Andrei proved this theorem by first reducing to the case when n is a prime number and A is a central simple algebra of degree n. Using calculations of certain K cohomology and étale cohomology groups of the Severi–Brauer variety of A, Andrei proved that an element $a \in F^\times$ is in the image of the reduced norm map if and only if $l(a) \cup [A] = 0$ in $H^3(F, \mu_n^{\otimes 2})$, where $[A]$ is the class of the algebra A in the subgroup $H^2(F, \mu_n) = \mathrm{Br}(F)[n]$ of the Brauer group of F.

Some years later, Theorem 2.3 was much improved by Eva Bayer-Fluckiger and Raman Parimala in [4]. A recent survey of work on Serre's Conjecture II by Phillipe Gille is given in [29].

Another outcome of [66] was progress on the Grothendieck–Serre conjecture that $H^1_{\mathrm{et}}(R, G) \to H^1_{\mathrm{et}}(K, G_K)$ is injective for any flat reductive group scheme G over a regular local ring R with field of fractions K. In [51], Andrei and Ivan Panin proved a special case of this conjecture (for $G = SL_{1,D}$ with D an Azumaya algebra over R). This was the starting point of the recent proof of the Grothendieck–Serre Conjecture by Roman Fedorov and Panin in [18].

3. Milnor K-theory versus algebraic K-theory

By Matsumoto's theorem, Milnor K-theory $K_*^M(F)$ of a field F agrees in degree 2 with $K_2(F)$. However, beginning in degree 3, $K_i^M(F)$ can be quite different from $K_i(F)$; for example, $K_3(\mathbb{F}_q)$ is nonzero for a finite field $\mathbb{F}_q$, whereas $K_3^M(\mathbb{F}_q) = 0$. Much of Andrei's work in the 1980s following the Merkurjev–Suslin theorem (which concerns $K_2(F)$) was dedicated to exploring the relationship between $K_*^M(F)$ and $K_*(F)$. In this section, we review some of Andrei's results obtained during this period.

We define the *indecomposable group* $K_3(F)_{\mathrm{nd}}$ as the cokernel of $K_3^M(F) \to K_3(F)$. Using the definition of $K_*^M(F)$ as an explicit quotient of the tensor algebra $T^*(F^\times)$, one can show that groups $K_2(F)$ and $K_3(F)_{\mathrm{nd}}$ may contain large, uniquely divisible subgroups that cannot be detected by torsion and cotorsion. On the other hand, the following theorem of Merkurjev and Suslin determines the torsion and cotorsion of $K_2(F)$, extending Theorem 2.2.

Theorem 3.1 ([44], see also [37]). *Let m be an integer prime to* $\mathrm{char}(F)$. *Then there is an exact sequence*

$$\begin{aligned} 0 \to H^0(F, \mu_m^{\otimes 2}) \to K_3(F)_{\mathrm{nd}} \xrightarrow{m} K_3(F)_{\mathrm{nd}} \to H^1(F, \mu_m^{\otimes 2}) \\ \to K_2(F) \xrightarrow{m} K_2(F) \to H^2(F, \mu_m^{\otimes 2}) \to 0. \end{aligned}$$

If $p = \mathrm{char}(F) > 0$, then the group $K_3(F)_{\mathrm{nd}}$ is uniquely p-divisible.

Theorem 3.1 implies that the group $K_3(F)_{\mathrm{nd}}$ is never trivial! The theorem is proved using the analogue of Hilbert's Theorem 90 for relative K_2-groups of extensions of semilocal principal ideal domains. We remark that the motivic spectral sequence of Theorem 5.1 (constructed considerably later than the appearance of Theorem 3.1) yields an isomorphism $K_3(F)_{\mathrm{nd}} \simeq H^1_M(F, \mathbb{Z}(2))$.

One application of Theorem 3.1 is the computation that $K_3(\mathbb{Q})_{\mathrm{nd}}$ is a cyclic group of order n, where n is the largest integer such that $i^2 - 1$ is divisible by

n for all i prime to n. This integer n equals 24. One can deduce from this that $K_3(\mathbb{Q}) = \mathbb{Z}/48\mathbb{Z}$, a well-known result of Ronnie Lee and Robert Szczarba [35].

Let F be a field of characteristic not 2, and let $W(F)$ be the Witt ring of F. By definition, $W(F)$ is the factor ring of the Grothendieck ring of the category of nondegenerate quadratic forms over F by the ideal of hyperbolic forms. Write $I(F)$ for the *fundamental ideal* in $W(F)$ consisting of the classes of even-dimensional forms. For any n the nth power $I(F)^n$ of the fundamental ideal is generated by the classes of *Pfister forms*

$$\langle\langle a_1, a_2, \ldots, a_n\rangle\rangle = \langle 1, -a_1\rangle \otimes \langle 1, -a_2\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$$

with $a_1, a_2, \ldots, a_n \in F^\times$.

The Milnor K-theory of F is related to the Witt ring of F by the homomorphisms

$$s_n : K_n^M(F)/2K_n^M(F) \to I(F)^n/I(F)^{n+1}$$

taking a symbol $\{a_1, a_2, \ldots, a_n\}$ to the class of the Pfister form $\langle\langle a_1, a_2, \ldots, a_n\rangle\rangle$. Milnor conjectured in [47] that s_n is always an isomorphism. Another application of Theorem 3.1 is the verification of this conjecture of Milnor for $n = 3$, proved by showing that Andrei's map $K_3(F) \;\to\; K_3^M(F)$ (see Theorem 3.4) is trivial modulo 2.

Theorem 3.2 ([43], see also [2])**.** *Let F be a field whose characteristic is not 2. Then Milnor's map*

$$s_3 : K_3^M(F)/2K_3^M(F) \;\to\; I(F)^3/I(F)^4$$

is an isomorphism.

Here is the impressive theorem of Dimitri Orlov, Alexander Vishik, and Vladimir Voevodsky [50] proving the validity of Milnor's conjecture for all n.

Theorem 3.3 ([50])**.** *Let F be a field of characteristic not equal to 2, let $W(F)$ denote the Witt ring of equivalence classes of nondegenerate symmetric quadratic forms over F, and let $I(F) \subset W(F)$ denote the ideal of even forms. Then Milnor's mapping $(F^*)^n$ to $I(F)^n/I(F)^{n+1}$ sending $(a_1, \ldots, a_n)$ to $\langle\langle a_1, \ldots, a_n\rangle\rangle$ induces an isomorphism from the Milnor K-theory of F to the associated graded ring (with respect to $I(F)$) of the Witt ring of F,*

$$s_* : K_*^M(F)/2 \;\xrightarrow{\sim}\; \mathrm{gr}(W(F)).$$

Theorem 3.3 is a fairly direct consequence of Voevodsky's proof of the mod-2 Bloch–Kato conjecture (see Theorem 5.3) established by Voevodsky in [80] and [81].

The relationship between the Milnor K-theory $K_*^M(F)$ and the algebraic K-theory $K_*(F)$ of F is of great interest. The next theorem of Andrei's gives considerable information about this relationship, showing it is closely related to the homological stability for GL_n.

Theorem 3.4 ([61], [65])**.** *If F is an infinite field, then*

$$H_n(GL_n(F)) \;\xrightarrow{\sim}\; H_n(GL_\infty(F)),$$
$$H_n(GL_n(F))/\,\mathrm{im}\{H_n(GL_{n-1}(F))\} \;\simeq K_n^M(F).$$

Moreover, the natural composition

$$K_n^M(F) \to K_n(F) \to H_n(GL_\infty(F)) \;\simeq\; H_n(GL_n(F)) \;\to\; K_n^M(F)$$

is multiplication by $(n-1)!$.

This is a somewhat remarkable result, for Milnor K-theory is defined by generators and relations, whereas $K_*(F)$ is defined as homotopy groups of some infinite-dimensional space and thus seemingly very inaccessible to computations.

One consequence of the preceding theorems is another result of Andrei's.

Corollary 3.5. *The image of $K_3(F) \to K_3^M(F)$ coincides with the kernel of $K_3^M(F) \to I(F)^3/I(F)^4$, and hence coincides with $2K_3^M(F)$ by the bijectivity of s_3.*

Using the previous theorem, Andrei together with Yuri Nesterenko proved the following theorem relating Milnor K-theory to motivic cohomology. As discussed in Section 4, motivic cohomology was first formulated by Bloch as *higher Chow groups* in [8], and it was then reformulated and developed into a powerful theory by Suslin and Voevodsky. Shortly after the following theorem appeared, a new proof was given by Burt Totaro [78].

Theorem 3.6 ([49])**.** *Let F be a field, and let $H^p_M(F, \mathbb{Z}(q))$ denote Bloch's higher Chow group $\mathrm{CH}^q(F, 2q-p)$ of F. Then*

$$H^n_M(F, \mathbb{Z}(n)) \simeq K^M_n(F).$$

This too is a somewhat surprising theorem, for Milnor K-theory has an explicit, naive description, whereas motivic cohomology involves sophisticated constructions.

4. K-theory and cohomology theories

In this section we discuss two further important theorems of Andrei's from the 1980s: Theorem 4.1 computes the torsion and cotorsion of the algebraic K-theory in all degrees of an algebraically closed field, and Theorem 4.2 provides an alternate approach to producing *étale homology.* These theorems are of considerable interest in their own right. In addition, they introduced important insights for the formulation of Suslin–Voevodsky motivic cohomology; namely, Suslin rigidity and the consideration of (Suslin) complexes.

Beĭlinson, Bloch, Lichtenbaum, and Soulé all contributed to a grand vision of the role of algebraic K-theory in arithmetic algebraic geometry. Algebraic K-theory should be some sort of universal cohomology theory with *realizations* in familiar cohomology theories, it should carry much arithmetic information, it should be determined by étale cohomology in high degrees, and it should have properties analogous to those of topological K-theory. These have been codified as the *Beĭlinson conjectures* [5].

In the early 1970s, Quillen and Lichtenbaum proposed a close relationship between algebraic K-theory and étale cohomology. Of particular interest was $K_*(F)$ for a field F. Quillen formulated algebraic K-theory so that $K_{2i}(\mathbb{F}_q) = 0$, $K_{2i-1}(\mathbb{F}_q) = \mathbb{Z}/q^i - 1$ for any $i > 0$ and any prime power $q = p^d$; this determines the algebraic K-theory of the algebraic closure of a finite field which has close similarities to the topological K-theory of a point.

Since an algebraically closed field F appears as a *point* in the étale topology, the Quillen–Lichtenbaum conjecture for F predicts a similar computation for $K_*(F, \mathbb{Z}/n)$. Andrei proved this as stated below. (When he announced this theorem and gave its proof at a meeting in Paris, his mathematical audience vigorously applauded.)

Theorem 4.1 ([63], [64]). *Let F be an algebraically closed field, and let n be a positive integer invertible in F. Then*

$$K_{2i}(F, \mathbb{Z}/n) \ = \ \mathbb{Z}/n, \quad K_{2i+1}(F, \mathbb{Z}/n) = 0, \quad i \geq 0.$$

Andrei's proof of this theorem proceeded as follows, extending the known computation of $K_*(-, \mathbb{Z}/n)$ for the algebraic closure of $\mathbb{F}_p$ to any algebraically closed field F of characteristic p. Consider a smooth, connected curve over the algebraically closed field F with field of fractions $E = F(C)$. Andrei constructs a specialization map $K_*(E, \mathbb{Z}/n) \to K_*(F, \mathbb{Z}/n)$ using a local parameter at an F-rational point of the curve C. *Suslin's rigidity theorem*, which requires properties of the transfer map for algebraic K-theory and the divisibility of $\mathrm{Pic}^0(C)$, tells us that this specialization map is independent of the point $c \in C$. This leads quickly to the statement that if $X = \operatorname{Spec} A$ is a smooth, connected variety over F and x, x' are F-rational points of X, then the two induced maps $K_*(A, \mathbb{Z}/n) \to K_*(F, \mathbb{Z}/n)$ are equal. Andrei's proof that $K_*(F, \mathbb{Z}/n) \xrightarrow{\sim} K_*(L, \mathbb{Z}/n)$ is completed using a "trick" to base change from F to L and comparing maps induced by $A \to F \to L$ and $A \subset \mathrm{frac}(A) \subset L$.

Subsequently, Andrei verified in [64] the conjecture for algebraically closed fields of characteristic 0 using an argument of Ofer Gabber.

As Suslin first observed, and various other mathematicians have employed, this technique of Suslin rigidity applies to various other cohomology theories and K-theories, applies with the base field F replaced by a smooth scheme over F, and even applies with smoothness dropped if $K_*(-, \mathbb{Z}/n)$ is replaced by $K'_*(-, \mathbb{Z}/n)$. This proof typifies both Andrei's originality and his considerable algebraic prowess, motivated by geometric insight.

In 1987, Andrei introduced the Suslin complex $\mathrm{Sus}_*(X)$ associated to a variety X over a field F:

$$\mathrm{Sus}_*(X) \ = \ n \mapsto \mathrm{Hom}(\Delta^n, (\coprod_{d=0}^{\infty} S^d(X))^+).$$

Here, $S^d(X)$ is the d-fold symmetric product of X, the quotient of $X^{\times d}$ by the symmetric group Σ_d; $\Delta^n := \operatorname{Spec} k[t_0, \ldots, t_n]/(\sum_{i=0}^n t_i = 1)$ is the algebraic n-simplex over k; $\mathrm{Hom}(-,-)$ in this formula designates morphisms of varieties over F. Motivation for this definition comes from the Dold–Thom theorem [14] in algebraic topology, which asserts that the homotopy groups of the simplicial abelian group $\mathrm{Sing}(\coprod_{d=0}^{\infty} S^d(T))^+)$ can be naturally identified with the integral homology of a CW complex T. One might not expect this definition to be useful because many varieties X admit few maps from affine spaces; Andrei's insight was that symmetric powers of a variety X do admit many maps from the affine space $\Delta^d \simeq \mathbb{A}^d$.

Using this definition, Suslin and Voevodsky proved the following remarkable theorem. In some sense, they achieved a primary goal of Grothendieck for étale cohomology using Andrei's "naive" Suslin complex. They also prove a similar statement for varieties over an algebraic closed field of characteristic $p > 0$ provided that p does not divide n. Their proof uses Suslin rigidity and various Grothendieck topologies introduced by Voevodsky.

Theorem 4.2 ([73]). *If X is a quasi-projective variety over $\mathbb{C}$, then the natural map*

$$\pi_i(\mathrm{Sus}_*(X), \mathbb{Z}/n) \ \to \ H_i(X(\mathbb{C})^{an}, \mathbb{Z}/n), \quad i \geq 0,$$

is an isomorphism.

For the understanding of the algebraic K-theory of a scheme X which is "not equivalent to a point" in the étale topology, one must incorporate more information about X. This was made clear by Grothendieck at the inception of K-theory. A fundamental theorem of Grothendieck [9] (a consequence of Grothendieck's Riemann–Roch theorem) asserts that the Chern character

$$\mathrm{ch}_* : K_0(X) \otimes \mathbb{Q} \ \to \ \mathrm{CH}^*(X) \otimes \mathbb{Q} \tag{4}$$

is a ring isomorphism for any smooth, connected algebraic variety X over a field F, where $\mathrm{CH}^i(X)$ denotes the *Chow group* of rational equivalence classes of algebraic cycles on X of codimension i. These Chow groups are not simply étale cohomology groups.

A difficult theorem of Robert Thomason [77] established a convergent spectral sequence whose E_2-page involved the étale cohomology of a smooth variety with $\mathbb{Z}/n$-coefficients and which converged to $K_*(X, \mathbb{Z}/n)[1/\beta]$, the localization of algebraic K-theory obtained by inverting the *Bott element* $\beta \in K_2(X, \mathbb{Z}/n)$. Étale K-theory, formulated by Dwyer and Friedlander [16], provided the abutment for a spectral sequence whose E_2-term was étale cohomology. Nevertheless, it became clear that one required a more elaborate cohomology theory than étale cohomology to "approximate" algebraic K-theory. Beĭlinson's conjectures encompassed the existence of a well-behaved cohomology theory involving complexes of sheaves which could serve as a suitable refinement of étale cohomology and which would relate to algebraic K-theory through a *motivic spectral sequence*.

In [8], Bloch formulated bigraded *higher Chow groups* which are related to $K_*(X)$. For each $n \geq 0$, Bloch introduced $\mathrm{CH}^i(X, n)$, defined in terms of the nth homology group of the complex of codimension i cycles on $X \times \Delta^*$. Rationally, $\mathrm{CH}^i(X, n)$ is the ith weighted piece of $K_n(X)$; more generally, Bloch anticipated a spectral sequence relating his higher Chow groups to algebraic K-groups. Bloch's higher Chow groups proved to be a major step towards realizing Beĭlinson's vision.

Since groups of equivalence classes of algebraic cycles are so closely related to algebraic K-groups, we briefly outline Grothendieck's proof (see (4)) that the Chern character ch_* of (4) is an isomorphism.

For a scheme X, write $K_*(X)$ for the K-groups of the exact category of vector bundles over X. This is a graded ring cohomology theory, contravariant in X. If X is Noetherian, we define $K'_*(X)$ as the K-groups of the abelian category $\mathcal{M}(X)$ of coherent $\mathcal{O}_X$-modules. The assignment $X \mapsto K'_*(X)$ is a homology theory, covariant with respect to proper morphisms. For any $i \geq 0$, let $\mathcal{M}(X)^{(i)}$ be the full subcategory of $\mathcal{M}(X)$ consisting of all $\mathcal{O}_X$-modules with codimension of support at least i. The images $K'_*(X)^{(i)}$ of the natural homomorphisms $K_*(\mathcal{M}(X)^{(i)}) \to K'_*(X)$ form a *topological filtration* on $K'_*(X)$.

Assume now that X is a regular scheme. Then the natural homomorphism $K_*(X) \to K'_*(X)$ is an isomorphism. Thus, $K_*(X)$ is a graded ring together with the topological filtration by the ideals $K_*(X)^{(i)} := K'_*(X)^{(i)}$ with the subsequent factor groups $K_*(X)^{(i/i+1)}$. There is a well-defined surjective graded ring homomorphism

$$\varphi_* : \mathrm{CH}^*(X) \to K_0(X)^{(*/*+1)}$$

taking a class $[Z]$ of a codimension i closed subvariety $Z \subset X$ to $[\mathcal{O}_Z]$ in $K_0(X)^{(i/i+1)}$, the class of the structure sheaf of Z.

In order to construct homomorphisms in the opposite direction, Grothendieck constructed *Chern classes*:

$$c_i : K_0(X) \to \mathrm{CH}^i(X), \quad i \geq 0.$$

These are maps (not necessarily homomorphisms), functorial in X. The class c_0 sends all of $K_0(X)$ to $1 \in \mathrm{CH}^0(X)$. The map c_1 takes the class of a vector bundle E over X to $\det(E) \in \mathrm{Pic}(X) = \mathrm{CH}^1(X)$. These properties together with the Whitney sum formula $c_n(a+b) = \sum_{i+j=n} c_i(a)c_j(b)$ and the splitting principle uniquely determine the Chern classes. For every $i > 0$, the restriction of c_i to $K_*(X)^{(i)}$ is a group homomorphism trivial on $K_*(X)^{(i+1)}$. Hence, c_i yields a group homomorphism

$$\psi_i : K_0(X)^{(i/i+1)} \to \mathrm{CH}^i(X).$$

Grothendieck's Riemann–Roch theorem implies that both compositions $\varphi_i \circ \psi_i$ and $\psi_i \circ \varphi_i$ are multiplication by $(-1)^{i-1}(i-1)!$. For certain classes of varieties (for example for Severi–Brauer varieties of dimension $l-1$, l prime, used in the proof of Theorem 2.2) φ_* is an isomorphism, so that computations of the topological filtrations on the Grothendieck group $K_0(X)$ are particularly useful for the study of the Chow ring $\mathrm{CH}^*(X)$.

Chern classes with values in the Chow groups are a special case of more general constructions of Chern classes with values in an arbitrary oriented generalized cohomology theory. The Chern classes can also be extended to higher K-groups $K_*(X)$ with values in certain groups of étale cohomology (see [58] and [30]) or motivic cohomology (see [31] and [52]).

5. Motivic cohomology and K-theories

In the 1990s, Andrei enabled many of the foundational results for *Suslin–Voevodsky motivic cohomology* [40], whose origins can be traced to the Suslin complex $\mathrm{Sus}_*(X)$ and Suslin rigidity discussed in the previous section. As shown by Andrei in Theorem 5.4 below, Bloch's higher Chow groups $\mathrm{CH}^*(X,*)$ (further studied by Levine; see [38]) often agree with Suslin–Voevodsky motivic cohomology $H^*_M(X,\mathbb{Z})$. In contrast with Bloch's higher Chow groups, Suslin–Voevodsky motivic cohomology is more amenable to arguments using functoriality and local behavior; moreover, $H^*_M(X,\mathbb{Z})$ fits into the general framework of $\mathbb{A}^1$-homotopy theory of Fabien Morel and Voevodsky [48], enabling Voevodsky to prove many of the conjectures (now theorems) we have discussed: Milnor's conjecture, the Quillen–Lichtenbaum conjecture, the Beĭlinson–Lichtenbaum conjectures, and the Bloch–Kato conjecture.

In this section we discuss numerous foundational results for motivic cohomology proved by Andrei. We also return to the norm residue homomorphism, briefly discussing Voevodsky's dramatic results.

Voevodsky introduced important innovations into the study of algebraic varieties, continuing the historical development of the subject following work of Grothendieck. The first was to enlarge the set of morphisms from X to Y to include *finite correspondences* from X to Y. Another innovation was to focus on presheaves ϕ on the category of smooth varieties and finite correspondences which are homotopy invariant: the projection $X \times \mathbb{A}^1 \to X$ induces an isomorphism $\phi(X) \xrightarrow{\sim} \phi(X \times \mathbb{A}^1)$ for any smooth variety X. Yet another innovation was Voevodsky's introduction of new Grothendieck topologies, especially the Nisnevich topology.

The graph of a morphism $X \to Y$ of varieties over F can be viewed as a correspondence (a cycle on $X \times Y$) that projects isomorphically onto X. A *finite correspondence* from some smooth, connected X to Y is a cycle α on $X \times Y$ such that every prime component of α is finite and surjective over X. For example, if Y is also irreducible, then a finite, surjective morphism from Y to X can be viewed as a finite correspondence from X to Y. We have the category $\mathrm{Cor}(F)$ of finite correspondences: the objects are smooth varieties over F and morphisms from X to Y are finite correspondences from X to Y. A presheaf of abelian groups with transfers is a contravariant functor A from $\mathrm{Cor}(F)$ to abelian groups. Thus, if A is a presheaf with transfers, then a finite, surjective morphism $Y \to X$ with Y irreducible is equipped with a transfer (norm) homomorphism $A(Y) \to A(X)$. The role of Suslin rigidity arises in establishing the homotopy invariance of cohomological complexes associated to presheaves with transfers.

There are *motivic complexes* of étale sheaves with transfers $\mathbb{Z}(q)$ for $q \geq 0$. In fact, $\mathbb{Z}(0)$ and $\mathbb{Z}(1)$ are quasi-isomorphic to the sheaves $\mathbb{Z}$ and $\mathbb{G}_m$ placed in degree 0 and 1, respectively. One defines the *motivic* and *étale motivic* cohomology groups of a smooth variety X with coefficients in an abelian group A by

$$H^p_M(X, A(q)) := \mathbb{H}^p_{\mathrm{Zar}}(X, A \otimes \mathbb{Z}(q)), \quad H^p_{\mathrm{et}}(X, A(q)) := \mathbb{H}^p_{\mathrm{et}}(X, A \otimes \mathbb{Z}(q)).$$

The formulation of motivic cohomology by Suslin and Voevodsky led to much progress on conjectures made a decade earlier by Beĭlinson, Bloch, Lichtenbaum, and Soulé. For example, the following theorem provides the analogue for algebraic K-theory of the Atiyah–Hirzeburch spectral sequence for topological K-theory. Although many mathematicians contributed to the proof of this result, Andrei did most of the "heavy lifting."

Theorem 5.1 ([68], [27])**.** *Let X be a smooth quasi-projective variety over a field. Then there is a strongly convergent spectral sequence*

$$E_2^{p,q} = H_M^{p-q}(X, \mathbb{Z}(-q)) \;\Rightarrow\; K_{-p-q}(X). \tag{5}$$

One sees more clearly the interplay between Milnor K-theory and algebraic K-theory with the help of motivic cohomology (and the above spectral sequence; see Theorem 3.6). We start with the observation that $H^1_M(F, \mathbb{Z}(1)) = \mathbb{G}_m(F) = F^\times$. The product in motivic cohomology yields a homomorphism $(F^\times)^{\otimes p} \to H^p_M(F, \mathbb{Z}(p))$. The image of a tensor $a_1 \otimes \cdots \otimes a_p$ is trivial if $a_i + a_j = 1$ for some $i \neq j$. Hence we get a homomorphism $K_p^M(F) \to H^p_M(F, \mathbb{Z}(p))$ which is an isomorphism by Theorem 3.6. The norm maps for Milnor K-groups correspond to the norm maps in motivic cohomology created by the structure of presheaves with transfers.

There is a natural homomorphism

$$H^p_M(X, \mathbb{Z}(q)) \;\to\; H^p_{\mathrm{et}}(X, \mathbb{Z}(q)).$$

The integral *Beĭlinson–Lichtenbaum conjecture* asserts that this is an isomorphism when $p \leq q$ and a monomorphism when $p = q+1$. Replacing $\mathbb{Z}(q)$ by $\mathbb{Z}/\ell(q)$ for some prime ℓ, we obtain the mod-ℓ Beĭlinson–Lichtenbaum conjecture which has an equivalent formulation (used in the statement of Theorem 5.2) asserting that for every prime ℓ the natural homomorphism

$$H^p_M(X, \mathbb{Z}/\ell(q)) \to\; H^p_{\mathrm{et}}(X, \mu_\ell^{\otimes q})$$

is an isomorphism if $p \leq q$ and a monomorphism if $p = q+1$. The mod-ℓ Beĭlinson–Lichtenbaum conjecture is now a theorem proved by Voevodsky (see Theorems 5.2 and 5.3).

The mod-ℓ Bloch–Kato conjecture asserts that the norm residue homomorphsm

$$h_n : K_n^M(F)/\ell \ \to \ H^n(F, \mu_\ell^{\otimes n})$$

is an isomorphism for all n, provided ℓ is invertible in the field F. We state an important theorem of Suslin and Voevodsky which closely links this conjecture to the Beĭlinson-Lichtenbaum conjecture. The paper [74] not only gives a carefully written, well organized presentation of the proof of this important link, but it also presents details of various key results of Voevodsky.

Theorem 5.2 ([74])**.** *Let F be a field, and let ℓ be a prime invertible in F. Then the following assertions are equivalent for any smooth, quasi-projective variety X over F:*

(1) *The* mod-l Bloch–Kato conjecture *for F in weight n asserts that the norm residue homomorphsm*

$$h_n : K_n^M(F)/\ell \ \to \ H^n(F, \mu_\ell^{\otimes n})$$

is an isomorphism;

(2) *The* mod-l Beĭlinson–Lichtenbaum conjecture *in weights $q \leq n$ asserts that*

$$H^p_M(X, \mathbb{Z}/\ell(q)) \ \simeq \ H^p_{\mathrm{et}}(X, \mu_\ell^{\otimes q}), \ p \leq q; \quad H^{q+1}_M(X, \mathbb{Z}/\ell(q)) \ \hookrightarrow \ H^{q+1}_{\mathrm{et}}(X, \mu_\ell^{\otimes q}).$$

We remark that the mod-ℓ Bloch–Kato conjecture is essentially the diagonal portion of the mod-ℓ Beĭlinson–Lichtenbaum conjecture, yet the inductive argument for the mod-ℓ Bloch–Kato conjecture requires the verification of earlier nondiagonal cases of the mod-ℓ Beĭlinson–Lichtenbaum conjecture. The Beĭlinson–Lichtenbaum conjecture admits a precise formulation in terms of truncations of complexes; with this formulation, the conjecture is a statement that a certain map of complexes is a quasi-isomorphism.

With considerable input from Andrei, Markus Rost, Charles Weibel and others, Voevodsky proved the following spectacular result, the mod-ℓ Bloch–Kato conjecture. Theorem 5.3 partially realizes the vision of Beĭlinson, closely related to conjectures of Bloch, Lichtenbaum, and Soulé, a vision which has served as a template for much of the work on motivic cohomology. A detailed exposition of the proof of this theorem is given in the book [32] by Christian Haesemeyer and Weibel.

Theorem 5.3 ([82])**.** *Let F be a field, and let ℓ be a prime invertible in F. For all $n \geq 0$,*

$$K_n^M(F)/\ell \ \simeq \ H^n(F, \mu_\ell^{\otimes n}).$$

Consequently, for any smooth variety over F,

$$H^p_M(X, \mathbb{Z}/\ell(q)) \ \simeq \ H^p_{\mathrm{et}}(X, \mu_\ell^{\otimes q}), \ p \leq q; \quad H^{q+1}_M(X, \mathbb{Z}/\ell(q)) \ \hookrightarrow \ H^{q+1}_{\mathrm{et}}(X, \mu_\ell^{\otimes q}).$$

Voevodsky proves Theorem 5.3 with an argument which proceeds by induction on n. A significant component of Voevodsky's proof of Theorem 5.3 is the existence and properties of suitable splitting varieties for symbols in Milnor K-groups of F. This is foreshadowed by the role of the Severi–Brauer variety for a symbol $\alpha = \{a_1, a_2\} \in K_2^M(F)$ appearing in the proof of Theorem 2.2.

The existence of norm varieties (generic splitting varieties of dimension $\ell^{n-1} - 1$) for an arbitrary symbol $\alpha = \{a_1, a_2, \ldots, a_n\} \in K_n^M(F)$ modulo ℓ was proved by Markus Rost. In a series of clear, detailed lectures (notes of which by Seva Joukhovitski served as the basis for [72]), Andrei established the basic properties of these norm varieties needed for Voevodsky's proof. The proofs Andrei gives follow Rost's unpublished results, notably Rost's degree formula and Rost's chain lemma.

Let X be a norm variety of an n-symbol α modulo ℓ. Consider the simplicial scheme $\mathcal{X}$ with $\mathcal{X}_n = X^{n+1}$ whose face maps are given by various projections. The motive of $\mathcal{X}$ is independent of the choice of the norm variety of α modulo ℓ. Voevodsky proved triviality of the motivic cohomology group $H^{n+1}_M(\mathcal{X}, \mathbb{Z}(n))$; he then used this vanishing to deduce the validity of the Bloch–Kato conjecture.

Somewhat surprisingly, the triviality of $H^{n+1}(\mathcal{X}, \mathbb{Z}(n))$ (together with the tools used in Voevodsky's proof) yields a computation of the motivic cohomology groups $H^i_M(\mathcal{X}, \mathbb{Z}(j))$ for all i and j (see [45] and [84]).

The following theorem of Andrei's makes explicit the close relationship of Bloch's higher Chow groups and Suslin–Voevodsky motivic cohomology.

Theorem 5.4 ([67])**.** *Let X be an equidimensional quasi-projective scheme over an algebraically closed field F of characteristic* 0*. Assume that $i \geq d := \dim X$. Then*

$$\mathrm{CH}^i(X, n; \mathbb{Z}/\ell) \;=\; H_c^{2(d-i)+n}(X, \mathbb{Z}/\ell(d-i))^{\#}; \tag{6}$$

in other words, the mod-ℓ *bigraded higher Chow groups of Bloch equal the* mod-ℓ *Suslin–Voevodsky bigraded motivic cohomology groups with compact supports.*

Andrei's last published paper extended results of Suslin–Voevodsky motivic cohomology for smooth varieties over a perfect field F of characteristic $p > 0$ by showing how to avoid the assumption that F is perfect. Andrei proves that one can simply base change to the separable closure F_∞ of F and apply the existing theory for varieties over F_∞. Step by step, Andrei verifies that the theory developed by Suslin and Voevodsky applies without the assumption that F be perfect, provided that one considers presheaves with transfers of $\mathbb{Z}[1/p]$-modules. His primary goal is to prove that every homotopy sheaf with transfers of $\mathbb{Z}[1/p]$-modules is strictly homotopy invariant, a key result for the Suslin–Voevodsky theory.

We state Andrei's final theorem in his final paper, giving the flavor of the mathematics involved. For those who wish precision, we mention that $DM_p^-(F)$ appearing in the statement of Theorem 5.5 is the full subcategory of the derived category of bounded above complexes of Nisnevich sheaves with transfers of $\mathbb{Z}[1/p]$-modules consisting of those complexes whose cohomology sheaves are homotopy invariant.

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Theorem 5.5 ([69])**.** *Let E/F be an arbitrary field extension, and consider an arbitrary $A^\bullet \in DM_p^-(F)$. For any smooth scheme X over F, there is a natural isomorphism*

$$\mathrm{Hom}(M_p(X), A^\bullet)_E \;\simeq\; \mathrm{Hom}(M_p(X_E), A^\bullet_E),$$

where the left-hand side is the base change to E of the internal Hom *of $DM_p^-(F)$, and the right-hand side is the internal* Hom *of $DM_p^-(E)$.*

It is interesting to observe that during the development of Suslin–Voevodsky motivic cohomology there was a parallel development of *semitopological theories* initiated by H. Blaine Lawson in [34] and continued in various papers by numerous authors. We point out the formulation of morphic cohomology by Friedlander and Lawson in [21] and the work of Friedlander and Mark Walker in [28]. In a paper by Friedlander, Haesemeyer, and Walker [20] an interesting conjecture by Andrei was stated and investigated with the aim of relating morphic cohomology and singular cohomology of complex algebraic varieties in the spirit of the Bloch–Lichtenbaum conjecture. This conjecture is related to many classical conjectures. One result concerning such relationships is given by Beĭlinson [7].

6. Modular representation theory

This mathematical subject is one that attracted Andrei's attention late in his career, but its appeal to him is natural. Andrei gave qualitative information about the (Hochschild) cohomology of finite group schemes over a field k of characteristic $p > 0$, extending known results for finite groups, and he investigated the actions of these finite group schemes on finite-dimensional vector spaces over k (in other words, modular representations). On the one hand, Andrei answered general structural questions by developing new tools and by extending known techniques in a highly nontrivial manner. On the other hand, Andrei's algebraic insights provided computations and examples that were previously inaccessible.

Andrei's most cited paper, joint with Friedlander, proves the following theorem. This is a generalization of a classical theorem of Leonard Evens [17] and Boris Venkov [79].

Theorem 6.1 ([26]). *Let G be a finite group scheme over a field k. Then $H^*(G,k)$ is a* finitely generated *algebra over k.*

Moreover, if M is a G-module finite-dimensional over k, then $H^*(G,M)$ is a finitely generated module over $H^*(G,k)$.

This is a first suggestion that one can find a common context for finite groups, restricted enveloping algebras of finite-dimensional restricted Lie algebras, and other finite group schemes. The outline of proof for this theorem has been used in other contexts (for example, in the recent paper by Friedlander and Cris Negron [22]). At its heart, it requires a proof of the existence of certain cohomology classes which can serve as generators. The existence proof of Theorem [26] explicitly constructs these classes (in high degree) using extensions in the category of strict polynomial functors (which are not actually functors).

These strict polynomial functors have led to numerous explicit calculations of Ext groups by Andrei and others (e.g., [19]). Furthermore, Antoine Touzé and Wilberd van der Kallen in [76] used this technology to prove that the subalgebra of G-invariants of $H^*(G,A)$ is finitely generated, where G is a reductive group over a field and A is a finitely generated G-algebra; this extends the classical result that the algebra of G-invariants $H^0(G,A)$ of A is finitely generated. As another example, Christopher Drupeiski in [15] extended the arguments of the above theorem in order to prove its generalization to finite supergroup schemes.

Theorem 6.1 is the foundational result enabling a theory of supports for representations of finite group schemes, providing a geometric interpretation of cohomological invariants for such representations. Among the most geometric and informative results in this theory of supports are those proved in two papers by Suslin, Friedlander, and Christopher Bendel [70], [71] concerning infinitesimal groups schemes. The following theorem states central results of these two papers.

We remind the reader that an infinitesimal group scheme H over k is an affine group scheme represented by a finitely dimensional, local k-algebra $k[G]$ (so that $k[G]$ is equipped with the structure of a Hopf algebra over k). An important example of infinitesimal group schemes is the rth Frobenius kernel of the additive group $\mathbb{G}_a$ for some $r > 0$, usually denoted $\mathbb{G}_{a(r)}$. The coordinate algebra $k[\mathbb{G}_{a(r)}]$ equals $k[T]/t^{p^r}$ with dual algebra $k\mathbb{G}_{a(r)} := (k[\mathbb{G}_{a(r)}])^{\#}$ isomorphic to $k[u_0, \ldots, u_{r-1}]/(u_i^{p^r})$.

Theorem 6.2 ([70], [71]). *Let G be a connected affine group scheme over a field k of positive characteristic, and let r be a positive integer. Then the morphisms $\mathbb{G}_{a(r)} \to G$ of group schemes over k (i.e., the height r, one-parameter subgroups of G) are the k-points of an affine scheme $V_r(G)$.*

There is a natural map of finitely generated commutative k-algebras

$$\psi : k[V_r(G)] \ \to \ H^*(G_{(r)}, k)$$

which induces a homeomorphism *on prime ideal spectra.*

Theorem 6.2 is reminiscent of Quillen's description of the spectrum of the cohomology of a finite group in [53]; in the special case $r = 1$, this recovers a theorem of Friedlander and Brian Parshall [23] and eliminates the condition on the prime p required in that earlier paper.

Andrei's computational power is clearly evident in [70] and [71], which provide a qualitative description of the cohomology of infinitesimal group schemes. The arguments required to prove the various results of Theorem 6.2 involve questions already considered for G a finite group (detection of cohomology classes modulo nilpotents, characteristic classes) but formulated now in the more general context of group schemes. Computations with characteristic classes become elaborate, but fortunately one does not always need these computations in closed form. One can see the origins of the theory produced in the foundational work of Jon Carlson for *rank varieties* for elementary abelian groups [11] and of Friedlander and Parshall [23] for restricted Lie algebras.

Considering the example of $G = GL_N$ gives a flavor of the information provided by Theorem 6.2. The scheme $V_r(GL_N)$ is the scheme of r-tuples $(B_0, \ldots, B_{r-1})$ of pairwise commuting, p-nilpotent $N \times N$ matrices. Consequently, $k[V_r(GL_N]$ is generated by elements $\{X^{i,j}(\ell) : \ 1 \leq i, j \leq N, 0 < r\}$ with explicit relations given by the conditions that the B_i's are p-nilpotent and pairwise commuting.

The two papers [70] and [71] also provided a *geometric* interpretation of (cohomological) support varieties of finite-dimensional modules for an infinitesimal group scheme G. Namely, the support of M is given as the closed subscheme of one-parameter subgroups $\psi : \mathbb{G}_{(r)} \to G$ such that $\psi^*(M)$ has an explicit non-projectivity property. One surprising aspect of these results is that no condition is placed on p, the residue characteristic of the ground field k.

These papers led to the formulation of π-points of finite group schemes by Friedlander and Julia Pevtsova [24] which further extended certain aspects of the representation theory of finite groups to all arbitrary finite group schemes.

In the next theorem, Andrei (together with Friedlander and Pevtsova) introduced refined invariants of G-modules, new even for finite groups. These invariants involve Jordan types, the decomposition of a p-nilpotent operator into blocks of sizes $\leq p$.

Theorem 6.3 ([25]). *Let G be a finite group scheme, let M be a finite-dimensional G-module, and let $x \in \operatorname{Proj} H^*(G, k)$ correspond to a minimal homogeneous prime ideal of $H^*(G, k)$. Then this data naturally determines a natural partition of*

$m := \dim(M)$,

$$m \; = \; \sum_{i=1}^{p} a_i \cdot i, \quad a_i \geq 0.$$

This partition arises from the Jordan type of any representative of the p-nilpotent action of G on M at the generic point $x \in \operatorname{Proj} H^*(G,k)$.

The proof of this theorem is subtle, further evidence of Andrei's ingenious insights. Among other consequences, this theorem led to the interesting class of *modules of constant Jordan type* introduced by Carlson, Friedlander, and Pevtsova in [12]. The essential step of the proof is the following observation of Andrei's concerning Jordan types of commuting nilpotent elements $\alpha, \beta \in GL_N(F)$ for an infinite field F and a positive integer N: the Jordan type of α is greater than or equal to the Jordan type of $\alpha + t\beta$ for all $t \in \beta$ if and only if the kernel of α is contained in $\sum_{\mu \in F^*} \operatorname{Ker}(\alpha + \mu\beta)$.

We mention one further paper on modular representation theory that Andrei wrote with Carlson and Friedlander [13]. The title of the paper, "Modules for $\mathbb{Z}/p \times \mathbb{Z}/p$", is probably surprising to those unfamiliar with the complexities of modular representations. For $p > 2$, the category of finite-dimensional representations on k-vector spaces (with $\operatorname{char}(k) = p$) for the finite group $\mathbb{Z}/p \times \mathbb{Z}/p$ is *wild*, which implies that this category contains as a full subcategory the category of representations of every finite-dimensional k-algebra. The paper [13] investigated various special classes of $\mathbb{Z}/p \times \mathbb{Z}/p$-modules, providing a wealth of details.

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References

[1] M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Proc. Sympos. Pure Math., Vol. III, American Mathematical Society, Providence, R.I., 1961, pp. 7–38. MR0139181

[2] J. Kr. Arason, *Cohomologische invarianten quadratischer Formen* (French), J. Algebra **36** (1975), no. 3, 448–491, DOI 10.1016/0021-8693(75)90145-3. MR389761

[3] H. Bass, *Algebraic K-theory*, W. A. Benjamin, Inc., New York-Amsterdam, 1968. MR0249491

[4] E. Bayer-Fluckiger and R. Parimala, *Galois cohomology of the classical groups over fields of cohomological dimension* $\leq$ 2, Invent. Math. **122** (1995), no. 2, 195–229, DOI 10.1007/BF01231443. MR1358975

[5] A. A. Beĭlinson, *Higher regulators and values of L-functions* (Russian), Current problems in mathematics, Vol. 24, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984, pp. 181–238. MR760999

[6] A. A. Beĭlinson, *Height pairing between algebraic cycles*, K-theory, arithmetic and geometry (Moscow, 1984), Lecture Notes in Math., vol. 1289, Springer, Berlin, 1987, pp. 1–25, DOI 10.1007/BFb0078364. MR923131

[7] A. Beilinson, *Remarks on Grothendieck's standard conjectures*, Regulators, Contemp. Math., vol. 571, Amer. Math. Soc., Providence, RI, 2012, pp. 25–32, DOI 10.1090/conm/571/11319. MR2953406

[8] S. Bloch, *Algebraic cycles and higher K-theory*, Adv. in Math. **61** (1986), no. 3, 267–304, DOI 10.1016/0001-8708(86)90081-2. MR852815

[9] A. Borel and J.-P. Serre, *Le théorème de Riemann-Roch* (French), Bull. Soc. Math. France **86** (1958), 97–136. MR0116022

[10] K. S. Brown and S. M. Gersten, *Algebraic K-theory as generalized sheaf cohomology*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 266–292. Lecture Notes in Math., Vol. 341. MR0347943

[11] J. F. Carlson, *The varieties and the cohomology ring of a module*, J. Algebra **85** (1983), no. 1, 104–143, DOI 10.1016/0021-8693(83)90121-7. MR723070

[12] J. F. Carlson, E. M. Friedlander, and J. Pevtsova, *Modules of constant Jordan type*, J. Reine Angew. Math. **614** (2008), 191–234, DOI 10.1515/CRELLE.2008.006. MR2376286

[13] J. F. Carlson, E. M. Friedlander, and A. Suslin, *Modules for $\mathbb{Z}/p \times \mathbb{Z}/p$*, Comment. Math. Helv. **86** (2011), no. 3, 609–657, DOI 10.4171/CMH/236. MR2803855

[14] A. Dold and R. Thom, *Quasifaserungen und unendliche symmetrische Produkte* (German), Ann. of Math. (2) **67** (1958), 239–281, DOI 10.2307/1970005. MR0097062

[15] C. M. Drupieski, *Cohomological finite-generation for finite supergroup schemes*, Adv. Math. **288** (2016), 1360–1432, DOI 10.1016/j.aim.2015.11.017. MR3436410

[16] W. G. Dwyer and E. M. Friedlander, *Algebraic and etale K-theory*, Trans. Amer. Math. Soc. **292** (1985), no. 1, 247–280, DOI 10.2307/2000179. MR805962

[17] L. Evens, *The cohomology ring of a finite group*, Trans. Amer. Math. Soc. **101** (1961), 224–239, DOI 10.2307/1993372. MR137742

[18] R. Fedorov and I. Panin, *A proof of the Grothendieck-Serre conjecture on principal bundles over regular local rings containing infinite fields*, Publ. Math. Inst. Hautes Études Sci. **122** (2015), 169–193, DOI 10.1007/s10240-015-0075-z. MR3415067

[19] V. Franjou, E. M. Friedlander, A. Scorichenko, and A. Suslin, *General linear and functor cohomology over finite fields*, Ann. of Math. (2) **150** (1999), no. 2, 663–728, DOI 10.2307/121092. MR1726705

[20] E. M. Friedlander, C. Haesemeyer, and M. E. Walker, *Techniques, computations, and conjectures for semi-topological K-theory*, Math. Ann. **330** (2004), no. 4, 759–807, DOI 10.1007/s00208-004-0569-3. MR2102312

[21] E. M. Friedlander and H. B. Lawson Jr., *A theory of algebraic cocycles*, Ann. of Math. (2) **136** (1992), no. 2, 361–428, DOI 10.2307/2946609. MR1185123

[22] E. Friedlander and C. Negron, *Cohomology for Drinfeld doubles of some infinitesimal group schemes*, Algebra Number Theory **12** (2018), no. 5, 1281–1309, DOI 10.2140/ant.2018.12.1281. MR3840877

[23] E. M. Friedlander and B. J. Parshall, *Support varieties for restricted Lie algebras*, Invent. Math. **86** (1986), no. 3, 553–562, DOI 10.1007/BF01389268. MR860682

[24] E. M. Friedlander and J. Pevtsova, *Π-supports for modules for finite group schemes*, Duke Math. J. **139** (2007), no. 2, 317–368, DOI 10.1215/S0012-7094-07-13923-1. MR2352134

[25] E. M. Friedlander, J. Pevtsova, and A. Suslin, *Generic and maximal Jordan types*, Invent. Math. **168** (2007), no. 3, 485–522, DOI 10.1007/s00222-007-0037-2. MR2299560

[26] E. M. Friedlander and A. Suslin, *Cohomology of finite group schemes over a field*, Invent. Math. **127** (1997), no. 2, 209–270, DOI 10.1007/s002220050119. MR1427618

[27] E. M. Friedlander and A. Suslin, *The spectral sequence relating algebraic K-theory to motivic cohomology* (English, with English and French summaries), Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 6, 773–875, DOI 10.1016/S0012-9593(02)01109-6. MR1949356

[28] E. M. Friedlander and M. E. Walker, *Rational isomorphisms between K-theories and cohomology theories*, Invent. Math. **154** (2003), no. 1, 1–61, DOI 10.1007/s00222-003-0300-0. MR2004456

[29] P. Gille, *Groupes algébriques semi-simple en dimension cohomologique ≤ 2*, Lecture Notes in Mathematics, Springer-Verlag, 2019.

[30] H. Gillet, *Riemann-Roch theorems for higher algebraic K-theory*, Bull. Amer. Math. Soc. (N.S.) **3** (1980), no. 2, 849–852, DOI 10.1090/S0273-0979-1980-14828-4. MR578377

[31] H. Gillet and C. Soulé, *Motivic weight complexes for arithmetic varieties*, J. Algebra **322** (2009), no. 9, 3088–3141, DOI 10.1016/j.jalgebra.2009.07.015. MR2567412

[32] C. Haesemeyer and C. A. Weibel, *The norm residue theorem in motivic cohomology*, Annals of Mathematics Studies, vol. 200, Princeton University Press, Princeton, NJ, 2019. MR3931681

[33] G. Horrocks, *Projective modules over an extension of a local ring*, Proc. London Math. Soc. (3) **14** (1964), 714–718, DOI 10.1112/plms/s3-14.4.714. MR0169878

[34] H. B. Lawson Jr., *Algebraic cycles and homotopy theory*, Ann. of Math. (2) **129** (1989), no. 2, 253–291, DOI 10.2307/1971448. MR986794

[35] R. Lee and R. H. Szczarba, *The group $K_3(Z)$ is cyclic of order forty-eight*, Ann. of Math. (2) **104** (1976), no. 1, 31–60, DOI 10.2307/1971055. MR0442934

[36] S. Lichtenbaum, *Values of zeta-functions, étale cohomology, and algebraic K-theory*, Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 489–501. Lecture Notes in Math., Vol. 342. MR0406981

[37] M. Levine, *The indecomposable K_3 of fields*, Ann. Sci. École Norm. Sup. (4) **22** (1989), no. 2, 255–344. MR1005161

[38] M. Levine, *Bloch's higher Chow groups revisited*, Astérisque **226** (1994), 10, 235–320. K-theory (Strasbourg, 1992). MR1317122

[39] H. Matsumoto, *Sur les sous-groupes arithmétiques des groupes semi-simples déployés* (French), Ann. Sci. École Norm. Sup. (4) **2** (1969), 1–62. MR0240214

[40] C. Mazza, V. Voevodsky, and C. Weibel, *Lecture notes on motivic cohomology*, Clay Mathematics Monographs, vol. 2, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2006. MR2242284

[41] A. S. Merkur′ev, *On the norm residue symbol of degree* 2 (Russian), Dokl. Akad. Nauk SSSR **261** (1981), no. 3, 542–547. MR638926

[42] A. S. Merkur′ev and A. A. Suslin, *K-cohomology of Severi– Brauer varieties and the norm residue homomorphism* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), no. 5, 1011–1046, 1135–1136. MR675529

[43] A. S. Merkur′ev and A. A. Suslin, *Norm residue homomorphism of degree three* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **54** (1990), no. 2, 339–356; English transl., Math. USSR-Izv. **36** (1991), no. 2, 349–367. MR1062517

[44] A. S. Merkur′ev and A. A. Suslin, *The group K_3 for a field* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **54** (1990), no. 3, 522–545; English transl., Math. USSR-Izv. **36** (1991), no. 3, 541–565. MR1072694

[45] A. Merkurjev and A. Suslin, *Motivic cohomology of the simplicial motive of a Rost variety*, J. Pure Appl. Algebra **214** (2010), no. 11, 2017–2026, DOI 10.1016/j.jpaa.2010.02.006. MR2645334

[46] J. Milnor, *Introduction to algebraic K-theory*, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1971. Annals of Mathematics Studies, No. 72. MR0349811

[47] J. Milnor, *Algebraic K-theory and quadratic forms*, Invent. Math. **9** (1969/1970), 318–344, DOI 10.1007/BF01425486. MR0260844

[48] F. Morel and V. Voevodsky, *$\mathbf{A}^1$-homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math. **90** (1999), 45–143 (2001). MR1813224

[49] Yu. P. Nesterenko and A. A. Suslin, *Homology of the general linear group over a local ring, and Milnor's K-theory* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 1, 121–146, DOI 10.1070/IM1990v034n01ABEH000610; English transl., Math. USSR-Izv. **34** (1990), no. 1, 121–145. MR992981

[50] D. Orlov, A. Vishik, and V. Voevodsky, *An exact sequence for $K_*^M/2$ with applications to quadratic forms*, Ann. of Math. (2) **165** (2007), no. 1, 1–13, DOI 10.4007/annals.2007.165.1. MR2276765

[51] I. A. Panin and A. A. Suslin, *On a conjecture of Grothendieck concerning Azumaya algebras* (Russian, with Russian summary), Algebra i Analiz **9** (1997), no. 4, 215–223; English transl., St. Petersburg Math. J. **9** (1998), no. 4, 851–858. MR1604322

[52] O. Pushin, *Higher Chern classes and Steenrod operations in motivic cohomology*, K-Theory **31** (2004), no. 4, 307–321, DOI 10.1023/B:KTHE.0000031355.85701.d8. MR2068875

[53] D. Quillen, *The spectrum of an equivariant cohomology ring. I, II*, Ann. of Math. (2) **94** (1971), 549–572; ibid. (2) 94 (1971), 573–602, DOI 10.2307/1970770. MR0298694

[54] D. Quillen, *On the cohomology and K-theory of the general linear groups over a finite field*, Ann. of Math. (2) **96** (1972), 552–586, DOI 10.2307/1970825. MR0315016

[55] D. Quillen, *Higher algebraic K-theory. I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341. MR0338129

[56] D. Quillen, *Projective modules over polynomial rings*, Invent. Math. **36** (1976), 167–171, DOI 10.1007/BF01390008. MR0427303

[57] J.-P. Serre, *Faisceaux algébriques cohérents* (French), Ann. of Math. (2) **61** (1955), 197–278, DOI 10.2307/1969915. MR0068874

[58] C. Soulé, *K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale* (French), Invent. Math. **55** (1979), no. 3, 251–295, DOI 10.1007/BF01406843. MR553999

[59] A. A. Suslin, *Projective modules over polynomial rings are free* (Russian), Dokl. Akad. Nauk SSSR **229** (1976), no. 5, 1063–1066. MR0469905

[60] A. A. Suslin, *The cancellation problem for projective modules and related topics*, Ring theory (Proc. Conf., Univ. Waterloo, Waterloo, 1978), Lecture Notes in Math., vol. 734, Springer, Berlin, 1979, pp. 323–338. MR548136

[61] A. A. Suslin, *Mennicke symbols and their applications in the K-theory of fields*, Algebraic K-theory, Part I (Oberwolfach, 1980), Lecture Notes in Math., vol. 966, Springer, Berlin-New York, 1982, pp. 334–356. MR689382

[62] A. A. Suslin, *Quaternion homomorphism for the field of functions on a conic* (Russian), Dokl. Akad. Nauk SSSR **265** (1982), no. 2, 292–296. MR665314

[63] A. Suslin, *On the K-theory of algebraically closed fields*, Invent. Math. **73** (1983), no. 2, 241–245, DOI 10.1007/BF01394024. MR714090

[64] A. A. Suslin, *On the K-theory of local fields*, Proceedings of the Luminy conference on algebraic K-theory (Luminy, 1983), J. Pure Appl. Algebra **34** (1984), no. 2-3, 301–318, DOI 10.1016/0022-4049(84)90043-4. MR772065

[65] A. A. Suslin, *Homology of GL_n, characteristic classes and Milnor K-theory*, Algebraic K-theory, number theory, geometry and analysis (Bielefeld, 1982), Lecture Notes in Math., vol. 1046, Springer, Berlin, 1984, pp. 357–375, DOI 10.1007/BFb0072031. MR750690

[66] A. A. Suslin, *Algebraic K-theory and the norm residue homomorphism* (Russian), Current problems in mathematics, Vol. 25, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984, pp. 115–207. MR770942

[67] A. A. Suslin, *Higher Chow groups and etale cohomology*, Cycles, transfers, and motivic homology theories, Ann. of Math. Stud., vol. 143, Princeton Univ. Press, Princeton, NJ, 2000, pp. 239–254. MR1764203

[68] A. Suslin, *On the Grayson spectral sequence*, Tr. Mat. Inst. Steklova **241** (2003), no. Teor. Chisel, Algebra i Algebr. Geom., 218–253; English transl., Proc. Steklov Inst. Math. **2(241)** (2003), 202–237. MR2024054

[69] A. Suslin, *Motivic complexes over nonperfect fields*, Ann. K-Theory **2** (2017), no. 2, 277–302, DOI 10.2140/akt.2017.2.277. MR3590347

[70] A. Suslin, E. M. Friedlander, and C. P. Bendel, *Infinitesimal* 1*-parameter subgroups and cohomology*, J. Amer. Math. Soc. **10** (1997), no. 3, 693–728, DOI 10.1090/S0894-0347-97-00240-3. MR1443546

[71] A. Suslin, E. M. Friedlander, and C. P. Bendel, *Support varieties for infinitesimal group schemes*, J. Amer. Math. Soc. **10** (1997), no. 3, 729–759, DOI 10.1090/S0894-0347-97-00239-7. MR1443547

[72] A. Suslin and S. Joukhovitski, *Norm varieties*, J. Pure Appl. Algebra **206** (2006), no. 1-2, 245–276, DOI 10.1016/j.jpaa.2005.12.012. MR2220090

[73] A. Suslin and V. Voevodsky, *Singular homology of abstract algebraic varieties*, Invent. Math. **123** (1996), no. 1, 61–94, DOI 10.1007/BF01232367. MR1376246

[74] A. Suslin and V. Voevodsky, *Bloch-Kato conjecture and motivic cohomology with finite coefficients*, The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), NATO Sci. Ser. C Math. Phys. Sci., vol. 548, Kluwer Acad. Publ., Dordrecht, 2000, pp. 117–189. MR1744945

[75] J.-P. Tignol, *Algèbres indécomposables d'exposant premier* (French), Adv. in Math. **65** (1987), no. 3, 205–228, DOI 10.1016/0001-8708(87)90022-3. MR904723

[76] A. Touzé and W. van der Kallen, *Bifunctor cohomology and cohomological finite generation for reductive groups*, Duke Math. J. **151** (2010), no. 2, 251–278, DOI 10.1215/00127094-2009-065. MR2598378

[77] R. W. Thomason, *Algebraic K-theory and étale cohomology*, Ann. Sci. École Norm. Sup. (4) **18** (1985), no. 3, 437–552. MR826102

[78] B. Totaro, *Milnor K-theory is the simplest part of algebraic K-theory*, K-Theory **6** (1992), no. 2, 177–189, DOI 10.1007/BF01771011. MR1187705

[79] B. B. Venkov, *Cohomology algebras for some classifying spaces* (Russian), Dokl. Akad. Nauk SSSR **127** (1959), 943–944. MR0108788

[80] V. Voevodsky, *Reduced power operations in motivic cohomology*, Publ. Math. Inst. Hautes Études Sci. **98** (2003), 1–57, DOI 10.1007/s10240-003-0009-z. MR2031198

[81] V. Voevodsky, *Motivic cohomology with $\mathbf{Z}/2$-coefficients*, Publ. Math. Inst. Hautes Études Sci. **98** (2003), 59–104, DOI 10.1007/s10240-003-0010-6. MR2031199

[82] V. Voevodsky, *On motivic cohomology with $\mathbf{Z}/l$-coefficients*, Ann. of Math. (2) **174** (2011), no. 1, 401–438, DOI 10.4007/annals.2011.174.1.11. MR2811603

[83] C. A. Weibel, *The K-book*, Graduate Studies in Mathematics, vol. 145, American Mathematical Society, Providence, RI, 2013. An introduction to algebraic K-theory. MR3076731

[84] N. Yagita, *Algebraic BP-theory and norm varieties*, Hokkaido Math. J. **41** (2012), no. 2, 275–316, DOI 10.14492/hokmj/1340714416. MR2977048

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