# HAUSDORFF DIMENSION, LAGRANGE AND MARKOV DYNAMICAL SPECTRA FOR GEOMETRIC LORENZ ATTRACTORS 

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#### Abstract

In this paper, we show that geometric Lorenz attractors have Hausdorff dimension strictly greater than 2 . We use this result to show that for a "large" set of real functions, the Lagrange and Markov dynamical spectrum associated to these attractors has persistently nonempty interior.


## 1. Introduction

In 1963 the meteorologist E. Lorenz published in the Journal of Atmospheric Sciences Lor63 an example of a parametrized polynomial system of differential equations,

$$
\begin{array}{rlrl}
\dot{x} & =a(y-x), & a & =10, \\
\dot{y} & =r x-y-x z, & & r=28 \\
\dot{z} & =x y-b z, & b=8 / 3, \tag{1}
\end{array}
$$

as a very simplified model for thermal fluid convection, motivated by an attempt to understand the foundations of weather forecasting. Numerical simulations for an open neighborhood of the chosen parameters suggested that almost all points in phase space tend to a strange attractor, called the Lorenz attractor. However Lorenz's equations proved to be very resistant to rigorous mathematical analysis and also presented very serious difficulties to rigorous numerical study.

A very successful approach was taken by Afraimovich, Bykov, and Shil'nikov ABS77] and by Guckenheimer and Williams GW79] independently: they constructed the so-called geometric Lorenz models for the behavior observed by Lorenz (see section 2 for a precise definition). These models are flows in three-dimensions for which one can rigorously prove the coexistence of an equilibrium point accumulated by regular orbits. Recall that a regular solution is an orbit where the flow does not vanish. Most remarkably, this attractor is robust: it cannot be destroyed by a small perturbation of the original flow. Taking into account that the divergence of the vector field induced by system (1) is negative, it follows that the Lebesgue measure of the Lorenz attractor is zero. Henceforth, it is natural to ask about its Hausdorff dimension. Numerical experiments give that this value is approximately equal to 2.062 (cf. Vis04) and also, for some parameter, the dimension of the physical invariant measure lies in the interval [1.24063, 1.24129] (cf. [GN16]).


Figure 1. Lorenz attractor

In this paper we address the problem to prove that the Hausdorff dimension of a geometric Lorenz attractor is strictly greater that 2. In AP83] and [Ste00, this dimension is characterized in terms of the pressure of the system and in terms of the Lyapunov exponents and the entropy with respect to a good invariant measure associated to the geometric model. But, in both cases, the authors prove that the Hausdorff dimension is greater than or equal to 2 , but it is not necessarily strictly greater than 2. A first attempt to obtain the strict inequality was given in ML08, where the authors achieve this result in the particular case that both branches of the unstable manifold of the equilibrium meet the stable manifold of the equilibrium. But this condition is quite strong and extremely unstable. One of our goals in this paper is to prove the strict inequality for the Hausdorff dimension for any geometric Lorenz attractor; see Figure 1 Thus, our first result is the following.

Theorem A. The Hausdorff dimension of a geometric Lorenz attractor is strictly greater than 2 .

To achieve this, since it is well known that the geometric Lorenz attractor is the suspension of a skew product map with contracting invariant leaves, defined in a cross-section, we start studying the one-dimensional map $f$ induced in the space of leaves. We are able to prove the existence of an increasing nested sequence of fat (Hausdorff dimension almost 1) regular Cantor sets of the one-dimensional map (Theorem (1). This fact implies that the maximal invariant set $\Lambda_{P}$ for the skew product (or, to first return map $P$ associated to the flow) has Hausdorff dimension strictly greater than 1, and this, in its turn, implies that the Hausdorff dimension of a geometric Lorenz attractor is strictly greater than 2 . In another words, Theorem A is a consequence of the following result.

Theorem 1. There is an increasing family of regular Cantor sets $C_{k}$ for $f$ such that

$$
H D\left(C_{k}\right) \rightarrow 1 \quad \text { as } k \rightarrow+\infty .
$$

The proof of this theorem, although nontrivial, is relatively elementary, and it combines techniques of several subjects of mathematics, such as ergodic theory, combinatorics, and dynamical systems (fractal geometry).

To announce the next goal of this paper, let us recall the classical notions of Lagrange and Markov spectra (see CF89 for further explanation and details).

The Lagrange spectrum $\mathcal{L}$ is a classical subset of the extended real line, related to Diophantine approximation. Given an irrational number $\alpha$, the first important result about upper bounds for Diophantine approximation is Dirichlet's approximation theorem, stating that for all $\alpha \in \mathbb{R} \backslash \mathbb{Q},\left|\alpha-\frac{p}{q}\right|<\frac{1}{q^{2}}$ has an infinite number of solutions $\frac{p}{q} \in \mathbb{Q}$.

Markov and Hurwitz improved this result by verifying that, for all irrational $\alpha$, the inequality $|\alpha-p / q|<\frac{1}{\sqrt{5} \cdot q^{2}}$ has an infinite number of rational solutions $p / q$, and $\sqrt{5}$ is the best constant that works for all irrational numbers. Indeed, for $\alpha=\frac{1+\sqrt{5}}{2}$, the gold number, Markov and Hurwitz also proved that, for every $\epsilon>0,\left|\alpha-\frac{p}{q}\right|<\frac{1}{(\sqrt{5}+\epsilon) \cdot q^{2}}$ has a finite number of solutions in $\mathbb{Q}$. Searching for better results for a fixed $\alpha \in \mathbb{R} \backslash \mathbb{Q}$ we are lead to define
$k(\alpha)=\sup \left\{k>0:|\alpha-p / q|<1 /\left(k q^{2}\right)\right.$ has infinitely many rational solutions $\left.p / q\right\}$. Note that the results by Markov and Hurwitz imply that $k(\alpha) \geq \sqrt{5}$ for all $\alpha \in \mathbb{R} \backslash \mathbb{Q}$, and $k\left(\frac{1+\sqrt{5}}{2}\right)=\sqrt{5}$. It can be proved that $k(\alpha)=\infty$ for almost every $\alpha \in \mathbb{R} \backslash \mathbb{Q}$.

We are interested in $\alpha \in \mathbb{R} \backslash \mathbb{Q}$ such that $k(\alpha)<\infty$ (which forms a set of Hausdorff dimension 1).
Definition 1. The Lagrange spectrum $\mathcal{L}$ is the image of the map $k$ :

$$
\mathcal{L}=\{k(\alpha), \alpha \in \mathbb{R} \backslash \mathbb{Q} \text { and } k(\alpha)<\infty\} .
$$

In 1921, Perron gave an alternative expression for the map $k$, as below. Write $\alpha$ in continued fractions: $\alpha=\left[a_{0}, a_{1}, a_{2}, \ldots\right]$. For each $n \in \mathbb{N}$, define

$$
\alpha_{n}=\left[a_{n}, a_{n+1}, a_{n+2}, \ldots\right], \quad \beta_{n}=\left[0, a_{n-1}, a_{n-2}, \ldots\right] .
$$

Then

$$
\begin{equation*}
k(\alpha)=\limsup _{n \rightarrow \infty}\left(\alpha_{n}+\beta_{n}\right) \tag{2}
\end{equation*}
$$

For a proof of equation (2) see, for instance, [CM, Proposition 21].
Markov proved (Mar80]) that the initial part of the Lagrange spectrum is discrete: $\mathcal{L} \cap(-\infty, 3)=\left\{k_{1}=\sqrt{5}<k_{2}=2 \sqrt{2}<k_{3}=\frac{\sqrt{221}}{5}<\cdots\right\}$ with $k_{n} \rightarrow 3, k_{n}^{2} \in \mathbb{Q}$, for all $n$.

In 1947, Hall proved (Hal47) that the regular Cantor set $C(4)$ of the real numbers in $[0,1]$ in whose continued fraction only appear coefficients $1,2,3,4$ satisfies $C(4)+C(4)=[\sqrt{2}-1,4(\sqrt{2}-1)]$. Using expression (22) and this result by Hall it follows that $[6, \infty) \subset \mathcal{L}$. That is, the Lagrange spectrum contains a whole half-line, nowadays called a Hall's ray.

Here we point out $\Lambda=C(4) \times C(4)$ is a horseshoe for a local diffeomorphism related to the Gauss map, which has Hausdorff dimension $H D(\Lambda)>1$. Hall's result says that its image $f(\Lambda)=C(4)+C(4)$ under the projection $f(x, y)=x+y$ contains an interval. This is a key point to get nonempty interior in $\mathcal{L}$. In 1975, Freiman proved ([Fre75]) some difficult results showing that the arithmetic sum of
certain (regular) Cantor sets, related to continued fractions, contain intervals, and he used them to determine the precise beginning of Hall's ray (the biggest half-line contained in $\mathcal{L}$ ) which is

$$
\frac{2221564096+283748 \sqrt{462}}{491993569} \cong 4,52782956616 \cdots
$$

Another interesting set related to Diophantine approximation is the classical Markov spectrum defined by
$\mathcal{M}=\left\{\left(\inf _{(x, y) \in \mathbb{Z}^{2} \backslash(0,0)}|f(x, y)|\right)^{-1}: f(x, y)=a x^{2}+b x y+c y^{2}\right.$ with $\left.b^{2}-4 a c=1\right\}$.
Notably, the Lagrange and Markov spectrum have a dynamical interpretation. Indeed, the expression of the map $k(\alpha)$ in terms of the continued fraction expression of $\alpha$ given in (2) allows one to characterize the Lagrange and Markov spectrum in terms of a shift map in a proper space. Let $\Sigma=\left(\mathbb{N}^{*}\right)^{\mathbb{Z}}$ be the set of bi-infinite sequences of integer numbers and consider the shift map $\sigma: \Sigma \rightarrow \Sigma, \sigma\left(\left(a_{n}\right)_{n}\right)=$ $\left(a_{n+1}\right)_{n}$, and define

$$
f: \Sigma \rightarrow \mathbb{R}, \quad f\left(\left(a_{n}\right)_{n}\right)=\alpha_{0}+\beta_{0}
$$

where $\alpha_{0}=\left[a_{0}, a_{1}, a_{2}, \ldots\right]$ and $\beta_{0}=\left[0, a_{-1}, a_{-2}, \ldots\right]$.
The Lagrange and the Markov spectra are characterized as (cf. CF89] for more details)

$$
\mathcal{L}=\left\{\limsup _{k} f\left(\sigma^{k}\left(\left(a_{n}\right)_{n}\right)\right),\left(a_{n}\right)_{n} \in \Sigma\right\}, \quad \mathcal{M}=\left\{\sup _{k} f\left(\sigma^{k}\left(\left(a_{n}\right)_{n}\right)\right),\left(a_{n}\right)_{n} \in \Sigma\right\}
$$

These characterizations lead naturally to a natural extension of these concepts to the context of dynamical systems.

For our purposes, let us consider a more general definition of the Lagrange and Markov spectra. Let $M$ be a smooth manifold, let $T=\mathbb{Z}$ or $\mathbb{R}$, and let $\phi=\left(\phi^{t}\right)_{t \in T}$ be a discrete-time $(T=\mathbb{Z})$ or continuous-time $(T=\mathbb{R})$ smooth dynamical system on $M$; that is, $\phi^{t}: M \rightarrow M$ are smooth diffeomorphisms, $\phi^{0}=\mathrm{id}$, and $\phi^{t} \circ \phi^{s}=\phi^{t+s}$ for all $t, s \in T$.

Given a compact invariant subset $\Lambda \subset M$ and a function $f: M \rightarrow \mathbb{R}$, we define the dynamical Markov (resp., Lagrange) spectrum $M(\phi, \Lambda, f)$ (resp., $L(\phi, \Lambda, f)$ ) as

$$
M(\phi, \Lambda, f)=\left\{m_{\phi, f}(x): x \in \Lambda\right\}, \quad \text { resp., } \quad L(\phi, \Lambda, f)=\left\{\ell_{\phi, f}(x): x \in \Lambda\right\}
$$

where

$$
m_{\phi, f}(x):=\sup _{t \in T} f\left(\phi^{t}(x)\right), \quad \text { resp., } \quad \ell_{\phi, f}(x):=\limsup _{t \rightarrow+\infty} f\left(\phi^{t}(x)\right)
$$

It can be proved that $L(\phi, \Lambda, f) \subset M(\phi, \Lambda, f)$ (cf. RM17]). In the discrete case, we refer to RM17, where it was proved that for typical hyperbolic dynamics (with Hausdorff dimension greater than 1), the Lagrange and Markov dynamical spectra have nonempty interior for typical functions.

Moreira and Romaña also proved that Markov and Lagrange dynamical spectra associated to generic Anosov flows (including generic geodesic flows of surfaces of negative curvature) typically have nonempty interior (see RM15 and Rom16 for more details).

Now we are ready to state our next result. Let $X_{0}$ be the vector field that defines a geometric Lorenz attractor $\Lambda$, and let $U$ be an open neighborhood of $\Lambda$ where $X_{0}$ is defined.

Theorem B. Let $\Lambda$ be the geometric Lorenz attractor associated to $X_{0}^{t}$. Then arbitrarily close to $X_{0}^{t}$, there are a flow $X^{t}$ and a neighborhood $\mathcal{W}$ of $X^{t}$ such that, if $\Lambda_{Y}$ denotes the geometric Lorenz attractor associated to $Y \in \mathcal{W}$, there is an open and dense set $\mathcal{H}_{Y} \subset C^{1}(U, \mathbb{R})$ such that for all $f \in \mathcal{H}_{Y}$, we have

$$
\operatorname{int}\left(L\left(Y, \Lambda_{Y}, f\right)\right) \neq \emptyset, \quad \operatorname{int}\left(M\left(Y, \Lambda_{Y}, f\right)\right) \neq \emptyset
$$

where $\operatorname{int}(A)$ denotes the interior of $A$.
1.1. Organization of the text. This paper is organized as follows. In Section 2) we describe informally the construction of a geometric Lorenz attractor and announce the main proprieties used in the text. In Section 3 we prove the first main result in this paper, Theorem 1 and its consequences, Corollary $C$ and Theorem A, In Section 4 we proof our last result, Theorem B.

## 2. Preliminary results: geometrical Lorenz model

In this section we present informally the construction of the geometric Lorenz attractor, following GP10, AP10, where the interested reader can find a detailed exposition of this construction.

Let $(\dot{x}, \dot{y}, \dot{z})=\left(\lambda_{1} x, \lambda_{2} y, \lambda_{3} z\right)$ be a vector field in the cube $[-1,1]^{3}$, with a singularity at the origin $(0,0,0)$. Suppose the eigenvalues $\lambda_{i}, 1 \leq i \leq 3$, satisfy the relations

$$
\begin{equation*}
0<-\lambda_{3}<\lambda_{1}<-\lambda_{2}, \quad 0<\alpha=-\frac{\lambda_{3}}{\lambda_{1}}<1<\beta=-\frac{\lambda_{2}}{\lambda_{1}} . \tag{3}
\end{equation*}
$$

Consider $S=\{(x, y, 1):|x| \leq 1 / 2,|y| \leq 1 / 2\}$ and $S^{-}=\{(x, y, 1) \in S: x<0\}$, $S^{+}=\{(x, y, 1) \in S: x>0\}$, and $S^{\star}=S \backslash \Gamma$, with $\Gamma=\{(x, y, 1) \in S: x=0\}$.

Assume that $S$ is a transverse section to the flow so that every trajectory eventually crosses $S$ in the direction of the negative $z$ axis as in Figure 2 Consider also $\widetilde{\Sigma}^{ \pm}=\{(x, y, z): x= \pm 1\}$ and put $\Sigma:=\widetilde{\Sigma}^{-} \cup \widetilde{\Sigma}^{+}=\{(x, y, z):|x|=1\}$. For each $\left(x_{0}, y_{0}, 1\right) \in S^{\star}$ the time $\tau$ such that $X^{\tau}\left(x_{0}, y_{0}, 1\right) \in \Sigma$ is given by $\tau\left(x_{0}\right)=-\frac{1}{\lambda_{1}} \log \left(\left|x_{0}\right|\right)$, which depends on $x_{0} \in S^{\star}$ only and is such that $\tau\left(x_{0}\right) \rightarrow+\infty$ when $x_{0} \rightarrow 0$. Hence we get ( where $\operatorname{sgn}(x)=\frac{x}{|x|}$ for $x \neq 0$ )

$$
X^{\tau}\left(x_{0}, y_{0}, 1\right)=\left(\operatorname{sgn}\left(x_{0}\right), y_{0} e^{\lambda_{2} \tau\left(x_{0}\right)}, e^{\lambda_{3} \tau\left(x_{0}\right)}\right)=\left(\operatorname{sgn}\left(x_{0}\right), y_{0}\left|x_{0}\right|^{-\frac{\lambda_{2}}{\lambda_{1}}},\left|x_{0}\right|^{-\frac{\lambda_{3}}{\lambda_{1}}}\right) .
$$

Let $L: S^{\star} \rightarrow \Sigma$ be given by

$$
\begin{equation*}
L(x, y, 1)=\left(\operatorname{sgn}(x), y x^{\beta}, x^{\alpha}\right) . \tag{4}
\end{equation*}
$$

It is easy to see that $L\left(S^{ \pm}\right)$has the shape of a triangle without the vertex $( \pm 1,0,0)$, which are cusps points of the boundary of each of these sets. From now on we denote by $\Sigma^{ \pm}$the closure of $L\left(S^{\star}\right)$. Note that each line segment $S^{\star} \cap\left\{x=x_{0}\right\}$ is taken to another line segment $\Sigma \cap\left\{z=z_{0}\right\}$ as sketched in Figure 2. Outside the cube, to imitate the random turns of a regular orbit around the origin and obtain a butterfly shape for our flow, we let the flow return to the cross section $S$ through a flow described by a suitable composition of a rotation $R_{ \pm}$, an expansion $E_{ \pm \theta}$, and a translation $T_{ \pm}$. Note that these transformations take line segments $\Sigma^{ \pm} \cap\left\{z=z_{0}\right\}$ into line segments $S \cap\left\{x=x_{1}\right\}$ as shown in Figure 2, and so does the composition $T_{ \pm} \circ E_{ \pm \theta} \circ R_{ \pm}$. This composition of linear maps describes a vector field $Y$ in a region outside $[-1,1]^{3}$, such that the time-one map of the associated flow realizes $T_{ \pm} \circ E_{ \pm \theta} \circ R_{ \pm}$as a map $\Sigma^{ \pm} \rightarrow S$. We note that the flow on the attractor we are constructing will pass though the region between $\Sigma^{ \pm}$and $S$ in a


Figure 2. Behavior near the origin
relatively small time with respect the linearized region. The above construction enables us to describe, for $t \in \mathbb{R}^{+}$, the orbit $X^{t}(x)$ for all $x \in S$ : the orbit starts following the linear flow $L$ until $\Sigma^{ \pm}$and then it will follow $Y$ coming back to $S$ and so on. Now observe that $\Gamma=\left\{(x, y, 1 \in S: x=0\} \subset W^{s}((0,0,0))\right.$ and so the orbit of all $x \in \Gamma$ converges to $(0,0,0)$. Let us denote by $W=\left\{X^{t}(x): x \in S ; t \in \mathbb{R}^{+}\right\}$ the set where this flow acts. The geometric Lorenz flow is the couple ( $W, X^{t}$ ) and the geometric Lorenz attractor is the set

$$
\begin{equation*}
\Lambda=\overline{\bigcap_{t \geq 0} X^{t}\left(\Lambda_{P}\right)}, \text { where } \Lambda_{P}=\overline{\bigcap_{i \geq 1} P^{i}\left(S^{\star}\right)} \tag{5}
\end{equation*}
$$

where $P: S^{\star} \rightarrow S$ is the Poincaré map.
Composing the expression in (4) with $R_{ \pm}, E_{ \pm \theta}$, and $T_{ \pm}$and taking into account that points in $\Gamma$ are contained in $W^{s}((0,0,0))$, we can write an explicit formula for the Poincaré map $P$ by

$$
\begin{gathered}
P(x, y)=(f(x), g(x, y)), \\
f(x)=\left\{\begin{array}{ll}
f_{0}\left(x^{\alpha}\right), & \text { if } x>0, \\
f_{1}\left(x^{\alpha}\right), & \text { if } x<0,
\end{array} \quad \text { with } f_{i}=(-1)^{i} \theta \cdot x+b_{i} \quad i=1,2\right.
\end{gathered}
$$

and

$$
g(x, y)= \begin{cases}g_{0}\left(x^{\alpha}, y \cdot x^{\beta}\right), & \text { if } x>0 \\ g_{1}\left(x^{\alpha}, y \cdot x^{\beta}\right), & \text { if } x<0\end{cases}
$$

where $g_{1}: L_{1} \times I \rightarrow I$ and $g_{0}: L_{2} \times I \rightarrow I$ are suitable affine maps, with $L_{1}=$ $[-1 / 2,0)$ and $L_{2}=(0,1 / 2]$. Figure 3 displays the main features of $f$ and $P$ on $[-1 / 2,1 / 2]$ and $S$, respectively.
2.1. Properties of the one-dimensional map $f$. Here we specify the properties of the one-dimensional map $f$ described in Figure 3:
$(f 1) f$ is discontinuous at $x=0$ with lateral limits $f\left(0^{-}\right)=\frac{1}{2}$ and $f\left(0^{+}\right)=-\frac{1}{2}$;
( $f 2$ ) $f$ is differentiable on $I \backslash\{0\}$ and $f^{\prime}(x)>\sqrt{2}$, where $I=[-1 / 2,1 / 2]$;
(f3) the lateral limits of $f^{\prime}$ at $x=0$ are $f^{\prime}\left(0^{-}\right)=+\infty=$ and $f^{\prime}\left(0^{+}\right)=+\infty$.


Figure 3. The one-dimensional Lorenz map $f$ and the image $P\left(S^{*}\right)$

The properties $(f 1)-(f 3)$ above imply another important feature for the map $f$, as it is shown in Lemma 2.1 below. We will present the proof of R. Williams (cf. Wil79, Proposition 1]) for Lemma 2.1] which we will use to construct "almost locally eventually onto" avoiding the singularity 0 of $f$ (cf. Section 2.3).

Lemma 2.1. Put $I=\left[-\frac{1}{2}, \frac{1}{2}\right]$. If $J \subset I$ is a subinterval, then there is an integer $n$ such that $f^{n}(J)=I$. That is, $f$ is locally eventually onto.

Proof. Let $J_{0}=J$, if $0 \notin J$; otherwise let $J_{0}$ be the bigger of the two intervals into which 0 splits $J$. Similarly, for each $i$ such that $J_{i}$ is defined, set

$$
J_{i+1}= \begin{cases}f\left(J_{i}\right), & \text { if } 0 \notin f\left(J_{i}\right) \\ \text { bigger of two parts } 0 \text { splits } f\left(J_{i}\right) \text { into, } & \text { if } 0 \in f\left(J_{i}\right)\end{cases}
$$

Note that $\left|f\left(J_{i+1}\right)\right|>\eta\left|J_{i+1}\right|$, where $\eta=\inf \left|f^{\prime}\right|>\sqrt{2}$ and $|\cdot|$ denotes length. Thus unless 0 is in both $f\left(J_{i}\right)$ and $f\left(J_{i+1}\right)$, we have

$$
\left|J_{i+2}\right| \geq \frac{\eta^{2}}{2}\left|J_{i}\right|
$$

But as $\eta^{2}>2$, this last cannot always hold, say

$$
0 \in f\left(J_{n-2}\right) \text { and } 0 \in f\left(J_{n-1}\right)
$$

Then $f\left(J_{n-1}\right)$ contains 0 and one end point of $I$, so that $J_{n}$ is one "half" of $I$. Note that $f\left(J_{n}\right)$ contains the other half, and finally $f^{3}\left(J_{n}\right)=I$.

The next lemma gives us the following ergodic property for $f$ as above (Via97, Corollary 3.4]).

Lemma 2.2. Let $f:[-1 / 2,1 / 2] \backslash\{0\} \rightarrow[-1 / 2,1 / 2]$ be a $C^{2}$-function, satisfying properties $(f 1)-(f 3)$ in Section 2.1. Then $f$ has some absolutely continuous invariant probability measure (with respect to Lebesgue measure m). Moreover, if $\mu$ is any such measure, then $\mu=\varphi m$ where $\varphi$ has bounded variation.
2.2. Properties of the map $g$. By definition $g$ is piecewise $C^{2}$ and the following bounds on its partial derivatives hold:
(a) For all $(x, y) \in S^{\star}(x \neq 0)$, we have $\left|\partial_{y} g(x, y)\right|=|x|^{\beta}$. As $\beta>1$ and $|x| \leq 1 / 2$ there is $0<\lambda<1$ such that

$$
\left|\partial_{y} g\right|<\lambda
$$

(b) For $(x, y) \in S^{\star}(x \neq 0)$, we have $\partial_{x} g(x, y)=\beta|x|^{\beta-\alpha}$. Since $\beta>\alpha$ and $|x| \leq 1 / 2$, we get $\left|\partial_{x} g\right|<\infty$.

We note that from the first item above it follows the uniform contraction of the foliation given by the lines $S \cap\{x=$ constant $\}$. The foliation is contracting in the following sense: there is a constant $C>0$ such that, for any given leaf $\gamma$ of the foliation and for $y_{1}, y_{2} \in \gamma$, then

$$
\operatorname{dist}\left(P^{n}\left(y_{1}\right), P^{n}\left(y_{2}\right)\right) \leq C \lambda^{n} \operatorname{dist}\left(y_{1}, y_{2}\right) \quad \text { as } \quad n \rightarrow \infty .
$$

We notice that the geometric Lorenz attractor constructed above is robust, that is, it persists for all nearby vector fields. More precisely, there exists a neighborhood $U$ in $\mathbb{R}^{3}$ containing the attracting set $\Lambda$, such that for all vector fields $Y$ which are $C^{1}$-close to $X$, the maximal invariant subset in $U, \Lambda_{Y}=\bigcap_{t \geq 0} Y^{t}(U)$, is still a transitive $Y$-invariant set. This is a consequence of the domination of the contraction along the $y$-direction over the expansion along the $x$-direction (see, e.g., AP10, Session 3.3.4]). Moreover, for every $Y$ that is $C^{1}$-close to $X$, the associated Poincaré map preserves a contracting foliation $\mathcal{F}_{Y}$ with $C^{1}$ leaves. It can be shown that the holonomies along the leaves are in fact Hölder- $C^{1}$; see AP10. Moreover, if we have a strong dissipative condition on the equilibrium $O$, that is, if $\beta>\alpha+k$ for some $k \in \mathbb{Z}^{+}$(see the definitions of $\alpha, \beta$ as functions of the eigenvalues of 0 in (3)), it can be shown that $\mathcal{F}_{Y}$ is a $C^{k}$-smooth foliation SV16, and so the holonomies along the leaves of $\mathcal{F}_{Y}$ are $C^{k}$-maps. In particular, for strongly dissipative Lorenz attractors with $\beta>\alpha+k$, the one-dimensional quotient map is $C^{k}$-smooth away from the singularity (cf. SV16]).

We finish this section noting that putting together the observations above and the results proved in AP10, Section 3.3.4], we easily deduce the following result.

Proposition 1. There is a neighborhood $\mathcal{U} \supset X$ such that for all $Y \in \mathcal{U}$, if $f_{Y}$ is the quotient map $f_{Y}: S^{*} / \mathcal{F}_{Y} \rightarrow S / \mathcal{F}_{Y}$ associated to the corresponding Poincaré map $P_{Y}$, then the properties $(f 1)-(f 3)$ from Section 2.1 are still valid. Moreover, there are constants $C, C_{1}>1$ uniformly on a $C^{2}$-neighborhood of $X$ such that if $\alpha(Y)=-\frac{\lambda_{3}(Y)}{\lambda_{1}(Y)}$ is the continuation of $\alpha=-\frac{\lambda_{3}}{\lambda_{1}}$ obtained for the initial flow $X^{t}$ it holds that

$$
\begin{equation*}
\frac{1}{C} \leq \frac{D f_{Y}(x)}{|x|^{\alpha(Y)-1}} \leq C \quad \text { and } \quad \frac{\left|D f_{Y}^{2}(x)\right|}{|x|^{\alpha(Y)-2}} \leq C_{1} . \tag{6}
\end{equation*}
$$

Furthermore, condition (f3) ensures that $f_{Y}$ has enough expansion to easily prove that every $f_{Y}$ is locally eventually onto for all $Y$ close to $X$.
2.3. Almost locally eventually onto. In this section we shall use an argument similar to the one given in Lemma [2.1, to achieve a property of $f$ fundamental for the construction of the family of Cantor sets in Theorem Roughly speaking, we
shall prove the existence of a number $a$ arbitrarily close to one, depending only on $f$, such that for any interval $J \subset I$, we have the following:
(1) an interval $J^{\prime} \subset J$ such that $0 \notin J^{\prime}$ and with size equal to a fixed proportion of the size of $J$.
(2) a number $n=n(J)$ such that the restriction $f^{n} \mid J^{\prime}, f^{n}: J^{\prime} \rightarrow L_{1}^{a}$ is a diffeomorphism, where $L_{1}^{a}=[f(1-a), 0)$. Moreover, we obtain a control on the distortion at each step $f^{j}$, for all $1 \leq j \leq n-1$.
To do that, we start with an auxiliary result.
Lemma 2.3. There is a constant $\kappa>0$ such that for all interval $J \subset I \backslash\{0\}$ such that $0 \in f(J)$ and $0 \in f^{2}(J)$, then

$$
|J| \geq \kappa
$$

Proof. We denote $0_{1} \in\left[-\frac{1}{2}, 0\right]$ and $0_{2} \in\left[0, \frac{1}{2}\right]$ the preimage of 0 in each branch of $f$, that is, $f\left(0_{i}\right)=0, i \in\{1,2\}$. Consider also the two preimages of $0_{i}, 0_{i}^{1}$, $0_{i}^{2}, i=1,2$, in $\left[-\frac{1}{2}, 0\right]$ and $\left[0, \frac{1}{2}\right]$, respectively. As $0 \in f(J)$ and $0 \in f^{2}(I)$, then $0,0_{i} \in f(J)$ for some $i$, and thus we get that some of the intervals $J_{1}=$ $\left[0_{1}^{1}, 0_{1}\right], J_{2}=\left[0_{1}, 0_{2}^{1}\right], J_{3}=\left[0_{1}^{2}, 0_{2}\right]$, and $J_{4}=\left[0_{2}, 0_{2}^{2}\right]$ is contained in $J$. Thus, taking $\kappa=\min \left\{\left|J_{1}\right|,\left|J_{2}\right|,\left|J_{3}\right|,\left|J_{4}\right|\right\}$, we finish the proof.

Recall that $\eta^{2}>2$. Now we consider a number $0<a<1$ satisfying

$$
\begin{equation*}
a^{2} \eta^{2}>2 \text { and } 1-a<\kappa \tag{7}
\end{equation*}
$$

For an interval $J \subset I$, we will use the number $a$ satisfying equation (7) to define an interval $\tilde{J} \subset J$ avoiding the singularity 0 and that is obtained by cutting a small part of $J$ with length $(1-a)|J|$. In this direction, we proceed as follows.

Given any interval $J=(b, c) \subset I \backslash\{0\}$, we denote by $J_{a}$ the subinterval of $J$ cutting an interval of size $(1-a)|J|$ on the closest side to zero, that is,

$$
J_{a}= \begin{cases}(b, a c+(1-a) b), & \text { if } c<0, \\ (a b+(1-a) c, c), & \text { if } b>0\end{cases}
$$

Note that, if $c=0$, then $J_{a}=(b,(1-a) b)$ and if $b=0, J_{a}=((1-a) c, c)$. It is clear that $\left|J_{a}\right|=a|J|$ and $0 \notin J_{a}$.

When an interval $J=(b, 0)$ or $J=(0, c)$ has size large enough $(|J|>(1-a))$, we define the subinterval ${ }_{a} J$ of $J$ cutting an interval of size $(1-a)$ of $J$ in the side of the point 0 ; in other words,

$$
{ }_{a} J=(b,-(1-a)) \text { or }{ }_{a} J=((1-a), c) .
$$

It is clear that both kinds of intervals, $J_{a}$ and ${ }_{a} J$, avoid the singularity.
Recall that $0_{1} \in\left(-\frac{1}{2}, 0\right)$ and $0_{2} \in\left(0, \frac{1}{2}\right)$ are the preimages of 0 , that is, $f\left(0_{i}\right)=0$, $i=1,2$. For the next lemma assume that $0_{1} \in J_{1}=(b, 0)$ and $0_{2} \in J_{2}=(0, c)$. So, for $a$ sufficiently close to 1 , we have that $0_{1} \in{ }_{a} J_{1}=(b, a-1)$ and $0_{2} \in{ }_{a} J_{2}=$ $(1-a, c)$. Therefore $0 \in f\left({ }_{a} J_{1}\right)$ and $0 \in f\left({ }_{a} J_{2}\right)$. Denote $f\left({ }_{a} J_{1}\right)^{+}$and $f\left({ }_{a} J_{2}\right)^{+}$the bigger of two parts into which 0 splits $f\left({ }_{a} J_{1}\right)$ and $f\left({ }_{a} J_{2}\right)$, respectively.

The next lemma says that if $a$ is sufficiently close to 1 , then it is easy to determine $f\left({ }_{a} J_{1}\right)^{+}$and $f\left({ }_{a} J_{2}\right)^{+}$explicitly.

Lemma 2.4. Keeping the notation of above, if a is close enough to 1 , then

$$
f\left({ }_{a} J_{1}\right)^{+}=f\left(\left[0_{1}, a-1\right]\right) \text { and } f\left({ }_{a} J_{2}\right)^{+}=f\left(\left[1-a, 0_{2}\right]\right)
$$

Proof. Since $0_{1} \in{ }_{a} J_{1}=(b, a-1)$ and $0_{2} \in{ }_{a} J_{2}=(1-a, c)$, we need only to prove that

$$
\left|f\left(\left[b, 0_{1}\right]\right)\right|<\left|f\left(\left[0_{1}, a-1\right]\right)\right| \text { and }\left|f\left(\left[0_{2}, c\right]\right)\right|<\left|f\left(\left[1-a, 0_{2}\right]\right)\right|
$$

for $a$ sufficiently close to 1 . Let us prove the left-hand inequality; the other one is analogous. Note that $f\left(-\frac{1}{2}\right) \neq-\frac{1}{2}$, then by definition of $f$ we have that $\left|f\left(\left[-\frac{1}{2}, 0_{1}\right]\right)\right|<\left|f\left(\left[0_{1}, 0\right)\right)\right|$. In particular, for all $b \in\left[-\frac{1}{2}, 0_{1}\right]$, it holds that $\left|f\left(\left[b, 0_{1}\right]\right)\right|<\left|f\left(\left[0_{1}, 0\right)\right)\right|$. Now consider the number $\psi:=\left|f\left(\left[0_{1}, 0\right)\right)\right|-\left|f\left(\left[-\frac{1}{2}, 0_{1}\right]\right)\right|>$ 0 , which only depends of $f$. So, we can taken $a$ sufficiently close to 1 such that

$$
\left|f\left(\left[0_{1}, 0\right)\right)\right|-\left|f\left(\left[0_{1}, a-1\right]\right)\right|<\frac{\psi}{2}
$$

and therefore we conclude that $\left|f\left[b, 0_{1}\right]\right|<\left|f\left[0_{1}, a-1\right]\right|$, as we wished.
From now on, we will denote $L_{1}^{a}:=[f(1-a), 0)$.
To prove the next lemma, we use the same idea as in the proof of Lemma [2.1, to get control on the number of iterations required to increase the size of any interval avoiding the singularity in each step.

Lemma 2.5. If $J \subset I$ is a subinterval, then there are a subinterval $J^{\prime} \subset J$ and an integer $n(J)$ such that $f^{n(J)}: J^{\prime} \rightarrow L_{1}^{a}$ is a diffeomorphism such that $d\left(f^{i}\left(J^{\prime}\right),\{0\}\right)>0, i=0, \ldots, n(J)-1$, and

$$
n(J) \leq 3+\frac{\log \frac{1}{2|J|}}{\log \frac{a^{2} \eta^{2}}{2}}
$$

Proof. Given $J \subset I$, let $J_{0}:=J_{a}$ if $0 \notin J$; otherwise let $J_{0}:=J_{a}^{+}$, where $J^{+}$is the biggest connected component of $J \backslash\{0\}$. Similarly, for each $i$ such that $J_{i}$ is defined, set

$$
J_{i+1}= \begin{cases}f\left(J_{i}\right)_{a}, & \text { if } 0 \notin f\left(J_{i}\right)  \tag{8}\\ f\left(J_{i}\right)_{a}^{+}, & \text {if } 0 \in f\left(J_{i}\right)\end{cases}
$$

Note that $\left|f\left(J_{i+1}\right)\right|>\eta\left|J_{i+1}\right|$, where $\eta=\inf \left|f^{\prime}\right|>\sqrt{2}$. Thus, unless 0 is in both $f\left(J_{i}\right)$ and $f\left(J_{i+1}\right)$, we have

$$
\left|J_{i+2}\right| \geq \frac{a^{2} \eta^{2}}{2}\left|J_{i}\right| .
$$

But as $a^{2} \eta^{2}>2$, this last inequality cannot always hold. Let $n$ be the minimum number such that

$$
\begin{equation*}
0 \in f\left(J_{n-2}\right) \text { and } 0 \in f\left(J_{n-1}\right) \tag{9}
\end{equation*}
$$

Thus equation (9) implies that $J_{n-2}$ satisfies the hypothesis of Lemma 2.3. Therefore, $\left|J_{n-2}\right| \geq \kappa$, and as $1-a<\kappa$ (see equation (7)), we define the interval ${ }_{a} f\left(J_{n-2}\right)^{+}:=\tilde{J}_{n-1} \subset f\left(J_{n-2}\right)^{+}$. To finish the proof of the lemma, we have to consider two cases, depending on the relative position of $\tilde{J}_{n-1}$ in the connected components of $I \backslash\{0\}$.

Case 1. Assume that $\tilde{J}_{n-1} \subset\left(0, \frac{1}{2}\right]$. Thus by definition of ${ }_{a} f\left(J_{n-2}\right)^{+}$we have that $\tilde{J}_{n-1}=[1-a, b]$ for some $b>0$. Moreover, as $0 \in f\left(J_{n-1}\right)$, then $0 \in f^{2}\left(J_{n-2}\right)$, therefore as $\tilde{J}_{n-1} \subset\left(0, \frac{1}{2}\right]$, then arguing as in the proof of Lemma 2.3, we get that $\left[0_{1}, 0_{2}^{1}\right] \subset f\left(J_{n-2}\right)$ or $\left[0_{2}, 0_{2}^{2}\right] \subset f\left(J_{n-1}\right)$, consequently since $f\left(0_{2}^{i}\right)=0_{2}$, then $0_{2} \in \tilde{J}_{n-1}$, which implies by Lemma 2.4 that $\left|f\left(\left(1-a, 0_{2}\right)\right)\right|>\left|f\left(\left(0_{2}, b\right)\right)\right|$ or equivalently

$$
f\left(\tilde{J}_{n-1}\right)^{+}=f\left(\left[1-a, 0_{2}\right)\right)=[f(1-a), 0)=L_{1}^{a}
$$

In this case, we define the following sequence of intervals $I_{n-2}=f^{-1}\left[1-a, 0_{2}\right] \subset$ $J_{n-2}$ and $I_{i}=f^{-1}\left(I_{i+1}\right) \subset J_{i}, 0 \leq i \leq n-2$. Hence, by construction, the interval $J^{\prime}:=I_{0} \subset J$ satisfies

$$
f^{i}\left(J^{\prime}\right) \subset J_{i-1}= \begin{cases}f\left(J_{i-2}\right)_{a}, & \text { if } 0 \notin f\left(J_{i-2}\right), \\ f\left(J_{i-2}\right)_{a}^{+}, & \text {if } 0 \in f\left(J_{i-2}\right)\end{cases}
$$

Therefore, we conclude that

$$
d\left(f^{i}\left(J^{\prime}\right),\{0\}\right) \geq \begin{cases}(1-a) \cdot\left|f\left(J_{i-2}\right)\right| \geq(1-a) \cdot\left|J_{i-2}\right|, & \text { if } 0 \notin f\left(J_{i-2}\right) \\ (1-a) \cdot\left|f\left(J_{i-2}\right)^{+}\right| \geq \frac{1-a}{2}\left|J_{i-2}\right|, & \text { if } 0 \in f\left(J_{i-2}\right)\end{cases}
$$

So, taking $n(J)=n$, we have that $f^{n(J)}: J^{\prime} \rightarrow L_{1}^{a}$ is a diffeomorphism and it is easy to see that $d\left(f^{i}\left(J^{\prime}\right),\{0\}\right)>0, i=0, \ldots, n(J)-1$. This concludes the proof of Case 1 .

Case 2. Assume that $\tilde{J}_{n-1} \subset\left[-\frac{1}{2}, 0\right)$. Then, $\tilde{J}_{n-1}=[c, a-1]$. Thus by the same argument as in Case 1 , we have that $0_{1} \in \tilde{J}_{n-1}$, and by Lemma 2.4 we have $\left|f\left(\left(0_{1}, a-1\right)\right)\right|>\left|f\left(\left(c, 0_{1}\right)\right)\right|$ or, equivalently, $f\left(\tilde{J}_{n-1}\right)^{+}=f\left(\left(0_{1}, a-1\right]\right)=$ $(0, f(a-1)]$, then, $\tilde{J}_{n}:={ }_{a} f\left(\tilde{J}_{n-1}\right)^{+}=[1-a, f(a-1)]$. Note that for $a$ sufficiently close to $1, f(a-1)>0_{2}$ and therefore $0_{2} \in \tilde{J}_{n}$. To conclude our arguments, we note that by Lemma 2.4

$$
f\left(\tilde{J}_{n}\right)^{+}=[f(1-a), 0)=L_{1}^{a}
$$

In this case, we define the following sequence of intervals $I_{n-1}=f^{-1}\left[1-a, 0_{2}\right] \subset$ $\tilde{J}_{n-1}$ and $I_{i}=f^{-1}\left(I_{i+1}\right) \subset J_{i}$. Then using a similar argument as in Case 1 , we have that the interval $J^{\prime}:=I_{0} \subset J$ satisfies the condition $d\left(f^{i}\left(J^{\prime}\right),\{0\}\right)>0$, $i=0, \ldots, n(J)-1$, for $n(J)=n+1$. The proof of Case 2 is complete.

To finish the proof of lemma, it is only left to estimate $n(J)$. For this, note that in any case, by construction $\left|J_{n-2}\right| \geq\left(\frac{a^{2} \eta^{2}}{2}\right)^{n-2}|J|$ and since $\left|J_{n-2}\right| \leq \frac{1}{2}$, the estimate required for $n(J)$ follows immediately.

Remark 1. Note that we also proved the estimate

$$
d\left(f^{i}\left(J^{\prime}\right),\{0\}\right) \geq \frac{(1-a)}{2}\left|J_{i-2}\right|, \quad i=0, \ldots, n(J)-1
$$

where $J_{i-2}$ are given by (8) above.
The next corollary will be a fundamental tool for the proof of Theorem 1, more specifically, see Claim 4 in the proof of Theorem 1

Corollary 1. Let $m_{k} \in \mathbb{N}$ be a sequence such that $\lim _{k \rightarrow \infty} m_{k}=\infty$. Then if $J_{k} \subset I$ with $\left|J_{k}\right| \geq \frac{1}{3 m_{k}^{3}}$, we have the following.
(a) There is a constant $D$ such that $n\left(J_{k}\right) \leq D \log m_{k}$.
(b) There are constants $E>0$ and $\xi>0$ such that for each $i=0,1, \ldots$, $n\left(J_{k}\right)-1$ hold that $\sup _{x \in f^{i}\left(J_{k}^{\prime}\right)}\left|f^{\prime}\right|=E \cdot m_{k}^{\xi}$, where $J_{k}^{\prime}$ is as in Lemma 2.5., Proof.
(a) If $\left|J_{k}\right| \geq \frac{1}{3 m_{k}^{3}}$, Lemma 2.5 implies that

$$
n\left(J_{k}\right) \leq \frac{\log 3-\log 2+3 \log m_{k}+3 \log \frac{a^{2} \eta^{2}}{2}}{\log \frac{a^{2} \eta^{2}}{2}} \leq D \log m_{k}
$$

where $D=5 / \log \frac{a^{2} \eta^{2}}{2}$, and we finish the proof of item (a).
(b) Let $J_{k}^{\prime}$ be the interval given by Lemma 2.5. Then Remark 1 provides

$$
d\left(f^{i}\left(J_{k}^{\prime}\right),\{0\}\right) \geq \frac{1-a}{2}\left|\left(J_{k}\right)_{i-2}\right|,
$$

where $\left(J_{k}\right)_{i-2}$ are defined in (8). The construction of $\left(J_{k}\right)_{i-2}$ gives $\left|\left(J_{k}\right)_{i-2}\right| \geq\left|J_{k}\right| \geq \frac{1}{3 m_{k}^{3}}$. Thus

$$
d\left(f^{i}\left(J_{k}^{\prime}\right),\{0\}\right) \geq \frac{1-a}{2} \cdot \frac{1}{3 m_{k}^{3}}
$$

The next step is to estimate the derivative of $f: f^{i}\left(J_{k}^{\prime}\right) \rightarrow f^{i+1}\left(J_{k}^{\prime}\right)$. For this purpose, we use the inequalities (10) and (6) which provides that

$$
\sup _{x \in f^{i}\left(J_{k}^{\prime}\right)}\left|f^{\prime}\right| \leq C\left(\frac{1-a}{6 m_{k}^{3}}\right)^{\alpha-1}=C \cdot(1-a)^{\alpha-1} \cdot 6^{1-\alpha} m_{k}^{3 \cdot(1-\alpha)} .
$$

We take $E=C \cdot(1-a)^{\alpha-1} \cdot 6^{1-\alpha}$ and $\xi=3 \cdot(1-\alpha)$. This concludes the proof.

## 3. Fat Cantor sets for $f$ and the proof of Theorem 1

The main goal in this section is to prove Theorem that is, that there are infinitely many regular Cantor sets for the one-dimensional map associated to a geometric Lorenz attractor, with Hausdorff dimension ( $H D$ ) very close to 1.

Before we announce precisely this result, let us recall the definition of the Hausdorff dimension of a Cantor set and the notion of a regular Cantor set. We refer the reader to the book [PT93, Chapter 4] for a nice exposition of the main properties of these kinds of Cantor sets. We proceed as follows.

Let $K \subset \mathbb{R}$ be a Cantor set, and let $\mathcal{U}=\left\{U_{i}\right\}_{1 \leq i \leq n}$ be a finite covering of $K$ by open intervals in $\mathbb{R}$. We define the diameter $\operatorname{diam}(\mathcal{U})$ as the maximum of $\ell\left(U_{i}\right), 1 \leq$ $i \leq n$, where $\ell_{i}:=\ell\left(U_{i}\right)$ denotes the length of $U_{i}$. Define $H_{\alpha}(\mathcal{U})=\sum_{1 \leq i \leq n} \ell_{i}^{\alpha}$. Then the Hausdorff $\alpha$-measure of $K$ is

$$
m_{\alpha}(K)=\lim _{\epsilon \rightarrow 0}\left(\inf _{\substack{\mathcal{U} \supset K, \operatorname{diam}(\mathcal{U})<\epsilon}} H_{\alpha}(\mathcal{U})\right)
$$

One can show that there is an unique real number, the Hausdorff dimension of $K$, which we denote by $H D(K)$, such that for $\alpha<H D(K), m_{\alpha}(K)=\infty$ and for $\alpha>H D(K), m_{\alpha}(K)=0$.

Definition 2. A dynamically defined (or regular) Cantor set is a Cantor set $K \subset \mathbb{R}$, together with the following:
(i) disjoint compact intervals $I_{1}, I_{2}, \ldots, I_{r}$ such that $K \subset I_{1} \cup I_{2} \cup \cdots \cup I_{r}$ and the boundary of each $I_{j}$ is contained in $K$;
(ii) there is a $C^{1+\alpha}$ expanding map $\psi$ defined in a neighborhood of $I_{1} \cup I_{2} \cup \cdots \cup I_{r}$ such that, for each $j, \psi\left(I_{j}\right)$ is the convex hull of a finite union of some of these intervals $I_{s}$; moreover, $\psi$ satisfies

- for each $1 \leq j \leq r$ and $n$ sufficiently big, $\psi^{n}\left(K \cap I_{j}\right)=K$;
- $K=\bigcap \psi^{-n}\left(I_{1} \cup I_{2} \cup \cdots \cup I_{r}\right)$.

We say that $\left\{I_{1}, I_{2}, \ldots, I_{r}\right\}$ is a Markov partition for $K$ and that $K$ is defined by $\psi$.

A classical example of regular Cantor set in $\mathbb{R}$ is the ternary Cantor set $K_{\frac{1}{3}}$ of the elements of $[0,1]$ which can be written in base 3 using only digits 0 and 2 . The set $K_{\frac{1}{3}}$ is a regular Cantor set, defined by the map $\psi:\left[0, \frac{1}{3}\right] \cup\left[\frac{2}{3}, 1\right] \rightarrow \mathbb{R}$ given by

$$
\psi(x)=\left\{\begin{array}{cl}
3 x, & \text { if } x \in\left[0, \frac{1}{3}\right] \\
-3 x+3, & \text { if } x \in\left[\frac{2}{3}, 1\right]
\end{array}\right.
$$

There is a class of examples of regular Cantor sets, given by a nontrivial basic set $\Lambda$ associated to a $C^{2}$-diffeomorphism $\varphi: M \rightarrow M$ of a 2-manifold $M$, which appear in the proof of Corollary A. Recall that a basic set is a compact hyperbolic invariant transitive set of $\varphi$ which coincides with the maximal invariant set in a neighborhood of it. "Nontrivial" means that it does not consist of finitely many periodic orbits. These types of regular Cantor sets, roughly speaking, are given by the intersections $W^{s}(x) \cap \Lambda$ and $W^{u}(x) \cap \Lambda$, where $W^{s}(x)$ and $W^{u}(x)$ are the stable and unstable manifolds of $x \in \Lambda$, respectively. We denote by $K^{s}:=W^{s}(x) \cap \Lambda$ the stable Cantor set and by $K^{u}:=W^{u}(x) \cap \Lambda$ the unstable Cantor set (cf. [PT93, chap. 4] or [RM15, Appendix]).

If $\Lambda$ is a basic set associated to a $C^{2}$-diffeomorphism defined in a surface, then it is locally the product of two regular Cantor sets $K^{s}$ and $K^{u}$ (cf. [PT93, Appendix $2]$ ). We shall use the following properties of a regular Cantor set, whose proofs can be found in PT93:
Proposition 2 ([PT93, Proposition 4]). The Hausdorff dimension of a basic set $\Lambda$ satisfies

$$
H D(\Lambda)=H D\left(K^{s} \times K^{u}\right)=H D\left(K^{s}\right)+H D\left(K^{u}\right)
$$

Proposition 3 ([PT93, Proposition 7]). If $K$ is a regular Cantor set, then

$$
0<H D(K)<1
$$

Proof of Theorem 1. We construct the Cantor sets inductively. Denote $L_{1}=$ $[-1 / 2,0)$ and $L_{2}=(0,1 / 2]$, and pick any interval $I_{1} \subset L_{1}$. Lemma 2.1 implies that there is an iterate $f^{n^{1}}$ of $f$ such that $f^{n^{1}}: I_{1} \rightarrow L_{1}$ is a diffeomorphism. Let $\left\{J_{1}^{1}, J_{2}^{1}\right\}$ be the complementary intervals in $L_{1}$ of $I_{1}$. Again Lemma 2.1]implies that there are $I_{1}^{1} \subset J_{1}^{1}, I_{2}^{1} \subset J_{2}^{1}, n_{1}^{1}$, and $n_{2}^{1}$ such that $f^{n_{i}^{1}}: I_{i}^{1} \rightarrow L_{1}$ is a diffeomorphism.

Let $\left\{J_{1}^{11}, J_{1}^{12}\right\}$ be the complementary intervals of $I_{1}^{1}$ in $J_{1}^{1}$, and let $\left\{J_{2}^{11}, J_{2}^{12}\right\}$ be the complementary intervals $I_{2}^{1}$ in $J_{2}^{1}$.

Continuing with this process, in the $k$ th step, we obtain $r_{k}=2^{k}-1$ intervals $I_{1}, \ldots, I_{r_{k}}$ such that, for each $i \in\{1, \ldots, k\}$, there is $n_{i}$ so that $f^{n_{i}}: I_{r_{i}} \rightarrow L_{1}$ is a diffeomorphism.

Now let $\left\{J_{1}^{(k)}, \ldots, J_{r_{k}+1}^{(k)}\right\}$ be the complementary intervals of $\bigcup_{i=1}^{r_{k}} I_{i}$ in $L_{1}$, and let $\mu$ be an the invariant measure given by Lemma [2.2, which is absolutely continuous w.r.t. Lebesgue, and thus there is a constant $c$ such that

$$
\begin{equation*}
\mu(I) \leq c m(I)=c|I| \tag{11}
\end{equation*}
$$

for any interval $I$. Take $\epsilon_{k}=\frac{1}{c} \min _{i}\left\{\mu\left(J_{i}^{(k)}\right)\right\} \leq \min _{i}\left\{\left|J_{i}^{(k)}\right|\right\}$, and put $m_{k}=\left\lfloor\frac{1}{\epsilon_{k}}\right\rfloor$ the integer part of $\frac{1}{\epsilon_{k}}$, that is, $m_{k} \leq \frac{1}{\epsilon_{k}}<m_{k}+1$.

Next, split each interval $J_{i}^{(k)}$ in $2^{m_{k}}$ intervals $\left\{J_{i, j}^{(k)}: j=1, \ldots, 2^{m_{k}}\right\}$ that are pairwise disjoint of equal $\mu$-size. Then, for $j=1, \ldots, 2^{m_{k}}$, we have

$$
\begin{equation*}
\frac{1}{2^{m_{k}}} \geq \frac{\left|J_{i}^{(k)}\right|}{2^{m_{k}}}=\left|J_{i, j}^{(k)}\right| \geq \frac{1}{c} \mu\left(J_{i, j}^{(k)}\right)=\frac{1}{c} \frac{\mu\left(J_{i}^{(k)}\right)}{2^{m_{k}}} \geq \frac{\epsilon_{k}}{2^{m_{k}}}>\frac{1}{2^{m_{k}}\left(m_{k}+1\right)} \tag{12}
\end{equation*}
$$

Consider the interval $\left(-\frac{1}{m_{k}^{3}}, \frac{1}{m_{k}^{3}}\right)$. Since $\mu$ is $f$-invariant, inequality (11) implies that
$\mu\left(\bigcup_{j=1}^{4 m_{k}} f^{-j}\left(-\frac{1}{m_{k}^{3}}, \frac{1}{m_{k}^{3}}\right)\right) \leq \sum_{j=0}^{4 m_{k}} \mu\left(f^{-j}\left(-\frac{1}{m_{k}^{3}}, \frac{1}{m_{k}^{3}}\right)\right)=\sum_{j=0}^{4 m_{k}} \mu\left(-\frac{1}{m_{k}^{3}}, \frac{1}{m_{k}^{3}}\right)$

$$
\begin{equation*}
\leq 2 c \sum_{j=0}^{4 m_{k}} \frac{1}{m_{k}^{3}}=2 c\left(\frac{4 m_{k}+1}{m_{k}^{3}}\right) . \tag{13}
\end{equation*}
$$

In what follows, given $A \subset \mathbb{R}, \# A$ denotes the cardinality of $A$.
Claim 1. For any $k$ and any $1 \leq i \leq r_{k}+1$ there is a set $\mathcal{R}_{i} \subset\left\{1, \ldots, 2^{m_{k}}\right\}$ with $\# \mathcal{R}_{i}=2^{m_{k}-1}$, such that for each $r \in \mathcal{R}_{i}$ there is a point $x \in J_{i, r}^{(k)}$ such that

$$
x \notin \bigcup_{j=1}^{4 m_{k}} f^{-j}\left(-\frac{1}{m_{k}^{3}}, \frac{1}{m_{k}^{3}}\right)
$$

Proof. The idea of the proof is to count the number of intervals that does not satisfy this property. To do that, consider the set

$$
\mathcal{R}_{i}^{C}:=\left\{j: J_{i, j}^{(k)} \subset \bigcup_{j=0}^{4 m_{k}} f^{-j}\left(-\frac{1}{m_{k}^{3}}, \frac{1}{m_{k}^{3}}\right)\right\} .
$$

We want show that $\# \mathcal{R}_{i}^{C}<2^{m_{k}-1}$. For this we proceed as follows. Put $\# \mathcal{R}_{i}^{C}=$ $2^{m_{k}-n_{k}}+N_{k}$ with $0 \leq N_{k}<2^{m_{k}-n_{k}}$, and let $\underline{j} \in\left\{1, \ldots, 2^{m_{k}}\right\}$. Then, by the definition of $J_{i, j}^{(k)}$, we obtain $\mu\left(J_{i, \underline{j}}^{(k)}\right)=\mu\left(J_{i, j}^{(k)}\right)$ for all $j \in \mathcal{R}_{i}^{C}$. Hence, equations (12) and (13) imply that

$$
\frac{1}{2^{m_{k}}\left(m_{k}+1\right)}\left(2^{m_{k}-n_{k}}+N_{k}\right)<\mu\left(J_{i, \underline{j}}^{(k)}\right) \cdot \# \mathcal{R}_{i}^{C} \leq \mu\left(\bigcup_{j \in \mathcal{R}_{i}^{C}} J_{i, j}^{(k)}\right) \leq 2 c\left(\frac{4 m_{k}+1}{m_{k}^{3}}\right)
$$

Hence we have

$$
\frac{1}{2^{n_{k}}} \leq \frac{1}{2^{n_{k}}}+\frac{N_{k}}{2^{m_{k}}} \leq 2 c \cdot \frac{\left(4 m_{k}+1\right)\left(m_{k}+1\right)}{m_{k}^{3}} \leq \frac{20 c}{m_{k}}
$$

which implies that if $m_{k}$ is large enough $\left(m_{k}>40 c\right)$, then $n_{k}$ should be bigger than 1, i.e., $n_{k}>1$.

Now $N_{k}<2^{m_{k}-n_{k}}$ implies that $\frac{N_{k}}{2^{m_{k}}}<\frac{1}{2^{n}{ }^{n}}$, and as $n_{k}>1$, we get

$$
\frac{1}{2^{n_{k}}}+\frac{N_{k}}{2^{m_{k}}}<\frac{2}{2^{n_{k}}} \leq \frac{1}{2}
$$

Thus $\# \mathcal{R}_{i}^{C}=2^{m_{k}-n_{k}}+N_{k}<2^{m_{k}-1}$, and this concludes the proof of Claim 1 .
Claim 2. Consider the set $\mathcal{R}_{i}^{+}=\left\{r \in \mathcal{R}_{i}:\left|J_{i, r}^{(k)}\right| \geq \frac{1}{3 m_{k}^{3}}\right\}$. Then $\# \mathcal{R}_{i}^{+}<2^{m_{k}-2}$.
Proof. As the intervals $J_{i, j}^{(k)}$ are pairwise disjoint, if $\# \mathcal{R}_{i}^{+} \geq 2^{m_{k}-2}$, then

$$
1 \geq\left|\bigcup_{j \in \mathcal{R}_{i}^{+}} J_{i, j}^{(k)}\right| \geq \frac{2^{m_{k}-1}}{3 m_{k}^{3}}
$$

which implies a contradiction for $m_{k}$ large enough.
The above claim ensures that the set $\widetilde{\mathcal{R}}_{i}:=\mathcal{R}_{i} \backslash \mathcal{R}_{i}^{+}$has cardinality $\# \widetilde{\mathcal{R}}_{i} \geq$ $2^{m_{k}-2}$.
Claim 3. For all $r \in \widetilde{\mathcal{R}}_{i}$ there is $j(i, r) \in\left\{1, \ldots, 4 m_{k}\right\}$ minimal, such that

$$
\begin{equation*}
\left|f^{j(i, r)}\left(J_{i, r}^{(k)}\right)\right|>\frac{1}{3 m_{k}^{3}} . \tag{14}
\end{equation*}
$$

Proof. Let $r \in \widetilde{\mathcal{R}}_{i}$, then if $\left|f^{s}\left(J_{i, r}^{(k)}\right)\right| \geq \frac{1}{m_{k}^{3}}>\frac{1}{3 m_{k}^{3}}$ for some $s \in j=1, \ldots, 4 m_{k}$, we are done. Otherwise, assume that there is $s \in\left\{1, \ldots, 4 m_{k}\right\}$ such that $\left|f^{t}\left(J_{i, r}^{(k)}\right)\right|<$ $\frac{1}{m_{k}^{3}}$ for all $1 \leq t \leq s$. If $0 \in f^{t_{0}}\left(J_{i, r}^{(k)}\right)$ for some $1 \leq t_{0} \leq s$, Claim 1 implies that there is $x_{r} \in J_{i, r}^{(k)}$ such that $x_{r} \notin f^{-j}\left(-\frac{1}{m_{k}^{3}}, \frac{1}{m_{k}^{3}}\right)$ for $j=1, \ldots, 4 m_{k}$, and so we get $\left|f^{t_{0}}\left(J_{i, r}^{(k)}\right)\right|>\frac{1}{m_{k}^{3}}$, contradicting our hypothesis. Thus $0 \notin f^{t}\left(J_{i, r}^{(k)}\right)$ for all $1 \leq t \leq s$. Since $f^{s}$ acts as a diffeomorphism on $J_{i, r}^{(k)}$ with derivative $\left|\left(f^{s}\right)^{\prime}\right|>\eta^{s}>2^{s / 2}$ and equation (12) holds, we obtain

$$
\begin{aligned}
\frac{1}{m_{k}^{3}} & \geq\left|f^{s}\left(J_{i, j}^{(k)}\right)\right| \geq \frac{\eta^{s}}{2^{m_{k}}\left(m_{k}+1\right)} \geq \frac{2^{s / 2}}{2^{m_{k}}\left(m_{k}+1\right)} \\
& =\frac{2^{s / 2-m_{k}}}{m_{k}+1} \Longrightarrow s / 2-m_{k}<0 \Longrightarrow s<2 m_{k}
\end{aligned}
$$

If $\left|f^{s+1}\left(J_{i, r}^{(k)}\right)\right|>\frac{1}{3 m_{k}^{3}}$, then we are done. Otherwise, if $\left|f^{s+1}\left(J_{i, r}^{(k)}\right)\right| \leq \frac{1}{3 m_{k}^{3}}<\frac{1}{m_{k}^{3}}$, reasoning as before, we get that $0 \notin f^{s+1}\left(J_{i, r}^{(k)}\right)$. Since $0 \notin f^{s+1}\left(J_{i, r}^{(k)}\right)$, then $f$ acts as a diffeomorphism on $f^{s}\left(J_{i, r}^{(k)}\right)$ with derivative $\left|f^{\prime}\right|>\eta$, which allows us to state that $\left|f^{s+1}\left(J_{i, r}^{(k)}\right)\right|>\eta\left|f^{s}\left(J_{i, r}^{(k)}\right)\right|$. Again, if $\left|f^{s+2}\left(J_{i, r}^{(k)}\right)\right|>\frac{1}{3 m_{k}^{3}}$, we are done. Otherwise, if $\left|f^{s+2}\left(J_{i, r}^{(k)}\right)\right| \leq \frac{1}{3 m_{k}^{3}}<\frac{1}{m_{k}^{3}}$ and, reasoning as before, we get that $0 \notin f^{s+2}\left(J_{i, r}^{(k)}\right)$, then

$$
\left|f^{s+2}\left(J_{i, r}^{(k)}\right)\right|>\eta\left|f^{s+1}\left(J_{i, r}^{(k)}\right)\right|>\eta^{2}\left|f^{s}\left(J_{i, r}^{(k)}\right)\right| .
$$

Using this argument recursively, if $\left|f^{s+2 m_{k}-1}\left(J_{i, r}^{(k)}\right)\right| \leq \frac{1}{3 m_{k}^{3}}<\frac{1}{m_{k}^{3}}$, then $0 \notin$ $f^{s+2 m_{k}-1}\left(J_{i, r}^{(k)}\right)$ and it holds that

$$
\left|f^{s+2 m_{k}}\left(J_{i, r}^{(k)}\right)\right|>\eta\left|f^{s+2 m_{k}-1}\left(J_{i, r}^{(k)}\right)\right|>\cdots>\eta^{2 m_{k}}\left|f^{s}\left(J_{i, r}^{(k)}\right)\right|
$$

and

$$
\left|f^{s}\left(J_{i, r}^{(k)}\right)\right|>\eta^{s}\left|J_{i, r}^{(k)}\right| .
$$

Thanks to inequality (12) we conclude that

$$
\left|f^{s+2 m_{k}}\left(J_{i, r}^{(k)}\right)\right|>\frac{2^{m_{k}} \eta^{s}}{2^{m_{k}}\left(m_{k}+1\right)}=\frac{\eta^{s}}{m_{k}+1}>\frac{1}{3 m_{k}^{3}},
$$

finishing the proof of Claim 3.
Now consider the sequence of intervals $f^{j(i, r)}\left(J_{i, r}^{(k)}\right)$ given by Claim 3. Since $\left|f^{j(i, r)}\left(J_{i, r}^{(k)}\right)\right|>\frac{1}{3 m_{k}^{3}}$ for all $r \in \widetilde{\mathcal{R}}_{i}$, we can apply Lemma 2.5 and Corollary 1 to get the following.
Claim 4. For all $r \in \widetilde{\mathcal{R}}_{i}$, there is an interval $I_{i, r}^{(k)} \subset f^{j(i, r)}\left(J_{i, r}^{(k)}\right)$ and an integer $m_{i, r}^{(k)}$ such that $f^{m_{i, r}^{k}}: I_{i, r}^{(k)} \rightarrow L_{1}^{a}$ is a diffeomorphism, $0 \notin f^{s}\left(I_{i, r}^{(k)}\right)$ for $s=0,1, \ldots$, $m_{i, r}^{(k)}-1, m_{i, r}^{(k)} \leq D \log m_{k}$, and $\sup _{x \in f^{s}\left(I_{i, r}^{(k)}\right)}\left|f^{\prime}\right|=E \cdot m_{k}^{\xi}$.

Claim 5. Let $\tilde{I}_{i, r}^{(k)} \subset J_{i, r}^{(k)}$ with $f^{j(i, r)}\left(\tilde{I}_{i, r}^{(k)}\right)=I_{i, r}^{(k)}$, where $I_{i, r}^{(k)}$ is as in Claim 4. Then, there is a constant $H>0$, depending only of $f$, such that

$$
\begin{equation*}
\left|\tilde{I}_{i, r}^{k}\right| \geq H\left|I_{i, r}\right|\left|J_{i, r}^{(k)}\right| \tag{15}
\end{equation*}
$$

Proof. First note that the mean value theorem implies

$$
\begin{equation*}
\frac{\left|\tilde{I}_{i, r}^{(k)}\right|}{\left|J_{i, r}^{(k)}\right|}=\frac{\left|\left(f^{j(i, r)}\right)^{\prime}(y)\right|}{\left|\left(f^{j(i, r)}\right)^{\prime}(x)\right|} \cdot \frac{\left|I_{i, r}\right|}{\left|f^{j(i, r)}\left(J_{i, r}^{(k)}\right)\right|} \text { for some } x \in \tilde{I}_{i, r}^{(k)} ; y \in J_{i, r}^{(k)} \tag{16}
\end{equation*}
$$

It is enough to bound $\frac{\left|\left(f^{j(i, r)}\right)^{\prime}(y)\right|}{\left|\left(f^{j(i, r)}\right)^{\prime}(x)\right|}$, since equality (16) implies that inequality (15) holds. For this reason, we proceed as follows. As $j(i, r)$ is minimal satisfying (14), we get

$$
\begin{equation*}
\left|f^{s}\left(J_{i, r}^{(k)}\right)\right|<\frac{1}{3 m_{k}^{3}} \text { for } s=0, \ldots, j(i, r)-1 \tag{17}
\end{equation*}
$$

This implies, reasoning as in the proof of Claim 3, that $0 \notin f^{s}\left(J_{i, r}^{(k)}\right)$ for $s=$ $0, \ldots, j(i, r)-1$ and hence $\left.f^{s}\right|_{J_{i, r}^{(k)}}$ is a diffeomorphism for $s=0, \ldots, j(i, r)-1$.

Observe that by Claim for each $s \in\{0, \ldots, j(i, r)-1\}$, there is $x_{s} \in J_{i, r}^{(k)}$ such that $f^{s}\left(x_{s}\right) \notin\left(-\frac{1}{m_{k}^{3}}, \frac{1}{m_{k}^{3}}\right)$, and so, if $d(\cdot, \cdot)$ is the distance between sets, by equation (17), we conclude that

$$
\begin{equation*}
\inf _{x \in J_{i, r}^{(k)}}\left|f^{s}(x)\right|=: d\left(f^{s}\left(J_{i, r}^{(k)}\right),\{0\}\right)>\frac{1}{2 m_{k}^{3}} \tag{18}
\end{equation*}
$$

Now we have

$$
\begin{align*}
& \left\lvert\, \log \frac{\left(f^{j(i, r)}\right)^{\prime}(y)}{\left(f^{j(i, r))^{\prime}(x)} \mid\right.}\right. \\
& \quad=\mid \sum_{s=0}^{j(i, r)-1} \log \left(f^{\prime}\left(f^{s}(y)\right)-\log \left(f^{\prime}\left(f^{s}(x)\right) \mid\right.\right. \\
& \quad \leq \sum_{s=0}^{j(i, r)-1} \mid \log \left(f^{\prime}\left(f^{s}(y)\right)-\log \left(f^{\prime}\left(f^{s}(x)\right) \mid\right.\right. \\
& \quad \leq b y{ }_{b y} \sum_{s=0}^{j(i, r)-1} \frac{\mid f^{\prime \prime}\left(f^{s}\left(z_{s}\right) \mid\right.}{\left|f^{\prime}\left(f^{s}\left(z_{s}\right)\right)\right|}\left|f^{s}(y)-f^{s}(x)\right|  \tag{19}\\
& \quad \leq_{b y} \text { (6) some } z_{s} \in J_{i, r}^{(k)} \sum_{s=0}^{j(i, r)-1} C \cdot C_{1} \cdot \frac{1}{\left|f^{s}\left(z_{s}\right)\right|} \cdot\left|f^{s}\left(J_{i, r}^{(k)}\right)\right|
\end{align*}
$$

$$
\text { where } C, C_{1} \text { depend only on } f
$$

$$
\leq_{b y}\left\lfloor\sum_{s=0}^{j(18)} C \cdot C_{1} \cdot 2 m_{k}^{3} \cdot\left|f^{s}\left(J_{i, r}^{(k)}\right)\right| .\right.
$$

Recall that equation (6) implies that $\frac{\left|f^{\prime \prime}(x)\right|}{\left|f^{\prime}(x)\right|} \leq \frac{C \cdot C_{1}}{|x|}$, with $C_{1}, C$ depending only of $f$. Thus, since $\left.f^{s}\right|_{J_{i, r}^{(k)}}$ is a diffeomorphism for each $s \in\{0, \ldots, j(i, r)-1\}$, and satisfies property ( $f 2$ ) (see Section 2.1), we get

$$
\begin{aligned}
\left|f^{j(i, r)-1}\left(J_{i, r}^{(k)}\right)\right| & \geq \sqrt{2}\left|f^{j(i, r)-2}\left(J_{i, r}^{(k)}\right)\right| \\
& \geq \sqrt{2}^{2}\left|f^{j(i, r)-3}\left(J_{i, r}^{(k)}\right)\right| \geq \cdots>\sqrt{2}^{s}\left|f^{j(i, r)-(s+1)}\left(J_{i, r}^{(k)}\right)\right| .
\end{aligned}
$$

Making the change of variable $t=j(i, r)-(s+1)$, the last inequality provides

$$
\begin{equation*}
\left|f^{j(i, r)-1}\left(J_{i, r}^{(k)}\right)\right| \geq(\sqrt{2})^{j(i, r)-t-1}\left|f^{t}\left(J_{i, r}^{(k)}\right)\right| . \tag{20}
\end{equation*}
$$

Using the inequality (20) together with (17) and replacing in the last term of equation (19), we get that

$$
\begin{equation*}
\left|\log \frac{\left(f^{j(i, r)}\right)^{\prime}(y)}{\left(f^{j(i, r)}\right)^{\prime}(x)}\right| \leq C \cdot C_{1} \sum_{t=0}^{j(i, r)-1} 2 m_{k}^{3} \cdot \frac{1}{3 m_{k}^{3}}\left(\frac{1}{\sqrt{2}}\right)^{j(i, r)-t-1}<\frac{2}{3} \cdot C \cdot C_{1} \cdot \sqrt{2}(\sqrt{2}+1) \tag{21}
\end{equation*}
$$

 lows, implying that inequality (15) holds. The proof of Claim 5 is finished.

The next step is to construct the regular Cantor with Hausdorff dimension close to 1 . For this sake, we consider the collection of surjective maps

Let $g_{k i}: L_{k}^{i}=\bigcup_{r \in \widetilde{\mathcal{R}}_{i}} \tilde{I}_{i, r}^{(k)} \rightarrow L_{1}^{a}$ be defined by $g_{k i}=\left.g_{i, r}\right|_{\tilde{I}_{i, r}^{(k)}}$, and let $C_{k}^{i}$ be the regular Cantor set defined by the intervals $\tilde{I}_{i, r}^{(k)}$ and $g_{i, r}$, i.e.,

$$
C_{k}^{i}=\bigcap_{n \geq 1} g_{k i}^{-n}\left(L_{k}^{i}\right)
$$

The final step is to show that $H D\left(C_{k}^{i}\right) \rightarrow 1$ as $k \rightarrow+\infty$. For this, we use the same strategy given in [PT93, Theorem 3]. In fact, consider the number

$$
\Lambda_{1, \tilde{I}_{i, r}^{(k)}}=\sup _{x \in \tilde{I}_{i, r}^{(k)}}\left|g_{i, r}^{\prime}\right|
$$

and define $d_{1} \in[0,1]$ by

$$
\begin{equation*}
\sum_{r \in \widetilde{\mathcal{R}}_{i}}\left(\Lambda_{1, \tilde{I}_{i, r}^{(k)}}\right)^{-d_{1}}=1 \tag{22}
\end{equation*}
$$

It is shown in PT93, pp. 69-70] that $d_{1} \leq H D\left(C_{k}^{i}\right)$. Therefore, we can estimate
 simplify notation, denote $h_{1}=f^{m_{i, r}^{(k)}}$ and $h_{2}=f^{j(i, r)}$. Then

$$
\Lambda_{1, \tilde{I}_{i, r}^{(k)}} \leq \sup _{I_{i, r}^{(k)}}\left|h_{1}^{\prime}\right| \cdot \sup _{\tilde{I}_{i, r}^{(k)}}\left|h_{2}^{\prime}\right| .
$$

Corollary 1 gives that $\sup _{I_{i, r}^{(k)}}\left|h_{1}^{\prime}\right| \leq E^{D \log m_{k}} \cdot m_{k}^{F \log m_{k}}$, where $F=D \cdot \xi$. To estimate the the supremum of $\left|h_{2}^{\prime}\right|$, $\sup \left|h_{2}^{\prime}\right|$, in $\tilde{I}_{i, r}$, we note that by the proof of Claim 5 the function $\left.h_{2}\right|_{\tilde{I}_{i, r}^{(k)}}$ has bounded distortion. Thus

$$
\sup _{\tilde{I}_{i, r}^{(k)}}\left|h_{2}^{\prime}\right| \leq H^{-1} \inf _{\tilde{I}_{i, r}^{(k)}}\left|h_{2}^{\prime}\right|,
$$

where $H:=e^{-\frac{2}{3} \cdot C \cdot C_{1} \cdot \sqrt{2} \cdot(\sqrt{2}+1)}$ (see equation (21)). Since $h_{2}\left(\tilde{I}_{i, r}^{(k)}\right)=I_{i, r}^{(k)}$, the mean value theorem implies

$$
\inf _{\tilde{I}_{i, r}^{(k)}}\left|h_{2}^{\prime}\right| \leq \frac{\left|I_{i, r}^{(k)}\right|}{\left|\tilde{I}_{i, r}^{(k)}\right|} \stackrel{\text { by }}{\leq} \frac{H^{-1}}{\left|J_{i, r}^{(k)}\right|} \stackrel{\text { by }}{\leq} H^{-1} 2^{m_{k}}\left(m_{k}+1\right)
$$

The last two inequalities imply that

$$
\begin{equation*}
\Lambda_{1, \tilde{I}_{i, r}^{(k)}} \leq H^{-2} \cdot E^{D \log m_{k}} \cdot m_{k}^{F \log m_{k}} \cdot 2^{m_{k}}\left(m_{k}+1\right)=2^{(1+o(1)) m_{k}} \tag{23}
\end{equation*}
$$

since

$$
\lim _{k \rightarrow \infty} \frac{\log H^{-2}+D \log m_{k} \cdot \log E+F \cdot\left(\log m_{k}\right)^{2}+m_{k} \log 2+\log \left(m_{k}+1\right)}{m_{k}}=\log 2
$$

Therefore, since $\# \widetilde{\mathcal{R}}_{i} \geq 2^{m_{k}-2}$, inequalities (22) and (23) imply that

$$
2^{m_{k}-2} \cdot\left(\frac{1}{2^{(1+o(1)) m_{k}}}\right)^{d_{1}} \leq 1=\sum_{r \in \widetilde{\mathcal{R}}_{i}}\left(\Lambda_{1, \tilde{I}_{i, r}^{(k)}}\right)^{-d_{1}}
$$

Hence

$$
\begin{aligned}
\left(m_{k}\right. & -2) \log 2 \leq(1+o(1)) \cdot m_{k} \cdot d_{1} \cdot \log 2 \\
& \Longrightarrow 1-o(1)=\frac{m_{k}-2}{m_{k}} \leq(1+o(1)) \cdot d_{1} \\
& \Longrightarrow 1-o(1) \leq(1+o(1)) \cdot d_{1} .
\end{aligned}
$$

Thus, $1-o(1) \leq d_{1} \leq 1$. Now we define $C_{k}:=\bigcup_{i} C_{k}^{i}$ which satisfies the condition of the theorem, finishing the proof of Theorem 1 .

As an immediate consequence of Theorem [1 we have the following.
Corollary C. The Hausdorff dimension of the bidimensional attractor for the Poincaré map $P, \Lambda_{P}$, is strictly greater than 1.

Proof. For this, let $\Gamma=\{(x, y, 1): x=0\}$, and let

$$
\Lambda_{P}=\overline{\bigcap_{i \geq 1} P^{i}(S \backslash \Gamma)}
$$

be as in equation (5). For each $k>0$, let $C_{k}$ be the regular Cantor set given by Theorem 11 and define

$$
\begin{equation*}
\Lambda_{P}^{k}=\left\{(x, y) \in \Lambda_{P}: x \in C_{k}\right\} . \tag{24}
\end{equation*}
$$

Notice that by construction, each $C_{k}$ is a regular Cantor set (see comments after Definition (2) and so, for each $k, \Lambda_{P}^{k}$ is a basic set for $P$. Moreover, $\Lambda_{P}^{k} \subset \Lambda_{P}^{k+1}$, and by Proposition 2

$$
H D\left(\Lambda_{P}^{k}\right)=H D\left({ }_{u} K_{P}^{k}\right)+H D\left({ }_{s} K_{P}^{k}\right)=H D\left(C_{k}\right)+H D\left({ }_{s} K_{P}^{k}\right)
$$

where ${ }_{s} K_{P}^{k}$ and ${ }_{u} K_{P}^{k}=C_{k}$ are the stable and unstable Cantor sets associated to the basic set $\Lambda_{P}^{k}$. As ${ }_{s} K_{P}^{k}$ is a regular Cantor set, by Proposition 3, there is $\xi>0$ such that $H D\left({ }_{s} K_{P}^{1}\right)>\xi$. Hence

$$
H D\left(\Lambda_{P}^{k}\right)=H D\left(C_{k}\right)+H D\left({ }_{s} K_{P}^{k}\right) \geq H D\left(C_{k}\right)+H D\left({ }_{s} K_{P}^{1}\right)>H D\left(C_{k}\right)+\xi
$$

Thus, Theorem 1 implies that $H\left(\Lambda_{P}^{k}\right)>1$ for $k$ large enough. Since $\Lambda_{P}^{k} \subset \Lambda_{P}$, this finishes the proof of Corollary $\mathbb{C}$.
Proof of Theorem A. Note that the geometric Lorenz attractor $\Lambda$ satisfies

$$
\Lambda=\left(\bigcup_{t \in \mathbb{R}} X^{t}\left(\Lambda_{P}\right)\right) \cup O, \quad \text { where } O \text { is the singularity. }
$$

Thus,

$$
H D(\Lambda) \geq 1+H D\left(\Lambda_{P}\right)>2
$$

The proof of Theorem $A$ is complete.
We finish this section by announcing a corollary of the proof of Theorem that might be of interest to the reader.
Corollary D. If $f$ is a $C^{2}$-function that satisfies the properties $(f 1)-(f 3)$ described in Section 2.1 with $f\left(-\frac{1}{2}\right) \neq-\frac{1}{2}, f\left(\frac{1}{2}\right) \neq \frac{1}{2}$ and also satisfies equation (6), then there is an increasing family of regular Cantor sets $C_{k}$ for $f$ such that

$$
H D\left(C_{k}\right) \rightarrow 1 \text { as } k \rightarrow+\infty .
$$

## 4. Lagrange and Markov spectra: proof of Theorem B

In this section we prove Theorem B For this, we first prove that small perturbations of the Poincaré map $P$ restricted to $\Lambda_{P}^{k}$, with $\Lambda_{P}^{k}$ defined at (24), can be realized as Poincaré maps of small perturbations of the initial geometric Lorenz flow $X^{t}$ (Lemma 4.1). Then, taking $k$ such that $H D\left(\Lambda_{P}^{k}\right)>1$, we recover the properties described in [RM17 needed to apply [RM17, Main Theorem], obtaining nonempty interior in the Lagrange and Markov spectrum.

We start by giving the main theorem from RM17, which is a fundamental tool for obtaining Theorem Given $A \subset M$, $\operatorname{int}(A)$ denotes the interior of $A$.
Theorem (Main Theorem at RM17]). Let $\Lambda$ be a horseshoe associated to a $C^{2}$ diffeomorphism $\varphi$ such that $H D(\Lambda)>1$. Then there is, arbitrarily close to $\varphi, a$ diffeomorphism $\varphi_{0}$ and a $C^{2}$-neighborhood $\mathcal{W}$ of $\varphi_{0}$ such that, if $\Lambda_{\psi}$ denotes the continuation of $\Lambda$ associated to $\psi \in \mathcal{W}$, there is an open and dense set $H_{1}\left(\psi, \Lambda_{\psi}\right) \subset$ $C^{1}(M, \mathbb{R})$ such that for all $f \in H_{1}\left(\psi, \Lambda_{\psi}\right)$, we have

$$
\operatorname{int}\left(L\left(\psi, \Lambda_{\psi}, f\right)\right) \neq \emptyset \quad \text { and } \quad \operatorname{int}\left(M\left(\psi, \Lambda_{\psi}, f\right)\right) \neq \emptyset
$$

The set $H_{1}\left(\psi, \Lambda_{\psi}\right)$ is described by

$$
H_{\psi}=\left\{f \in C^{1}(M, \mathbb{R}): \# M_{f}\left(\Lambda_{\psi}\right)=1, z \in M_{f}\left(\Lambda_{\psi}\right), D \psi_{z}\left(e_{z}^{s, u}\right) \neq 0\right\}
$$

where $M_{f}\left(\Lambda_{\psi}\right):=\left\{z \in \Delta: f(z) \geq f(x)\right.$ for all $\left.x \in \Lambda_{\psi}\right\}$ is the set of maximum points of $f$ in $\Lambda_{\psi}$ and $e_{z}^{s, u}$ are unit vectors in $E^{s, u}(z)$, respectively.
4.1. Perturbations of the Poincaré map. Fix $k$ with $H D\left(\Lambda_{P}^{k}\right)>1$. By construction, there is $\epsilon>0$ small so that $d\left(\Lambda_{P}^{k}, \Gamma\right)>2 \epsilon$, where $\Gamma=\{(x, y, 1): x=0\}$. Let $\mathcal{U}_{P}$ be a $C^{2}$-neighborhood of $P$ such that, if $\tilde{P} \in \mathcal{U}_{P}$ and $\Lambda_{\tilde{P}}^{k}$ is the hyperbolic continuation of $\Lambda_{P}^{k}$, then $d\left(\Lambda_{\tilde{P}}^{k}, \Gamma\right)>\epsilon$.

The next lemma states that in a neighborhood of $\Lambda_{\tilde{P}}^{k}$, we can recover $\tilde{P} \in \mathcal{U}_{P}$ as a Poincaré map associated to a geometric Lorenz flow $\tilde{X}^{t}, C^{2}$-close to $X^{t}$.
Lemma 4.1. Given $\tilde{P} \in \mathcal{U}_{P}$, there is a geometric Lorenz flow $\tilde{X}^{t}, C^{2}$-close to $X^{t}$, such that the restriction to $\Lambda_{\tilde{P}}^{k}$ of the Poincaré map associated to $\tilde{X}^{t}$ coincides with the restriction of $\tilde{P}$ to $\Lambda_{\tilde{P}}^{k}$.
Proof. For the proof we construct explicitly a flow $\tilde{X}^{t}$, with the desired properties. For this, we proceed as follows. Let $\tilde{\mathcal{R}}=R_{1} \cup R_{2} \cup \cdots \cup R_{m}$ be a Markov partition of $\Lambda_{\tilde{P}}^{k}$, and let $U_{i} \subset S$ be an open set with $R_{i} \subset U_{i}, d\left(U_{i}, \Gamma\right)>\frac{\epsilon}{2}$ for all $i$, and such that if $\tilde{P}(x, y) \in R_{i}$, then $P(x, y) \in U_{i}$. The tubular flow theorem applied to $X$ gives local charts $\psi_{i}: U_{i} \times[-1,1] \rightarrow \mathbb{R}^{3}$ for $i \in\{1, \ldots, m\}$ satisfying

$$
\begin{equation*}
\psi_{i}\left(U_{i} \times\{0\}\right) \subset S \quad \text { and } \quad D\left(\psi_{i}\right)_{(x, y, t)}(0,0,1)=X\left(\psi_{i}(x, y, t)\right) \tag{25}
\end{equation*}
$$

Put $W_{i}:=\psi_{i}\left(U_{i} \times(-1,1)\right)$. Without loss of generality, we can assume that

$$
W_{i} \cap W_{j}=\emptyset \quad \text { if } \quad i \neq j
$$

We denote by $\tilde{P}_{i}$ and $P_{i}$ the maps $\tilde{P}$ and $P$, respectively, in these coordinates.
Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be a $C^{\infty}$-bump function such that $\varphi(t)=0$ for $t \leq-1$ and $\varphi(t)=1$ for $t \geq 1$. Define the following flow on $U_{i} \times[-1,1]$ :

$$
\phi_{i}^{t}(x, y, 0)=\left(P_{i}(x, y)+\varphi(3 t+1)\left(\tilde{P}_{i}(x, y)-P(x, y)\right), t\right) .
$$

Note that

$$
\phi_{i}^{t}(x, y, 0)= \begin{cases}\left(P_{i}(x, y), t\right), & \text { if } t \leq-\frac{2}{3}  \tag{26}\\ \left(\tilde{P}_{i}(x, y), t\right), & \text { if } t \geq 0 .\end{cases}
$$

Consider the vector field on $U_{i} \times[-1,1]$ given by

$$
\begin{equation*}
Z_{i}\left(\phi_{i}^{t}(x, y, 0)\right)=\frac{\partial}{\partial t} \phi_{i}^{t}(x, y, 0)=\left(3 \varphi^{\prime}(3 t+1)\left(\tilde{P}_{i}(x, y)-P(x, y)\right), 1\right) . \tag{27}
\end{equation*}
$$

By equation (26), this vector field satisfies

$$
Z_{i}\left(\phi_{i}^{t}(x, y, 0)\right)= \begin{cases}(0,0,1), & \text { if } t \leq-\frac{2}{3}  \tag{28}\\ (0,0,1), & \text { if } t \geq 0\end{cases}
$$

Let $Y_{i}$ be the vector field on $W_{i}=\psi_{i}\left(U_{i} \times(-1,1)\right)$ defined by

$$
Y_{i}\left(\psi_{i}(x, y, t)\right)=D\left(\psi_{i}\right)_{(x, y, t)}\left(Z_{i}\left(\phi_{i}^{t}(x, y, 0)\right)\right)
$$

By equations (25) and (28) we get that

$$
\begin{equation*}
Y_{i}\left(\psi_{i}(x, y, t)\right)=X\left(\psi_{i}(x, y, t)\right) \text { for } t \leq-\frac{2}{3} \text { and } t \geq 0 \tag{29}
\end{equation*}
$$

Let $\mathcal{W}$ be the open set $\mathcal{W}=\bigcup_{i=1}^{m} \psi_{i}\left(U_{i} \times(-1,1)\right)=\bigcup_{i=1}^{m} W_{i}$, and consider the vector field $Y: \mathcal{W} \rightarrow \mathbb{R}^{3}$ given by $Y=\left.Y_{i}\right|_{W_{i}}$. Finally, define the vector field $\tilde{X}$ by

$$
\tilde{X}:= \begin{cases}Y, & \text { on } \mathcal{W}, \\ X, & \text { outside of } \mathcal{W} .\end{cases}
$$

Since $\tilde{P} \in \mathcal{U}_{P}$, equation (27) implies that $\tilde{X}$ is $C^{2}$-close to $X$. If $\tilde{X}^{t}$ is the flow associated to the vector field $\tilde{X}$, equations (26) and (29) imply that the Poincaré map associated to $\tilde{Y}^{t}$ restricted to $\Lambda_{\tilde{P}}^{k}$ is equal to $\tilde{P}$ restricted to $\Lambda_{\tilde{P}}^{k}$. To finish the proof, note that $d\left(U_{i}, \Gamma\right)>\frac{\epsilon}{2}$ for all $i$, and thus, $\tilde{X}^{t}$ is a geometric Lorenz flow, as desired.
4.2. Regaining the spectrum. Recall that we are interested in studying the spectrum over a geometric Lorenz attractor $\Lambda$, that is not a hyperbolic set, as well as $\Lambda \cap S$. Thus, we cannot directly apply the techniques developed in the hyperbolic setting to analyze the spectrum in this case. So, the strategy we adopt is to profit from the fact that $\Lambda \cap S$ contains hyperbolic sets $\Lambda_{P}^{k}$ for the Poincaré map $P$ with Hausdorff dimension bigger than 1. Then we use similar arguments developed in RM17 to show that the Lagrange and Markov dynamical spectrum has nonempty interior for a set of $C^{1}$-real functions over the cross-section $S$ and with these functions regaining the spectrum over $\Lambda$. In this direction, we proceed as follows.

The dynamical Lagrange and Markov spectra of $\Lambda$ and $\Lambda_{P}^{k}$ are related in the following way. Given a function $F \in C^{s}(U, \mathbb{R}), s \geq 1$, let us denote by $f=$ $\max F_{\phi}: D_{P} \rightarrow \mathbb{R}$ the function

$$
\max F_{\phi}(x):=\max _{0 \leq t \leq t_{+}(x)} F\left(\phi^{t}(x)\right)
$$

where $D_{P}$ is the domain of $P$ and $t_{+}(x)$ is such that $P(x)=X^{t_{+}(x)}(x)$ and $U$ is a neighborhood of $\Lambda$ as in Theorem B.

Remark 2. The map $f=\max F_{\phi}$ might be not $C^{1}$ in general.

For all $x \in \Lambda_{P}^{k}$, we have

$$
\limsup _{n \rightarrow+\infty} f\left(P^{n}(x)\right)=\limsup _{t \rightarrow+\infty} F\left(X^{t}(x)\right) \quad \text { and } \quad \sup _{n \in \mathbb{Z}} f\left(P^{n}(x)\right)=\sup _{t \in \mathbb{R}} F\left(X^{t}(x)\right) .
$$

In particular, if $\Lambda^{k}=\bigcup_{t \in \mathbb{R}} X^{t}\left(\Lambda_{P}^{k}\right) \subset \Lambda$, we get

$$
L\left(X, \Lambda^{k}, F\right)=L\left(P, \Lambda_{P}^{k}, f\right) \text { and } M\left(X, \Lambda^{k}, F\right)=M\left(P, \Lambda_{P}^{k}, f\right)
$$

Remark 3. It is worth noting that, given a vector field $Y$ close to $X$, the flow of $Y$ still defines a Poincaré map $P_{Y}$ defined in the same cross-sections where $P$ is defined.

Thus, the last equality reduces Theorem $B$ to the following statement:
Theorem 4.1. In the setting of Theorem B arbitrarily close to $X$, there is an open set $\mathcal{W}$ of $C^{2}$-vector fields defined on $U$ such that for every $Y \in \mathcal{W}$ there is a $C^{2}$-open and dense subset $\mathcal{H}_{Y, \Lambda} \subset C^{2}(U, \mathbb{R})$, such that

$$
\operatorname{int} M\left(P_{Y}, \Lambda_{P_{Y}}^{k}, \max F_{Y}\right) \neq \emptyset \text { and } \operatorname{int} L\left(P_{Y}, \Lambda_{P_{Y}}^{k}, \max F_{Y}\right) \neq \emptyset
$$

whenever $F \in \mathcal{H}_{Y, \Lambda}$. Here $\Lambda_{P_{Y}}^{k}$ denotes the hyperbolic continuation of $\Lambda_{P}^{k}$.
4.2.1. Description of $\mathcal{H}_{Y, \Lambda}$. Given a compact hyperbolic set $\Delta$ for $P$ and a Markov partition $R$ of $\Delta$, we define the set

$$
\begin{equation*}
H_{1}(P, \Delta)=\left\{f \in C^{1}(S \cap R, \mathbb{R}): \# M_{f}(\Delta)=1, z \in M_{f}(\Delta), D P_{z}\left(e_{z}^{s, u}\right) \neq 0\right\} \tag{30}
\end{equation*}
$$

where $S$ is the cross-section, as in the Section 2 $M_{f}(\Delta):=\{z \in \Delta: f(z) \geq$ $f(x)$ for all $x \in \Delta\}$, the set of maximum points of $f$ in $\Delta$, and $e_{z}^{s, u}$ are unit vectors in $E^{s, u}(z)$, respectively (cf. RM17, section 3]).
Definition 3. We say that $F \in \mathcal{H}_{Y, \Lambda} \subset C^{2}(U, \mathbb{R})$ if there is a neighborhood $R_{F}$ of $\Lambda_{P_{Y}}^{k}$ such that
(i) $\left.\max F_{Y}\right|_{S \cap R_{F}} \in C^{1}\left(S \cap R_{F}, \mathbb{R}\right)$.
(ii) $\max F_{Y} \in H_{1}\left(P_{Y}, \Lambda_{P_{Y}}^{k}\right) \subset C^{1}\left(S \cap R_{F}, \mathbb{R}\right)$.

With arguments similar to RM15, section 4], we prove the following result.
Lemma 4.2. The set $\mathcal{H}_{Y, \Lambda}$ is a dense $C^{2}$-open set.
Remark 4. If $Y$ is $C^{2}$-close enough to $X$, then $H D\left(\Lambda_{P_{Y}}^{k}\right)>1$ (cf. [PT93, sec. 4.3]). Proof of Theorem 4.1. As $\Lambda_{P}^{k}>1$, by the main theorem of RM17, arbitrarily close to $P$ there exists a $C^{2}$-open set $\widetilde{\mathcal{W}}$, such that for $\tilde{P} \in \widetilde{\mathcal{W}}$ it holds that

$$
\operatorname{int} M\left(\tilde{P}, \Lambda_{\tilde{P}}^{k}, f\right) \neq \emptyset \text { and } \operatorname{int} M\left(\tilde{P}, \Lambda_{\tilde{P}}^{k}, f\right) \neq \emptyset
$$

whenever $f \in H_{1}\left(\tilde{P}, \Lambda_{\tilde{P}}{ }^{k}\right)$. Note also that Lemma 4.1 provides a neighborhood $\mathcal{W}$, $C^{2}$-close to $X$, such that for any $\tilde{P} \in \widetilde{\mathcal{W}}$ there is $Y \in \mathcal{W}$ such that $P_{Y}=\tilde{P}$ in a neighborhood of $\Lambda_{\tilde{P}}^{k}$. Thus, for $Y \in \mathcal{W}$ and $F \in \mathcal{H}_{Y, P}$ it holds that

$$
\operatorname{int} M\left(P_{Y}, \Lambda_{P_{Y}}^{k},\left.\max F_{Y}\right|_{S \cap R_{F}}\right) \neq \emptyset \text { and } \operatorname{int} L\left(P_{Y}, \Lambda_{P_{Y}}^{k},\left.\max F_{Y}\right|_{S \cap R_{F}}\right) \neq \emptyset,
$$

since $\left.\max F_{Y}\right|_{S \cap R_{F}} \in H_{1}\left(P_{Y}, \Lambda_{P_{Y}}^{k}\right)$. This finishes the proof of Theorem 4.1, thus concluding the proof of Theorem B

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