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ABOUT THE COVER: THE FINE–PETROVIĆ POLYGONS AND THE NEWTON–PUISEUX METHOD FOR ALGEBRAIC ORDINARY DIFFERENTIAL EQUATIONS

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Our study [7] of the genesis and evolution of geometric ideas and techniques related to movable singularities of ordinary differential equations (ODEs) has led us to the work of Mihailo Petrović on algebraic differential equations and his geometric ideas captured in his polygonal method from the last years of the nineteenth century. These ideas and results of Petrović have been left completely unnoticed by experts. It appeared that a similar concept was also developed independently by Henry Fine in a somewhat different direction. These results generalize the famous Newton–Puisseux polygonal method and apply to algebraic ODEs rather than algebraic equations. Mihailo Petrović (1868–1943) was an extraordinary person and the leading Serbian mathematician of his time. His results are, despite their significance, practically unknown to mathematicians working in the field, in the past and nowadays. The situation is less severe with Fine’s results. Thus, we emphasized in [7] the development of the ideas of Petrović and Fine from the point of view of contemporary mathematics.

In its essence, this is a story about two outstanding individuals, Henry Fine and Mihailo Petrović (see Figure 1). Along with doing science, both made transformational efforts in elevating of the mathematical research in their native countries to a remarkable new level. At the same time both left the deepest traces in the development of their own academic institutions at the moments of their transformation from being local colleges to renowned universities: Fine to American mathematics and Princeton University and Petrović to Serbian mathematics and the University of Belgrade. The list of striking similarities between the two scientists is not even

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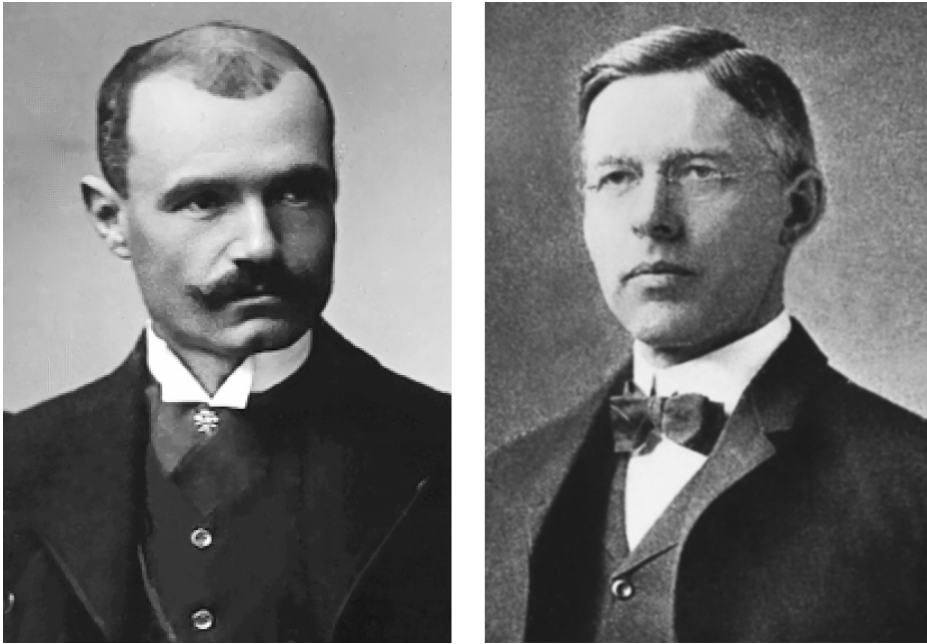


FIGURE 1. CLEAR COPY OF THE COVER IMAGES. Mihailo Petrović Alas on the left and Henry Burchard Fine on the right.¹

slightly exhausted here. Both mathematicians were sons of theologians. Both were in love with their own two beautiful rivers. Both actively enjoyed music playing their favorite instruments. Both went abroad, to western Europe, to work with the top mathematicians of their time as mentors during the preparation of their PhD theses. And both had state officials of the highest rank in their native countries as their closest friends. These friendships heavily shaped their lives. Fine published five scientific papers and had no known students. Petrović published more than 300 papers and had more than 800 scientific decedents. It seems surprising that the two did not know each other and did not know about each other's work.

In Belgrade a downtown street, an elementary school, a high school, and a fish restaurant are named after Mihailo Petrović Alas. The department of mathematics of Princeton University is housed in Fine Hall, the building named after its first chairman.

The main source of Petrović's results for us was his doctoral dissertation [12], written in French in 1894. It was reprinted along with a translation in Serbian, edited by academician Bogoljub Stanković in Volume 1 of [15] in 1999.

¹The image of Mihailo Petrović Alas, made ca. 1905–1910, is from the “Mihailo Petrović” Foundation, Primary school “Mihailo Petrović Alas”, a gift of Jovan Hans Ivanović, Serbian Academy of Sciences and Arts, signature SANU-F 261, 261/1g. The image of Henry Burchard Fine is from the *Find a Grave* website. *Find a Grave*, database and images (<https://www.findagrave.com>: accessed 24 October 2019), memorial page for Henry Burchard Fine (1858–22 Dec 1928), Find a Grave Memorial no. 60232952, citing Princeton Cemetery, Princeton, Mercer County, New Jersey, USA; Maintained by Scott Balyer (contributor 47094996).

Let us recall that an algebraic ODE has the following form:

$$(1) \quad f(x, y, y', \dots, y^{(n)}) = 0,$$

$$(2) \quad f = \sum_{i=1}^j \varphi_i(x) y^{m_{0i}} y'^{m_{1i}} \dots y^{(n)m_{ni}},$$

$j \in \mathbb{N}$, where $m_{0i}, \dots, m_{ni} \in \mathbb{Z}_+$, and φ_i are algebraic or analytic functions.

The key presumption of Petrović's main construction is that a given point x_0 is a nonsingular point of equation (1). In such a case, to each term in the sum (2) with the coefficient $\varphi_i(x_0) = \text{const} \neq 0$ corresponds a point in the $M0N$ plane, according the following formulae,

$$Q_i = (M_i, N_i), \quad M_i = m_{0i} + \dots + m_{ni}, \quad N_i = m_{1i} + 2m_{2i} + \dots + nm_{ni}.$$

It should be noticed that one and the same point in the plane can correspond to one or more terms in the sum (2). Let

$$S = \{Q_i, i = 1 \dots s, s \leq j\}$$

be the set of all points obtained in such a correspondence. We can draw these points in the $M0N$ plane. In Petrović's dissertation the set S was extended with two segments T_l and T_r which are orthogonal to the axis $0M$ and connect the leftmost and the rightmost point of the set S with their projections to the $0M$ axis, respectively. The boundary of the convex hull of the set $S \cup T_l \cup T_r$ is a polygon. Both that polygon and the concave part of the boundary of the convex hull of the set S will be called the *Petrović polygon* (see Figure 2).

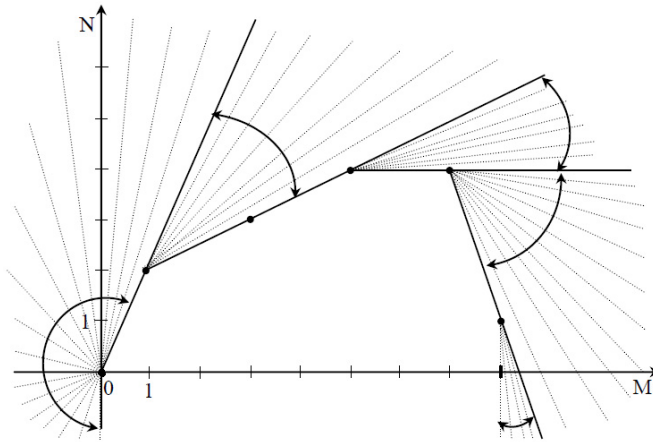


FIGURE 2. Construction of Petrović polygon [7]

At the beginning of Chapter 1 of Part 1 of his thesis, Petrović proved the following statements.

Proposition 1 (Petrović, 1894, [12]). *If $x = x_0$ is a nonsingular point of equation (1), and if $y = y(x)$ is a solution of the equation with initial conditions $y(x_0) = 0$ or $y(x_0) = \infty$, then the first term $c_0(x - x_0)^\lambda$ of the expansion into a Puiseux series of the solution is a solution of the approximate equation, which corresponds either to a vertex or to a slanted edge of its polygon.*

Theorem 1 (Petrović, 1894, [12]). *The necessary and sufficient condition for poles (zeros) of the general solution of a given algebraic ODE of the first order not to depend on the constants of integration is that its polygon does not contain right (left) slanted edges.*

Contrary to the necessary and sufficient conditions for the nonexistence of movable critical points of solutions of algebraic ODEs of the celebrated theorem of L. Fuchs [10], the conditions of Petrović's theorem do not require either the computation of solutions of the discriminant equation or to have the equation resolved with respect to the derivative. The conditions of Petrović's theorem can be checked easily and effectively by a simple construction of a geometric figure corresponding to the given equation. The first part of the dissertation also contains the theorems that provide a classification of rational, first-order ODEs explicitly resolved with respect to the derivative which have uniform (single-valued) solutions. Later on, these results of Petrović were essentially improved by J. Malmquist. In addition, in the first part of the thesis, the class of binomial ODEs of the first order is studied and the equations with solutions without movable singular points are described. Also, Petrović characterizes those binomial equations which possess uniform (single-valued) solutions. The results of Petrović are very similar to those obtained by K. Yosida more than 30 years later. The second part of the dissertation is devoted to the applications of the polygon method in the study of zeros and singularities of the algebraic higher-order ODEs. It is very important to stress that Petrović in his dissertation clearly pointed out the limitations of the applications of his polygonal method to the algebraic ODEs of higher orders. He showed that the method of planar polygons could be successfully applied to higher-order algebraic ODEs to study some types of movable singularities of the solutions. However, due to the lack of a Painlevé-type result for higher-order equations, Petrović understood that his method was powerless in proving absence of other types of movable singularities. Some of his results related to higher-order ODEs were published in *Acta Mathematica* in 1899 (see [13] and for a recent English translation [14], see also [6]).

Mihailo Petrović is one of the most respected and influential mathematicians in Serbia. Petrović's collected works in 15 volumes were published in 1999 [15]. The year 2018 was the Year of Mihailo Petrović in Serbia on the occasion of his 150th anniversary (see [16]). Certainly, some of the Petrović's results in that field were quite well known at the beginning of the twentieth century. Nevertheless, neither Golubev nor Picard, who together extensively used some other results from Petrović's thesis in their famous books, nor any other mathematician who used later analogous geometric methods in the study of the solutions of the algebraic ODEs, ever quoted Petrović's foundational results in this field.

A couple of years prior to Petrović, the American mathematician Henry Fine invented another modification of the Newton–Puiseux method for studying the formal solutions of algebraic ODEs [8]. Let us notice that although the Fine construction was similar to that of Petrović, they were not identical, and the questions they were considering were very much different. Fine generalized the polygonal method of Newton and Puiseux in order to study formal asymptotics of the solutions of algebraic ODEs (1) at the point $x = 0$. In his considerations he includes both cases, when the point $x = 0$ is a singular point of the equation and also when it is not a singular point of the equation. In [8], [9] Fine used Puiseux and Briot-Bouquet

results and generalized them. The Fine and Petrović methods of construction of approximate equations are based on the same principles. In the construction of Fine polygons, we correspond a point to every term of the equation of the type

$$c x^{l_{it}} y^{m_{0i}} y'^{m_{1i}} \dots y^{(n)m_{ni}}, \quad c \in \mathbb{C},$$

where the point (N_{it}, M_i) , is determined by the formula $N_{it} = l_{it} - N_i$, where N_i and M_i are defined in the same way as in Petrović's construction above. If the points (N_{it}, M_i) are depicted in the plane and if we consider the boundary of the convex hull of all the points (N_{it}, M_i) , then the *left part of that boundary* (consisting of the edges and vertices where the external normal is pointed to the left) captures the behavior of the solutions in a neighborhood of the point $x = 0$. We will call this left part of the boundary the *Fine polygon*. The vertices and edges of the Fine polygon correspond to the leading terms of the equation, i.e., those terms of the equation (1) which can form approximate equations. The candidates for the role of the asymptotics of the true solutions of the original equation lie among the solutions of the approximate equations. Let us observe that the Fine polygon takes into account the exponents l_{it} of the independent variable x in the coefficients $\varphi_i(x)$ of equation (1), because here $x = 0$ can be a singular point for equation (1), i.e., $\varphi_i(0)$ could be zero or undefined.

Let us also observe that by using the change $x = z + x_0$, the problem of analysis of the solutions in a neighborhood of an arbitrary point $x = x_0$ reduces to the problem of the analysis of solutions in a neighborhood of the point $z = 0$.

Theorem 2 (Dragović and Goryuchkina, [7]). *The Fine polygon of the equation*

$$f(z + x_0, y, y', \dots, y^{(n)}) = 0,$$

where x_0 is not a singular point of equation (1), coincides with the Petrović polygon of equation (1) rotated by $\pi/2$ in the counterclockwise direction.

Fine's paper [8] is mostly devoted to the question of calculations of terms in the expansion of formal solutions (which have a form of Puiseux series) of algebraic ODEs in a neighborhood of the point $x = 0$. Fine also treated the question of the convergence of formal series. Fine proved the following result.

Theorem 3 (Fine, 1889, [8]). *If every term of an algebraic ODE contains derivatives of all orders and the dependent variable, i.e., if every term*

$$\varphi_i(x) y^{m_{0i}} y'^{m_{1i}} \dots y^{(n)m_{ni}}$$

of (1) and (2) satisfies $m_{0i}, \dots, m_{ni} > 0$, then all the formal Taylor series which formally satisfy the given equation converge.

At the end of the twentieth century, the Fine method was developed further by J. Cano [4], [3]. As of today, contemporary methods based on different modifications of the Newton–Puiseux polygonal method allow wide classes of formal solutions to be computed for analytic differential equations and their systems by A. D. Bruno and his school [1], [2], and to prove their convergence and analysis of the rate of growth of terms of formal series by Cano, Malgrange, Ramis, Sybuya, and others.

Henry Burchard Fine (1858–1928) (see [5], [11]) was dean of faculty and the first and only dean of the departments of science at Princeton. He was one of a few who did the most to help Princeton develop from a college into a university. He made Princeton a leading center for mathematics and fostered the growth of

creative work in other branches of science as well. Professor Oswald Veblen, in his memorial article [17] described Fine’s contribution on the nationwide scale in his opening sentence by saying that “Dean Fine was one of the group of men who carried American mathematics forward from a state of approximate nullity to one verging on parity with the European nations.”

Despite great promise as a research mathematician, Fine moved very soon into other areas of academic life. He mainly devoted his time to teaching, administration, and the logical exposition of elementary mathematics.

Fine was one of the founders of the American Mathematical Society. He served as the AMS president in 1911 and 1912.

We hope that through [7] and this short note we have been able to bring the gems that were almost buried in the past to the attention of specialists in this actively developing field of mathematics as well as a more general audience. This is not only to restore the historic justice that these beautiful pioneering results and their outstanding authors deserve but even more—to propel these powerful ideas and put them in synergy with modern techniques and questions.

REFERENCES

- [1] A. D. Bruno, *Asymptotic behavior and expansions of solutions of an ordinary differential equation* (Russian, with Russian summary), Uspekhi Mat. Nauk **59** (2004), no. 3(357), 31–80, DOI 10.1070/RM2004v059n03ABEH000736; English transl., Russian Math. Surveys **59** (2004), no. 3, 429–480. MR2116535
- [2] A. D. Bruno and I. V. Goryuchkina, *Asymptotic expansions of solutions of the sixth Painlevé equation* (Russian, with Russian summary), Tr. Mosk. Mat. Obs. **71** (2010), 6–118, DOI 10.1090/S0077-1554-2010-00186-0; English transl., Trans. Moscow Math. Soc. (2010), 1–104. MR2760041
- [3] José Cano, *An extension of the Newton-Puiseux polygon construction to give solutions of Pfaffian forms* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **43** (1993), no. 1, 125–142. MR1209698
- [4] José Cano, *On the series defined by differential equations, with an extension of the Puiseux polygon construction to these equations*, Analysis **13** (1993), no. 1-2, 103–119, DOI 10.1524/anly.1993.13.12.103. MR1245746
- [5] J. J. O’Connor and E. F. Robertson “Henry Burchard Fine”, MacTutor History of Mathematics archive, University of St Andrews, 2005.
- [6] V. Dragović, “Mihailo Petrović, algebraic geometry and differential equations”, in Stevan Pilipović, Gradimir V. Milovanović, Žarko Mijajlović (editors), *Mihailo Petrović Alas: life work, times*, Serbian Academy of Sciences and Arts, Belgrade, 2019.
- [7] V. Dragović and I. Goryuchkina, *Polygons of Petrović and Fine, algebraic ODEs, and contemporary mathematics*, Arch. Hist. Exact Sci. (to appear). [arXiv:1908.03644](https://arxiv.org/abs/1908.03644) (2019).
- [8] Henry B. Fine, *On the functions defined by differential equations, with an extension of the Puiseux polygon construction to these equations*, Amer. J. Math. **11** (1889), no. 4, 317–328, DOI 10.2307/2369347. MR1505516
- [9] Henry B. Fine, *Singular solutions of ordinary differential equations*, Amer. J. Math. **12** (1890), no. 3, 295–322, DOI 10.2307/2369621. MR1505535
- [10] L. Fuchs, *Über Differentialgleichungen, deren Integrale feste Verzweigungspunkte besitzen*, Ges Werke (1885), Vol. II, p. 355.
- [11] A. Leitch, *A Princeton Companion*, Princeton University Press, 1978.
- [12] M. Petrowitch, *Thèses: Sur les zéro et les infinis des intégrales des équations différentielles algébriques. Propositions données par la Faculté*, Paris, 1894.
- [13] Michel Petrovitch, *Sur une propriété des équations différentielles intégrables à l’aide des fonctions méromorphes doublement p gdiroidiques* (French), Acta Math. **22** (1899), no. 1, 379–386, DOI 10.1007/BF02417881. MR1554911

- [14] M. Petrovitch, *On a property of differential equations integrable using meromorphic double-periodic functions*, Theoretical and Applied Mechanics, **45** (2018), no. 1, 121–127. English translation of Petrovitch’s *Acta Mathematicae* paper, [13].
- [15] Mihailo Petrović, *Collected Works*, 15 volumes (Serbian), The State Textbook Company, Belgrade, 1999.
- [16] Stevan Pilipović, Gradimir V. Milovanović, Žarko Mijajlović editors, *Mihailo Petrović Alas: life work, times*, Serbian Academy of Sciences and Arts, Belgrade, 2019.
- [17] Oswald Veblen, *Henry Burchard Fine—In memoriam*, Bull. Amer. Math. Soc. **35** (1929), no. 5, 726–730, DOI 10.1090/S0002-9904-1929-04815-8. MR1561798

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