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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
CHARLES FEFFERMAN, ALEX IONESCU, TERENCE TAO, STEPHEN WAINGER, ET AL.

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Calderon, A. P.; Zygmund, A.

On the existence of certain singular integrals.

Acta Mathematica **88** (1952), 85–139.

In his paper on conjugate functions [Math. Z. **27**, 218–244 (1927)] M. Riesz deals with Hilbert's operator

$$\tilde{f}(x) = \frac{1}{\pi}\mathrm{PV}\int_{-\infty}^{\infty} \frac{f(t)}{x-t}\,dt = \lim_{0<\lambda\to\infty} \tilde{f}_\lambda(x) \quad (-\infty < x < \infty)$$

where

$$\tilde{f}_\lambda(x) = \left(\int_{-\infty}^{x-1/\lambda} + \int_{x+1/\lambda}^{\infty}\right) K(x-t) f(t)\,dt,$$

the kernel $K(y) = (\pi y)^{-1}$. He uses complex variable methods. These are his basic results: Let $1 < p < \infty$, $f(t) \in L^p(-\infty,\infty)$; then (1) $\int_{-\infty}^{\infty} |\tilde{f}_\lambda(x)|^p\,dx \leq {A_p}^p \int_{-\infty}^{\infty} |f(t)|^p\,dt$ (that $A_2 = 1$ was known before; the best possible value of A_p for $p \neq 2$ is still unknown), (2) $\tilde{f}(x) = \lim \tilde{f}_\lambda(x)$ exists for almost all x, (3) $\tilde{f}_\lambda(x) \to \tilde{f}(x)$ also in the mean of order p; in (1), $\tilde{f}_\lambda(x)$ can equivalently be replaced by $\tilde{f}(x)$. The problem was subsequently solved by real variable methods, which are harder; again the difficult case $p = 1$ and related ones were treated by various writers.

The authors, dedicating their work to Riesz on his 65th birthday, give a complete theory of the corresponding problem in the n-dimensional space E^n, yet in a remarkably generalised form. They had to use real variable methods. This required auxiliary results which are of great interest in themselves and for which the reader must be referred to the paper. In three chapters, they discuss the mean convergence of the singular integral, its pointwise convergence, and the application of the results to some potentials, respectively. In an addendum results on periodic functions and on linear and quadratic forms are stated, and reference is given to related recent work. Let P and Q be points in E^n, $P - Q$ be the point whose coordinates are equal to the com ponents of the vector QP, $|P|$ the distance of P from the origin. Let $\Omega(P)$ be defined for the points P of Σ, the surface of the unit sphere, and be such that $\int_\Sigma \Omega(P)\,d\sigma = 0$, $|\Omega(P) - \Omega(Q)| \leq \omega(|P-Q|)$ (P and Q on Σ), where $\omega(t)$ is increasing $(0 \leq t < \infty)$, $\omega(t) \geq t$, $\int_0^1 \omega(t)t^{-1}\,dt < \infty$. Finally, for any P, let $K(P) = |P|^{-n}\Omega(P/|P|)$; $K_\lambda(P) = K(P)$ for $|P| \geq 1/\lambda$, $= 0$ otherwise; and

$$\tilde{f}_\lambda(P) = \int_{E^n} K_\lambda(P-Q) f(Q)\,dQ.$$

In chapters 1 and 2 the authors, starting with the case $p = 2$ which, in analogy to a familiar procedure, is dealt with by Fourier transforms in E^n, prove all the results corresponding to the above Riesz theorems on $\tilde{f}_\lambda(P)$ and $\tilde{f}(P)$ for $f(P) \in L^p(E^n)(1 < p < \infty)$; and also results on the cases $p = 1$ and ∞, e.g.: if $p = 1$, then

$\tilde{f}(P) = \lim \tilde{f}_\lambda(P)$ exists almost everywhere in E^n; also $\tilde{f}_\lambda(P) \to \tilde{f}(P)$ in the mean of order $1-\epsilon$ ($\epsilon > 0$) over any set S of finite measure, and

$$\int_S |\tilde{f}_\lambda(P)|^{1-\epsilon}\, dP \leq \epsilon^{-1} c |S|^\epsilon \left[\int_{E^n} |f(P)|\, dP\right]^{1-\epsilon} \quad (c = c(n, \Omega)),$$

which corresponds to the Kolmogoroff-Littlewood result in E^1. Other theorems concern cases like $|f(P)| \leq M$; $\int_{E^n} |f(P)|(\log^+ |f(P)| + 1)\, dP < \infty$ and similar ones; and the singular integral $\tilde{f}(P) = \int_{E^n} K(P-Q)\, d\mu(Q)$, where $\mu(P)$ is a completely additive function of Borel sets and is of bounded total variation over E^n.

The results are now applied to

$$u(x,y) = \int_{E^2} R^{-1} f(Q)\, dQ \qquad (P = (x,y), Q = (s,t),$$
$$R = |P - Q| = \{(x-s)^2 + (y-t)^2\}^{1/2}),$$

that is the Newtonian potential of masses with density $f(s,t)$ in E^2. Theorem 1. If $f \in L(E^2)$ and $|f| \log^+ |f|$ is integrable over any finite circle, then (a) over almost every line $y =$ constant the integral converges and is an absolutely continuous function of x; and $\partial u/\partial x = \iint_{E^2} R^{-3}(s-x) f(s,t)\, dsdt$ almost everywhere in E^2; (b) if $f \in L(E^2)$ and $|f|^q$ is integrable ($q > 2$) over any finite circle, then $u(x,y)$ has a complete differential almost everywhere. In the proof of (a) it is first shown that, for $z \to 0$ ($z > 0$), $\partial u(x,y,z)/\partial x - \tilde{f}_z(x,y)$ converges to 0 and $\tilde{f}_z(x,y)$ to a function $\tilde{f}(x,y)$ in the mean of order 1 over any set of finite measure; here

$$u(x,y,z) = \iint\limits_{E^2} (R^2 + z^2)^{-1/2} f(s,t)\, dsdt,$$
$$\tilde{f}_z(x,y) = \iint\limits_{R \geq z} R^{-3}(s-x) f(s,t)\, dsdt$$

(cf. Riesz's second method of proof). The theorem is shown not to admit of much improvement. Theorem 2 concerns its analogue in E^n, while Theorem 3 concerns the logarithmic potential $u(P) = \int_{E^2} \log |P-Q|^{-1} f(Q)\, dQ$ in E^2, and Theorem 4 the potential $\int_{E^n} |P-Q|^{-n+2} f(Q)\, dQ$ in E^n.

H. Kober

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Calderón, A.-P.; Zygmund, A.

Local properties of solutions of elliptic partial differential equations.

Polska Akademia Nauk. Instytut Matematyczny. Studia Mathematica **20** (1961), 171–225.

The paper studies linear elliptic partial differential equations of order $m > 0$ of the form $\mathcal{L}f = g$, where $f(x) = (f_1, \cdots, f_r)$, $g(x) = (g_1, \cdots, g_s)$, $r \leq s$, $\mathcal{L} = \sum a_\alpha(x)(\partial/\partial x)^\alpha f$, where the $a_\alpha(x)$ are $s \times r$ matrices, and all f_i, g_j, and the elements of the matrices a_α (the coefficients of $\mathcal{L}$), are complex-valued functions of x in E_n. As usual $\alpha = (\alpha_1, \cdots, \alpha_n)$ is any set of non-negative integers, $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$, $(\partial/\partial x)^\alpha f = \partial^{|\alpha|} f/\partial {x_1}^{\alpha_1} \cdots \partial {x_n}^{\alpha_n}$, and $\sum$ ranges over all $|\alpha| \leq m$. The ellipticity condition of $\mathcal{L}$ at a point x_0 of E_n is expressed as usual by assuming $\det[(\sum' a_\alpha{}^*(x_0)\xi^\alpha) \times (\sum' a_\alpha(x_0)\xi^\alpha)] \geq \mu(x_0) > 0$ for all vectors $\xi = (\xi_1, \cdots, \xi_n)$ of Euclidean length $|\xi| = 1$, and where $\sum'$ ranges over all α with $|\alpha| = m$. The important question is discussed as to whether differentiability properties of g and of the co-efficients of L at some point x_0 (or almost everywhere in a set Q) imply analogous properties of differentiability of the solution f at the same point x_0 (or almost everywhere in Q). The question is answered in the affirmative when the concepts T and t of differentiability in the mean at some point x_0 are introduced. The L^p-norm of a function $f(x)$, $x \in E^n$, $f \in L^p$, is denoted by $\|f\|_p$. A function $f \in L^p$ whose partial derivatives of all orders $|\alpha| \leq k$ in the sense of the theory of distributions are also in L^p is said to be of class $L_k{}^p$, and its norm $\|f\|_{pk}$ is the sum of the L^p-norms of f and of the same partial derivatives. A function $f(x)$, $x \in E_n$, $f \in L^p$, $1 \leq p \leq \infty$, is said to be of class $T_u{}^p(x_0)$ ($x_0 \in E_n$ a fixed point, $u \geq -n/p$) provided there is a polynomial $P(x - x_0)$ such that the L^p mean value $M(\rho)$ of the difference $|f(x) - P(x - x_0)|$ in the spherical bowl $\Gamma = \Gamma(x_0, \rho)$ of center x_0 and radius ρ is $O(\rho^u)$ as $\rho \to 0$. The same function f is said to be of class $t_u{}^p(x_0)$ if in addition $M(\rho) = o(\rho^u)$ as $\rho \to 0$. Here, $M(\rho) = (\rho^{-u} \cdot (\Gamma) \int |f(x) - P(x - x_0)|^p dx)^{1/p}$, $1 \leq p < \infty$, is replaced by the essential supremum of the difference $|f(x) - P(x - x_0)|$ in Γ if $p = \infty$. The functions $f \in T_u{}^p(x_0)$ form a linear space with norm $T_u{}^p(x_0, f)$ defined by the sum of $\|f\|_p$, of the absolute values of the coefficients of P, and of the infimum of the constants A for which $M(\rho) \leq A\rho^u$, $0 < \rho < \infty$. The main theorem states: I. If all coefficients of $\mathcal{L}$ are of class $T_u{}^p(x_0)$, if all components f_i and g_j of f and g are of class L^p, and $g_j \in T_v{}^p$, $1 < p < \infty$, $u \geq v \geq -n/p$, v non-integral, then

$$T^q_{v+m-|\alpha|}(x_0, (\partial/\partial x)^\alpha f_i) \leq C[\sum_j T_v{}^p(x_0, g_j) + \sum_i \|f_i\|_{pm}],$$

for all $i = 1, \cdots, r$, $|\alpha| \leq m$, where q is any number $p \leq q \leq \infty$ if $1/p < (m - |\alpha|)/n, p \leq q < \infty$ if $1/p = (m - |\alpha|)/n$, $1/p \leq 1/q \leq 1/p - (m - |\alpha|)/n$ if $1/p > (m - |\alpha|)/n$, and where C is a constant. In addition, if $g \in t_v{}^p(x_0)$, then $(\partial/\partial x)^\alpha f$ belongs to $t^q_{v+m-|\alpha|}(x_0)$. A further theorem states: II. If $\mathcal{L}$ is elliptic almost everywhere in a set of positive measure Q (and $\mu(x_0) \geq \mu > 0$, $x_0 \in Q$, for some constant μ), if all coefficients of $\mathcal{L}$ are in $T_u{}^p(x_0)$ and $g \in T_v{}^p(x_0)$ for almost all $x_0 \in Q$, and $f \in L_m{}^p$, then $(\partial/\partial x)^\alpha f$ belong to $t^q_{v+m-|\alpha|}(x_0)$ for almost all $x_0 \in Q$.

Stronger results concerning f are given for the case where g and the coefficients of $\mathcal{L}$ are all bounded on a closed set Q and their T-norms above are uniformly bounded in Q. These and other results are based on properties of non-integral derivatives and singular integral equations in the theory of distributions. The connection of these concepts with elliptic operators had been already studied in previous papers of the same authors [Amer. J. Math. **78** (1956), 289–309; MR0084633; ibid. **79** (1957), 901–921; MR0100768].

L. Cesari

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MR0163988 (29 #1287) 22.60; 22.57

Kunze, R. A.; Stein, E. M.

Uniformly bounded representations and harmonic analysis of the 2×2 real unimodular group.

American Journal of Mathematics **82** (1960), 1–62.

This paper contains a remarkably detailed study of representations of and harmonic analysis on the group G of 2×2 real unimodular matrices. The continuous irreducible unitary representations of G were found long ago by Gel′fand and Naĭmark [Acad. Sci. USSR J. Phys. **10** (1946), 93–94; MR0017282; Izv. Akad. Nauk SSSR Ser. Mat. **11** (1947), 411–504; MR0024440] and independently by V. Bargmann [Ann. of Math. (2) **48** (1947), 568–640; MR0021942]. Plancherel's formula for this group was established by Harish-Chandra [Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 337–342; MR0047055]. To explain the results of this paper, some notation is needed. A generic element g of G is a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $ad-bc=1$; $a,b,c,d \in \mathbf{R}$. For $g \in G$, $x \in \mathbf{R}$, and $s \in \mathbf{C}$, let $\phi^+(g,x,s) = |bx+d|^{2s-2}$, $\phi^-(g,x,s) = \operatorname{sgn}(bx+d)\phi^+(g,x,s)$. For $g \in G$ and $x \in \mathbf{R}$, let $g(x) = (ax+c)/(bx+d)$. For complex-valued functions f on $\mathbf{R}$, let

$$(v^\pm(g,s)f)(x) = \phi^\pm(g,x,s)f \circ g(x).$$

The linear operators $v^\pm(g,s)$ obviously provide representations of G. The representations $g \to v^\pm(g,\frac{1}{2}+it)$ $(t \in \mathbf{R})$ acting on $L_2(\mathbf{R})$ are continuous, irreducible and unitary, and are called the two principal continuous series. The representations $g \to v^+(g,\sigma)$ $(0 < \sigma < \frac{1}{2})$, acting on the Hilbert space H_σ of functions on $\mathbf{R}$ with inner product

$$a_\sigma \int_{\mathbf{R}} \int_{\mathbf{R}} f(x)\overline{f(y)}|x-y|^{-2\sigma}\,dxdy,$$

are also continuous, irreducible and unitary, and are called the complementary series. Finally, there are two discrete series of continuous irreducible unitary representations of G; these play only a minor rôle in the present paper.

A main result of the paper is that the representations of the principal and complementary series can be analytically continued into the set $\{s \in \mathbf{C}\colon 0 < \operatorname{Re}(s) < 1\} = \mathbf{S}$ in the following sense. There are a separable Hilbert space $\mathcal{H}$ and representations $g \to U^\pm(g,s)$ of G on $\mathcal{H}$ such that: (1) $g \to U^\pm(g,s)$ is a continuous representation of G for each $s \in \mathbf{S}$; (2) $g \to U^\pm(g,\frac{1}{2}+it)$ $(t \in \mathbf{R})$ is unitarily equivalent to the representation $g \to v^\pm(g,\frac{1}{2}+it)$ of the principal continuous series; (3) $g \to U^+(g,\sigma)$ $(0 < \sigma < \frac{1}{2})$ is unitarily equivalent to the representation $g \to v^+(g,\sigma)$ of the complementary series; (4) for g fixed in G, and ξ,η fixed in $\mathcal{H}$, the function $s \to \langle U^\pm(g,s)\xi,\eta\rangle$ is analytic in S;

$$\sup\{\|U^\pm(g,\sigma+it)\|\colon g \in G\} \le A_{\sigma,\varepsilon}(1+|t|)^{|\sigma-1/2|(1+\varepsilon)} \tag{5}$$

for $0 < \sigma < 1$ and $\varepsilon > 0$, where $A_{\sigma,\varepsilon} > 0$ and is bounded on every interval $0 < \alpha \le \sigma \le \beta < 1$.

The authors use the representations $U^\pm(g,s)$ $(s \in \mathbf{S})$ to obtain a complete analogue of the Hausdorff-Young theorem for G. A bounded linear operator A on the separable Hilbert space $\mathcal{H}$ is in $\mathcal{B}_p$ $(1 \le p < \infty)$ if the non-negative operator $(A^*A)^{1/2} = |A|$ has discrete spectrum $(\lambda_1,\lambda_2,\cdots,\lambda_n,\cdots)$ and $\|A\|_p = [\sum_{n=1}^\infty {\lambda_n}^p]^{1/p}$ is finite. For $f \in L_1(G)$ and $s \in \mathbf{S}$, let $\mathcal{F}^\pm(s) = \int_G U^\pm(g,s)f(g)\,dg$.

This is the [operator-valued] Fourier transform of f. Let $1 < p < 2$, and $q = p/(p-1)$. There is a measure $\mu_{q,\sigma}$ on $\mathbf{R}$ such that for all $f \in L_1(G) \cap L_p(G)$,

$$\left[\int_{-\infty}^{\infty} \|\mathcal{F}^{\pm}(\sigma + it)\|_q{}^q \, d\mu_{q,\sigma}(t)\right]^{1/q} \leq \|f\|_p. \tag{1}$$

The Fourier transform admits a [unique] linear bounded extension over all of $L_p(G)$ for which (1) holds. Furthermore, $\mathcal{F}^{\pm}(s)$ is analytic in s. Analogous results using only the discrete series hold for $p \in [1, 2]$.

The Hausdorff-Young inequality (1), involving the whole strip $\mathbf{S}$ instead of only the line $\operatorname{Re} s = \frac{1}{2}$, has some curious consequences. If $f \in L_2(G)$ and $g \in L_p(G)$, $1 \leq p < 2$, then $f * g \in L_2(G)$ and $\|f * g\|_2 \leq A_p \|f\|_2 \cdot \|g\|_p$. If $g \in L_2(G)$, then $f * g \in L_q(G)$ for all $q \in]2, \infty]$, and $\|f * g\|_q \leq A_q \|f\|_2 \cdot \|g\|_2$. Both of these assertions hold trivially for compact groups and fail for the simplest non-compact Abelian groups, e.g., for the group $\mathbf{R}$.

This paper is nearly self-contained, and it is very carefully written. It is an important step in realizing the possibilities for detailed harmonic analysis that are implicit in the Gel′fand-Raĭkov theorem.

Edwin Hewitt

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MR0447953 (56 #6263) 42A40; 30A78, 42A18, 42A92

Fefferman, C.; Stein, E. M.

***H^p* spaces of several variables.**

Acta Mathematica **129** (1972), *no.* 3–4, 137–193.

It is with this remarkable paper that the classical theory of H^p spaces freed itself from its heavy dependence on complex function theory and developed a rich life of its own in the field of real analysis, thus making possible a variety of important n-dimensional results and extensions. The authors point out that three main ideas contributed to these developments. First the realization that the boundedness of certain singular integral operators could be extended from the L^p spaces to the H^p for $0 < p \leq 1$ (Sections III and V). The second development is the crucial result of D. L. Burkholder, R. F. Gundy and M. L. Silverstein, proved in the context of Brownian motions, that an analytic function is in the class H^p of the upper half-plane when and only when the non-tangential maximal function of its real part is in L^p of the line, $0 < p < \infty$. (See Burkholder, Gundy and Silverstein [Trans. Amer. Math. Soc. **157** (1971), 137–153; MR0274767] and, for the n-dimensional extension, Burkholder and Gundy [Studia Math. **44** (1972), 527–544; MR0340557]; Sections IV and V.) The third idea is the identification of the dual of H^1 with BMO, the space of functions of bounded mean oscillation [the first author, Bull. Amer. Math. Soc. **77** (1971), 587–588; MR0280994] (Section II). A necessarily brief description of some of the results presented in this paper now follows. Section II, as we just pointed out, contains the duality of H^1 and BMO. It also describes the action of strongly singular and multipliers in H^1 and BMO. Section III contains further results on L^p boundedness of certain convolution operators. The main idea here is to apply the above duality through complex interpolation techniques. This requires a device that mediates between L^p and BMO. This device is the "sharp" maximal function $f^{\#}(x) = \sup\{|Q|^{-1} \int_Q |f(y) - \operatorname{av}_Q f| \, dy\}$, where $\operatorname{av}_Q f$ denotes the average of f over the cube Q and the sup is taken over all cubes containing x. Theorem 5 shows that if f is in $L^q(R^n)$ and $1 \le q \le p$, then $\|Mf\|_p \le A_p \|f^{\#}\|_p$, where $1 < p < \infty$ and Mf is the usual Hardy-Littlewood maximal function of f. Theorem 7 shows that if m is a bounded multiplier in H^1 and $|y|^\delta |m(y)| \le A$, $\delta > 0$, then $|y|^\gamma m(y)$ is a bounded multiplier in L^p for $1 < p < \infty$ and $|\frac{1}{2} - 1/p| \le \frac{1}{2} - \gamma/2\delta$ and $\gamma \ge 0$. Section IV contains the basic result relating the L^p norms, $0 < p < \infty$, of the non-tangential maximal function and the area integral of any harmonic function defined in ${R_+}^{n+1}$. The local analogues of these results were already known so that the right quantitative estimates in place of certain qualitative statements in the local case allow for the same line of proof. The section also contains a characterization of H^p in the sense of Stein and G. Weiss, to wit, a harmonic function in ${R_+}^{n+1}$ is in H^p if and only if its non-tangential maximal function is in L^p. The above-mentioned results concerning the L^p norms of area integrals (and other functions, such as the g-function, not discussed in this review) provide a variety of equivalent "norms" in H^p. One of the highlights of this section is to show that the Poisson kernel can be replaced by arbitrary "smooth" functions which are sufficiently small at infinity. In this way the authors are led to the fact that H^p classes can be characterized without any recourse to analytic functions, conjugacy of harmonic functions, etc. Thus H^p classes have an intrinsic real variable meaning of their own. What one roughly has is that a tempered distribution f is in H^p when and only when the non-tangential maximal function of $(f_* \varphi_\varepsilon)(x)$, where $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$, is in $L^p(R^n)$ for any sufficiently smooth φ (Theorems 10 and 11). This leads to Section V, where the equivalence of several definitions is given and provides a tool to prove the following a priori inequality for convolution kernels: Let K be a tempered distribution whose Fourier transform is bounded, and also assume that K is sufficiently smooth away from the origin (the precise degree depends on the value of p in the following statement) and that $|\partial^\alpha K/\partial x^\alpha| \le C|x|^{-n-|\alpha|}$ there. Then if f is a bounded C^∞ function on R^n we have that the mapping $f \mapsto K_* f$ is bounded in $H^p(R^n)$ (Theorem 12).

The authors communicated to the reviewer three spots where perhaps a further remark was indicated. On p. 143 the argument given for cubes with large diameter should be replaced by a direct crude estimate. On p. 177 the Lipschitz classes which appear (as duals of the H^p spaces) should be interpreted as the homogeneous versions of those classes to allow for polynomial growth at infinity. And on p. 179, although formula (10.3) is correct, its application to what follows is not. The formula to be used instead is the following, and the argument follows along similar lines: Let f be a tempered distribution, φ as in Lemma 7, $\psi = (1-\Delta)^{\alpha/2}$ and $u(x,y) = (f_* w_y)(x)$, where $w_y(x) = y^{-n} w(x/y)$ and $w(x) = \exp(-\pi|x|^2)$. Then $(f_*\varphi_\varepsilon)(x) = c\int_0^\infty \int_{R^n} u(x-t, \varepsilon y) y^{\alpha-1} e^{-\pi y^2} \varepsilon^{-n} \psi(t/\varepsilon)\, dt dy$.

Alberto Torchinsky

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Coifman, R. R.; Rochberg, R.; Weiss, Guido

Factorization theorems for Hardy spaces in several variables.

Annals of Mathematics. Second Series **103** (1976), *no.* 3, 611–635.

A basic result, which goes back to F. Riesz [Math. Z. **18** (1923), 87–95; Jbuch **49**, 225], states that every function in the Hardy space H^1 of the unit circle is the product of two H^2 functions. The result has a real variable formulation: a function in L^1 is the real part of an H^1 function if and only if it can be written as $g\tilde{h} + \tilde{g}h$, where g and h are real L^2 functions and $\sim$ denotes the conjugation operator. In the paper under review, the authors extend this factorization to several settings where a natural analogue of H^1 (or of $\operatorname{Re} H^1$) can be defined.

The authors first consider the situation in which the unit circle is replaced by Euclidean n-space. Following E. M. Stein [*Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N.J., 1970; MR0290095; Russian translation, Izdat. "Mir", Moscow, 1973; MR0348563], the space H^1 is defined as the class of functions that belong, together with their Riesz transforms, to L^1. An analogue of the real variable version of the Riesz factorization is established in which the conjugation operator is replaced by the Riesz transforms and the sum of two terms is replaced by an infinite sum. The result depends upon the duality between H^1 and BMO (the space of functions of bounded mean oscillation [see C. Fefferman and Stein, Acta Math. **129** (1972), no. 3–4, 137–193]) and upon a new result, very interesting in its own right, linking BMO and the L^p boundedness of certain commutator operators. The latter result is as follows. Let K be a Calderón-Zygmund singular integral operator with a smooth kernel, and let B be the operator of multiplication by the function b. If b is in BMO, then the commutator $[K,B]$ is bounded on L^p for $1<p<\infty$, its norm being no larger than $\|b\|_{\mathrm{BMO}}$ times a constant which depends only on K and p. Conversely, if, for some p in $(1,\infty)$, the commutator $[K,B]$ is bounded on L^p whenever K is one of the Riesz transforms, then b is in BMO. Two proofs are given for the first half (the difficult half) of this theorem, one involving direct estimates of certain maximal operators, the other involving B. Muckenhoupt's theory of weights [see the first author and Fefferman, Studia Math. **51** (1974), 241–250; MR0358205]. The latter proof is used at the end of the paper to obtain the following characterization of BMO in terms of the L^p boundedness of a certain maximal operator. For b and f locally integrable functions, define the function $M(b,f)$ on $\mathbf{R}^n$ by $M(b,f)(x) = \sup_{x\in Q} |b(x) - b_Q||f_Q|$, where the Q's are cubes with edges parallel to the coordinate axes, and b_Q and f_Q denote the averages of b and f over Q. Then b is in BMO if and only if the map $f \to M(b,f)$ is bounded on L^p $(1<p<\infty)$.

The authors next study the situation in which the unit circle is replaced by the unit sphere in complex n-space. Here, the definition of H^1 is the direct generalization of the classical definition for the case $n=1$. It is shown that every H^1 function can be written as an infinite sum of products of H^2 functions. The result depends upon the duality between H^1 and a suitably defined space BMO, and upon the so-called atomic decomposition of H^1 functions (see the first and third authors, "H^p spaces and harmonic analysis," to appear in Bull. Amer. Math. Soc.). As an application, an analogous factorization is established for holomorphic functions in the unit ball that are integrable with respect to the weight function $(1-|z|^2)^k$, k a positive integer.

The authors also establish a connection between BMO and Hankel operators on H^2 of the sphere. For b in H^2, define the operator K_b on H^2 by $K_b f = P(b\overline{f})$, where P is the orthogonal projection of L^2 onto H^2. It is shown that K_b is bounded if and only if b is in BMO, in which case $\|K_b\|$ is equivalent to $\|b\|_{\mathrm{BMO}}$. For the case $n=1$ this goes back to Z. Nehari [Ann. of Math. (2) **65** (1957), 153–162; MR0082945]. Moreover, K_b is compact if and only if b is in VMO, the space of functions of vanishing mean oscillation. The latter is the analogue of the space VMO on Euclidean n-space introduced by the reviewer [Trans. Amer. Math. Soc. **207** (1975), 391–405; MR0377518]. For $n = 1$, the criterion for the compactness of K_b goes back to P. Hartman [Proc. Amer. Math. Soc. **9** (1958), 862–866; MR0108684].

The authors indicate that much of their analysis applies in the general context of spaces of homogeneous type (see the first and third authors [op. cit.]). They single out an especially interesting case in which H^1 turns out to be the Bergman space, A^1, of functions that are holomorphic in the unit disk and integrable with respect to area measure. The dual of A^1 is identified as the space of holomorphic functions in the disk that have bounded mean oscillation in a suitable sense. It is also identified as the space of Bloch functions, that is, the space of functions f that are holomorphic in the disk and satisfy $|f'(z)| = O((1-|z|)^{-1})$. The factorization theorem and the theorems on boundedness and compactness of Hankel operators are true in this setting.

D. Sarason

From MathSciNet, August 2020

MR0508453 (80k:42023) 42B20; 28A15

Stein, Elias M.; Wainger, Stephen

Problems in harmonic analysis related to curvature.

Bulletin of the American Mathematical Society **84** (1978), *no.* 6, 1239–1295.

For what other sets, besides balls and cubes, does the standard real variable theorem of Lebesgue on the differentiation of integrals of functions of several variables hold? This subject has attracted a great deal of interest and this lucid article examines two specific instances, namely, that of spheres and pieces of curves emanating from points. The results are sharp, interesting and well motivated. Let us illustrate this by mentioning a very particular case of the positive results in the latter case, namely, that of a curve with an appropriate amount of curvature (the curvature assumption is crucial). Theorem B: Let f be locally in $L^p(\mathbf{R}^2)$, $p > 1$. Then $\lim_{h\to 0}\int_0^h f(x-\gamma(t))\,dt = f(x)$ a.e. provided that $\gamma(0) = 0$ and either γ has nonvanishing curvature at the origin or γ is real analytic in $\mathbf{R}^2$. Closely related to this problem is the study of singular integral operators along the curves $\gamma(t)$. These operators are defined for smooth f by $Kf(x) = \text{p.v.}\int_{-1}^1 f(x-\gamma(t))\,dt/t$. Theorem C: $\|Kf\|_p \le A_p\|f\|_p$, $1 < p < \infty$, provided γ is as in Theorem B. The article is divided into two parts and an epilogue. The first part reviews some of the classical ideas of maximal functions, differentiation theorems, singular integrals, their relationships and applications. An indication of the difficulty in carrying over these techniques to the present problems is included. Finally, the new ideas that enter in the analysis are discussed, namely, the authors show how to exploit the curvature of curves (and spheres) via the Fourier transform to obtain Theorems B and C as well as other related results. The second part contains the full proofs and the final part states some further results and poses some questions. The authors point out that A. Nagel and N. M. Rivière have worked with them in several of the problems and we finally mention that the Proceedings of the Williamstown Conference in Harmonic Analysis [Harmonic analysis in Euclidean spaces, Parts 1, 2, Providence, R.I., 1979;

MR0545233, b] contain several articles which complement the results given here, and which bring the subject up to date (i.e., up to the summer of 1978).

Alberto Torchinsky

From MathSciNet, August 2020

MR0676987 (84d:31005a) 31B25; 42B25

Jones, Peter W.

A geometric localization theorem.

Advances in Mathematics **46** (1982), *no.* 1, 71–79.

MR0676988 (84d:31005b) 31B25; 42B25

Jerison, David S.; Kenig, Carlos E.

Boundary behavior of harmonic functions in nontangentially accessible domains.

Advances in Mathematics **46** (1982), *no.* 1, 80–147.

Jerison and Kenig define a nontangentially accessible (N.T.A.) domain as a bounded open subset $D \subset \mathbf{R}^n$ such that any point $x_0 \in \partial D$ can be approached in some reasonable way by a regular chain of nontangential balls lying in D (and also by nontangential balls lying in the complement of $\overline{D}$). The corresponding regular chains are then used in order to define nontangential notions similar to the one used in the classical case. Nontangential balls are those for which diameter and distance to the boundary ∂D of D are of the same order of magnitude.

Jerison and Kenig prove the fundamental fact that the boundary ∂D of an N.T.A. domain D coincides with the so-called Martin boundary of D. In general this boundary is not rectifiable and will always be equipped with the harmonic measure ω. Then ω satisfies the doubling condition and $(\partial D, \omega)$ is a space of homogeneous type.

Jerison and Kenig prove the following crucial theorems for N.T.A. domains D: (1) Let u be harmonic in an N.T.A. domain D. The set of points of D where the area integral is finite equals ω-almost everywhere the set of points where u has nontangential limits. (2) Assume that u is harmonic in an N.T.A. domain D and is nontangentially bounded from below on $F \subset \partial D$. Then u has nontangential limits ω-almost everywhere on F. (3) Let D be an N.T.A. domain, $f \in L^1(d\omega)$, and for $x \in D$ define $u(x) = \int_{\partial D} f(y)\omega^x(dy)$, where $\omega^x(dy)$ is the harmonic measure. Then the nontangential maximal function $N_\alpha u$ of u at any point $x_0 \in \partial D$ is dominated by the Hardy and Littlewood maximal function $M_\omega f$ at x_0.

This last result implies that the so-called $\mathcal{H}^p$ theory of harmonic functions in D is similar to the classical case when $1 < p < +\infty$. This leads, when $0 < p \leq 1$, to defining $H^p(D, d\omega) =\{u$ harmonic in $D\colon N_\alpha(u) \in L^p(d\omega)\}$. (4) The main theorem is the identification of $H^p(D, d\omega)$ with the atomic $H^p_{\mathrm{at}}(\partial D, d\omega)$. Finally the authors study the Dirichlet problem with data in $\mathrm{BMO}(\partial D, d\omega)$ and characterize the corresponding harmonic functions.

The second paper is carefully organized and self-contained and should become a classic in the field.

Jerison and Kenig's theory relies on a crucial geometrical fact, proved by Jones in the first paper about the local structure of N.T.A. domains: If D is an N.T.A. domain, then for any $x_0 \in \partial D$ and $r < r_0$ there exists an N.T.A. domain $\Omega \subset D$

such that

$$B(x_0, r/M) \cap D \subset \Omega \subset B(x_0, Mr) \cap D.$$

Furthermore, the constant M in the N.T.A. definition for Ω is independent of x_0 and r.

Yves Meyer

From MathSciNet, August 2020

MR1315543 (96c:42028) 42B15; 42B20, 47B38

Bourgain, Jean

Some new estimates on oscillatory integrals.

Essays on Fourier analysis in honor of Elias M. Stein (Princeton, NJ, 1991), 83–112, *Princeton Math. Ser., 42*, *Princeton Univ. Press, Princeton, NJ*, 1995.

The author considers the following related problems: (i) restriction and extension problems; (ii) the Bochner-Riesz summation operators; (iii) the oscillatory integrals considered by L. Hörmander [Ark. Mat. **11** (1973), 1–11; MR0340924]. The Hörmander problem generalizes the ones defined in (i) and (ii). We describe briefly these problems. Let (A) $f \mapsto \hat{f}|_{S^{d-1}}$ be a restriction and let (A1) $\mu \in M(S^{d-1}) \mapsto \hat{\mu}$ be an extension map, where $S = S^{d-1}$ is the unit sphere in $\mathbf{R}^d$ with its invariant measure σ, and the measure μ in (A1) on S is assumed to be absolutely continuous with respect to σ.

Note that the operator in (A1) may be expressed by an operator of the form

$$Tf(x) = \int e^{i(x_1y_1 + \cdots + x_{d-1}y_{d-1} + x_d\psi(y))} f(y)dy \tag{1}$$

where $x \in \mathbf{R}^d$, and y is taken in a neighborhood of $0 \in \mathbf{R}^{d-1}$, ψ a smooth function in that neighborhood satisfying $\det(\partial^2\psi/\partial y_i \partial y_j) \neq 0$, and f is a function corresponding to $d\mu/d\sigma$.

$$Tf(x) = \int_{\mathbf{R}^d} \frac{f(y)e^{i|x-y|}}{|x-y|^{(d+1)/2+\lambda}} dy \quad x \in \mathbf{R}^d, \tag{B}$$

defines the Bochner-Riesz operators (and the corresponding multiplier operator). After fixing one coordinate of the y-variable, say $y_d = 1$, the phase function $|x-y|$ becomes (2) $[(x_1-y_1)^2 + \cdots + (x_{d-1}-y_{d-1})^2 + (x_d-1)^2]^{1/2}$. Rescaling along with an asymptotic expansion of (2) we obtain the generalizations considered in Hörmander's paper [op. cit.]. Consider the operators

$$T_N f(x) = \int_{\mathbf{R}^{d-1}} e^{iN\phi(x,y)} a(x,y) f(y) dy, \quad x \in \mathbf{R}^d,$$

where

$$\phi(x,y) = x_1y_1 + \cdots + x_{d-1}y_{d-1} + x_d\langle Ay, y\rangle + O(|x| \cdot |y|(|x|^2 + |y|^2)) \tag{3}$$

with $a \in C_0^\infty(\mathbf{R}^{2d-1})$, ϕ real-valued, $\phi \in C^\infty(\mathbf{R}^{2d-1})$, and ϕ also satisfying some further technical conditions to insure non-degenerate critical points. For $N > 1$ one seeks uniform estimates of the form (C) $\|T_N f\|_q \leq CN^{-d/q}\|f\|_r$ for certain pairs (q, r). It is conjectured that the operators defined in (B) map $L^p(\mathbf{R}^d)$ into itself iff $\lambda > 0$ and

$$\frac{2d}{d+1+2\lambda} < p < \frac{2d}{d-1-2\lambda}. \tag{4}$$

In Section 5, the author shows the following proposition. Let $p^* = p \vee p'$. Then the operators T defined in (B) map $L^p(\mathbf{R}^d)$ into itself if p^* satisfies (4) and some additional (but explicitly determined) lower bound. For dimension $d = 3$ he gets the result in the case $p^* \geq \frac{58}{15}$.

It is conjectured that the operator defined in (A) maps boundedly from $L^{q'}(\mathbf{R}^d)$ to $L^{p'}(S^{d-1})$ or that the operator T defined in (1) satisfies $\|Tf\|_{L^q(\mathbf{R}^d)} \leq C\|f\|_{L^r(S^{d-1})}$ for

$$q > \frac{2d}{d-1} \quad \text{and} \quad \frac{d+1}{(d-1)q} + \frac{1}{r} \leq 1. \tag{5}$$

One considers the same range (5) for the inequality in (C). For $d \geq 3$ the inequality (C) may fail under the conditions (5). In fact if $d > 2$ is odd, (C) may only hold if $q \geq 2(d+1)/(d-1)$, even if $r = \infty$.

For the case $d = 3$ see another paper by the author [Geom. Funct. Anal. **1** (1991), no. 4, 321–374; MR1132294]. The author points out that if the dimension d is even, then inequality (C) holds when $r = \infty$ for certain $q < 2(d+1)/d - 1$ for ϕ described in (3) (including the additional technical conditions). In Section 6, the author outlines the argument in case $d = 4$.

In Section 7, the author gives an application to the two-dimensional Schrödinger equation.

Gary Sampson

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Seeger, Andreas

Singular integral operators with rough convolution kernels.

Journal of the American Mathematical Society **9** (1996), *no.* 1, 95–105.

In the classical Calderón-Zygmund theory [A. P. Calderón and A. Zygmund, Acta Math. **88** (1952), 85–139; MR0052553], singular integral operators of the form

$$Tf(x) = \text{p.v.} \int_{\mathbf{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy \tag{1}$$

were shown to be bounded on L^p, $1 < p < \infty$. Here, Ω is positively homogeneous of degree zero, has mean value zero on the sphere S^{n-1}, and, in the original theory, satisfied some mild regularity condition. Given the results obtained in the paper under review, we need not insist on this latter point too much, but it is more than enough, for example, that $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$, for any positive α. L^p estimates for such singular integral operators were originally of interest because, for example, they imply (interior) $W^{2,p}$ regularity of solutions of certain constant coefficient elliptic equations.

One observes that, by duality, it is enough to prove L^p bounds for operators of the type (1) in the case that $1 < p \leq 2$. The method of proof in Calderón and Zygmund's 1952 paper [op. cit.] involved interpolation between an L^2 bound (relatively easy to obtain by Plancherel's theorem) and a weak-type $(1,1)$ estimate. To establish the latter was really the heart of the "Calderón-Zygmund method". Let us describe this method in some detail. One wishes to show that, for all $\lambda > 0$,

$$|E_\lambda| \equiv |\{x \in \mathbf{R}^n \colon\ |Tf(x)| > \lambda\}| \leq C\lambda^{-1}\|f\|_1. \tag{2}$$

The famous "Calderón-Zygmund decomposition lemma" states that, given $f \in L^1$ and $\lambda > 0$, there exists a non-overlapping family of cubes, $\{Q_k\}_{k=1}^{\infty}$, together with a decomposition of f into its "good" and "bad" parts $f = g + b$, with the following properties: (3) $\|g\|_2 \leq C\lambda\|f\|_1$, (4) $\|b\|_1 \leq C\|f\|_1$, (5) $b \equiv \sum_{k=1}^{\infty} b_k$, where $b_k \equiv b\chi_{Q_k}$, (6) $\int b_k = 0$ and $\|b_k\|_1 \leq C\lambda|Q_k|$, and (7) $\sum_{k=1}^{\infty} |Q_k| \leq C\lambda^{-1}\|f\|_1$. Thus, by (3), (4) and (7), it is enough to prove (2) with f replaced by b and with E_λ replaced by $F_\lambda \equiv \mathbf{R}^n \setminus \bigcup_{k=1}^{\infty} Q_k^*$, where Q_k^* is the "double" of Q_k. In Calderón and Zygmund's 1952 paper [op. cit.], properties (5) and (6) were used, together with the smoothness of Ω, to show that one actually has something stronger: (8) $\int_{F_\lambda} |Tb| \leq C\|b\|_1$. Calderón and Zygmund [Amer. J. Math. **78** (1956), 289–309; MR0084633] extended the L^p bounds of their 1952 paper to the case that Ω satisfies only a size condition on the sphere ($\Omega \in L\log L$, which cannot, in general, be relaxed), with no hypothesis of smoothness. The method of proof (the "method of rotations") was quite different from that of the 1952 paper: in particular, it does not yield the weak-type $(1,1)$ bound (2). Indeed, to prove (2) in the absence of any smoothness assumption on Ω remained a completely open problem for 30 years. The first significant progress toward such a weak-type $(1,1)$ bound was made by M. Christ [Ann. of Math. (2) **128** (1988), no. 1, 19–42; MR0951506], who considered the following variant of the maximal operator:

$$M_\Omega f(x) \equiv \sup_{r>0} r^{-n} \int_{|x-y|<r} |\Omega(x-y)f(y)|\, dy. \tag{9}$$

It is a consequence of the methods of Calderón and Zygmund [op. cit., 1956] that M_Ω is bounded on $L^p, 1 < p < \infty$, for all $\Omega \in L^1$. Christ showed that, when the dimension $n = 2$, the operator M_Ω is of weak type $(1,1)$ whenever $\Omega \in L^q$, $q > 1$ (this result was subsequently extended to all dimensions, and to the case $\Omega \in L\log L$, by Christ and J. L. Rubio de Francia [Invent. Math. **93** (1988), no. 1, 225–237; MR0943929]—the case $\Omega \in L^1$ remains open). The method of proof in the work of Christ [op. cit.] and of Christ and Rubio de Francia [op. cit.] involves a variant of the Calderón-Zygmund method in which the L^1 estimate (8) is replaced by the L^2 bound: (10) $\int_{F_\lambda} |Tb|^2 \leq C\lambda\|b\|_1$. This approach to weak-type $(1,1)$ inequalities had previously appeared in work of C. Fefferman [Acta Math. **124** (1970), 9–36; MR0257819]. Using heavily the ideas of Christ [op. cit.], Christ and Rubio de Francia [op. cit.], and, independently, the reviewer [Proc. Amer. Math. Soc. **103** (1988), no. 1, 260–264; MR0938680], were able to extend Christ's 2-dimensional result for M_Ω to the case of the singular integral operator T defined in (1). The former pair of authors even improved this result to the case $\Omega \in L\log L$, and, in unpublished work, proved the corresponding theorem in dimensions 3, 4, 5, 6 and 7. But still, the problem remained open, in general. Part of the difficulty was that the technique of Christ [op. cit.], while beautifully suited to the geometry of the plane, was less suitable in higher dimensions. In spite of this, Christ and Rubio de Francia [op. cit.] were able to push this method to arbitrary dimension in the case of M_Ω, which is a somewhat simpler object than the singular integral operator T. Already, though, the proof is quite difficult technically. And for the singular integral, these technical difficulties become extreme with increasing dimension.

Finally, the problem has been laid to rest in all dimensions, in the present beautiful paper by A. Seeger. Seeger's proof is a variant of Christ's method, combined with a microlocal decomposition of the kernel which rather painlessly yields an analogue of the L^2 estimate (10) in all dimensions. To be a little more precise, Ω

is chopped up in such a way that the corresponding pieces of the kernel are nearly orthogonal. Indeed, they are almost orthogonal, if one ignores errors which arise by chopping off "tails" on the Fourier transform side. The rapidly decaying tails reflect enough smoothness on the kernel side so that, roughly speaking, the classical theory can be applied to these error terms, which thus satisfy an L^1 estimate in the spirit of (8). This leaves the main terms, whose orthogonality alone yields, essentially, (10). It is remarkable that it is only orthogonality which is at work here, and not, in contrast to the papers by Christ [op. cit.], Christ and Rubio de Francia [op. cit.] and the reviewer [op. cit.], some weak form of smoothness. In particular, for this part of the operator, one requires in condition (6) only the size estimate for $\|b_k\|_1$, and not the fact that b_k has mean value zero.

Steve Hofmann

From MathSciNet, August 2020

MR1625056 (99f:42026) 42B10; 42B25

Tao, Terence; Vargas, Ana; Vega, Luis

A bilinear approach to the restriction and Kakeya conjectures.

Journal of the American Mathematical Society **11** (1998), *no.* 4, 967–1000.

In this paper, bilinear versions of two of the most classical problems of modern harmonic analysis are considered, and results about them are then applied to the usual versions. These two problems are the restriction problem for the Fourier transform and the Kakeya maximal problem. The (dual) form of the restriction problem asks for which $1 \le p, q \le \infty$ one has the estimate

$$\|\widehat{gd\sigma}\|_{L^q(\mathbf{R}^n)} \le C\|g\|_{L^p(d\sigma)} \tag{1}$$

(say for all test functions g), where $d\sigma$ denotes surface measure on, for example, the unit sphere or the base of a paraboloid P^{n-1} in $\mathbf{R}^n$. The Kakeya maximal problem asks for the precise $L^p \to L^q$ mapping properties (in terms of the parameter δ) of the mapping

$$f \mapsto \sup_{x\in R} \frac{1}{|R|}\int_R |f|, \tag{2}$$

where the sup is taken over (say) the family of all $\delta \times \delta \times \cdots \times \delta \times 1$ rectangles in $\mathbf{R}^n$.

A great deal about these problems is known, especially in low dimensions, as is the fact that the two problems are inextricably linked and also provide a quantitative framework for estimating lower bounds on dimensions of Kakeya sets (that is, sets containing unit line segments in each direction) in $\mathbf{R}^n$.

The bilinear restriction problem is the problem of determining those p_1, p_2 and q such that

$$\|\widehat{gd\sigma}\widehat{hd\sigma}\|_{L^q(\mathbf{R}^n)} \le C\|g\|_{L^{p_1}(d\sigma)}\|h\|_{L^{p_1}(d\sigma)} \tag{3}$$

for all g and h supported on 1-separated 1-cells on P^{n-1}, and the ideas connected with this problem have a long (if rather unexplicit) history going back to works of Sjölin, Beals and the present reviewer in the early 1980s.

The first result presented by the authors is that problems (1) and (3) are essentially equivalent in the range where it is conjectured that (1) holds. This is essentially an advanced form of a scaling argument. On the other hand, the range of indices for which one might expect (3) to be valid is larger than what simple considerations of (1) might lead one to imagine. Indeed, the next result characterises the p's for which (3) holds for $q = 2$ and $p_1 = p_2 = p$ as being $[4n/(3n-2), \infty]$; this is based upon "a routine modification" of a previous argument (in the case $n = 3$) of Moyua, Vargas and Vega. Examples giving necessary conditions on $p_1 = p_2 = p$ and q such that (3) holds are also given. An immediate consequence of the aforementioned results is the first sharp (dual) restriction theorem on the line $1/q = (n-1)/(n+1)p'$ when $n = 3$ which goes beyond the Stein-Tomas point $q = 4,\ p = 2$. A further improvement on this occurs later in the paper.

With a view to this further improvement, the authors next turn to formulating and proving bilinear variants of the dual of the Kakeya problem (2) with a suitable separation hypothesis on the supports of the pair of functions. Once again they prove an equivalence between the usual and the bilinear forms of the problem in the natural ranges of indices. They then prove a bilinear estimate which in the case $n = 3$ improves upon Wolff's Kakeya maximal theorem—but only on the scale of bilinear, not linear estimates.

In the last part of the paper the authors obtain their sharpest results for the restriction problem by modifying the "Bourgain machine" to make it suitable for the bilinear estimates considered earlier. This is by far the most technical part of the paper, but the final result is that when $n = 3$, (1) holds for $p > \frac{170}{77}$ and $q > \frac{34}{9}$ while on the line $1/q = (n-1)/(n+1)p'$ $(n = 3)$ (1) holds for $q > 4 - \frac{5}{27}$. The authors also show that the so-called Bochner-Riesz conjecture holds when $n = 3$ and $p > \frac{34}{9}$ (≥ 2).

This paper represents an exciting new development in the study of the restriction phenomenon.

Anthony Carbery

From MathSciNet, August 2020

MR1745019 (2001b:42015) 42B20; 42B25

Lacey, Michael T.

The bilinear maximal functions map into L^p for $2/3 < p \leq 1$.

Annals of Mathematics. Second Series **151** (2000), *no.* 1, 35–57.

This article is concerned with boundedness properties of the maximal function

$$M(f,g)(x) = \sup_{N>0} \frac{1}{2N} \int_{-N}^{+N} |f(x-\alpha t)g(x-t)|\, dt$$

where x and α are real numbers. The interesting cases arise when α is not equal to 0 or 1; otherwise the problem is trivial.

It is expected that the bisublinear operator M maps products of Lebesgue spaces into another Lebesgue space, when the indices are related as in Hölder's inequality. This is easily seen when the index of the target space is a number strictly larger than 1. A. Calderón conjectured that the same conclusion is valid when this index is equal to 1, in particular, that M maps $L^2 \times L^2$ into L^1. The author proves the validity of this conjecture plus more. More precisely, he proves the following: For all $1 < p, q \leq \infty$ with $2/3 < r = pq/(p+q) < \infty$ and all $\alpha \neq 1$, there is a constant $C(p,q,r,\alpha)$ such that the estimate $\|M(f,g)\|_{L^r} \leq C\|f\|_{L^p}\|g\|_{L^q}$ is valid for all Schwartz functions f and g. Moreover, Lacey shows that the same estimate holds for the maximal singular integral

$$T^*(f,g)(x) = \sup_{\epsilon,\delta>0} \left| \int_{\epsilon<|y|<\delta} f(x-\alpha y)g(x-y)K(y)\,dy \right|,$$

where $K(x)$ is a function on the line which satisfies the derivative estimates $|K^{(m)}(x)| \le C_m|x|^{-1-m}$, $x \ne 0$, for all $0 \le m \le N$ up to a sufficiently large number N. This maximal operator controls the bilinear Hilbert transform when $K(y) = 1/y$ and is related to the maximal function introduced above.

The method of proof is quite close to earlier work of Lacey and C. Thiele in which boundedness of the bilinear Hilbert transforms was obtained in the same range of the indices p, q, r. The author reduces the study of the operators M and T^* to a class of operators called the "model sums" which are easily shown to control them. It is only here that the author uses the high degree of smoothness of the kernel K. The restriction $r > 2/3$ in the theorems above appears since the model sums are shown to be unbounded otherwise. It is still an open question whether M and T^* map into L^r when $1/2 < r \le 2/3$.

The key ingredient of the proof of the boundedness of the model sums is an L^2 inequality regarding maximal multipliers due to J. Bourgain. The proof has a flavor of orthogonality, typical of Lacey's work, manifested in several maximal variants of earlier orthogonality estimates obtained by Lacey and Thiele. These results are brilliantly put together to yield the proof, like pieces of a jigsaw puzzle. The author's understanding of the subject is undoubtedly very profound. The mastery of technique and profusion of ideas in this work are remarkable. To put it simply, Lacey has produced a very noteworthy paper.

Loukas Grafakos

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