BOURGAIN’S WORK IN FOURIER RESTRICTION

CIPRIAN DEMETER

ABSTRACT. This note will show how Bourgain’s work in Fourier restriction theory has built bridges to number theory, partial differential equations, and additive combinatorics.

1. A BRIEF INTRODUCTION

As far as the topics presented in this note are concerned, I am tempted to split Bourgain’s work into two categories: prior to and after 2011. There are a few reasons why this division is not entirely artificial. Over the last years of his life Jean fought an aggressive form of cancer. Yet, he managed to write an inconceivably large number of papers in that period, and breathed mathematical ideas until his last few days. This latter part of his work speaks volumes about his endurance and unconditional passion for doing mathematics.

Also, many survey papers have been written that describe his contributions to Fourier restriction and related areas, made prior to 2011. An excellent and rather comprehensive article is by Laba [32]. Jean himself took part of his precious time to write a few expository notes ([17], [15], [11]) about this segment of his work. Below is his opening paragraph in [15].

When I was asked to write an essay for mathematics for the year 2000, I felt that, as an individual contributor to this project, it made the most sense to discuss only issues in which I had a direct research involvement over the past years. The expertise needed to measure and accurately value progress in a given area requires indeed more than a few superficial encounters with that subject. As such, I faced initially the possibility of writing a paper advertising a large number of unsolved mathematical problems I had thought about for a significant amount of time and would engage others to pursue. But I quickly realized that carrying out such an enterprise in a meaningful way would be quite demanding in space and especially time. I therefore decided to postpone this, like many other things, to some later stage of my career.

One can only imagine what this paper with problems would have looked like, had his dream not been cut short by an untimely death.

This note will start by summarizing his main contributions prior to 2011. Following the philosophy laid out in the cited paragraph, I will give more details about his later work, where I trust my expertise a bit better. This is largely due to the fortunate collaboration I entertained with Jean over the last five years of his life.

Received by the editors September 2, 2020.

2020 *Mathematics Subject Classification*. Primary 42B37.

It is no secret to anybody who has read at least a few of Bourgain's papers, that his work is rather technical. To maximize the clarity of the narrative, I will occasionally oversimplify technical details and some of the definitions.

2. THE INITIAL BREAKTHROUGH ON THE KAKEYA AND THE RESTRICTION CONJECTURES

In the early 1990s Bourgain expanded his research horizon in new directions. Most notably, he became interested in dispersive PDEs and also in an area of harmonic analysis called *Fourier restriction*. It took him very little time to uncover fundamental connections between these fields. See Section 3 in the companion paper [31] for more details.

Let us start by recalling two central conjectures that have motivated and have been fundamentally transformed by Bourgain's work in Fourier restriction. At the heart of the first problem lies an object that looks deceptively simple. A *Kakeya* (or *Besicovitch*) set in $\mathbb{R}^n$ is a set containing a unit line segment in every direction. A trivial example is any unit ball. It has been known for about a century that there are Kakeya sets with zero Lebesgue measure. This is of course rather nonintuitive, but the following conjecture partly restores the expectation that Kakeya sets need to be "big".

Conjecture 2.1 (Kakeya conjecture). *Each Kakeya set in $\mathbb{R}^n$ has Hausdorff dimension equal to n .*

There are a few different, almost equivalent ways to formulate such a conjecture. For example, one can ask the slightly weaker question about the Minkowski dimension. Another version involves $(\delta, 1)$ -tubes—cylinders of length 1 and radius δ in $\mathbb{R}^n$. If we consider an arbitrary collection $\mathcal{T}$ consisting of roughly δ^{-n+1} such tubes with δ -separated directions, we may conjecture that they are essentially disjoint:

$$|\bigcup_{T \in \mathcal{T}} T| \gtrsim (\log \frac{1}{\delta})^{-C} \sum_{T \in \mathcal{T}} |T|.$$

To describe the second problem, we let $(\mathbb{S}^{n-1}, d\sigma)$ be the unit sphere in $\mathbb{R}^n$ equipped with the surface measure. Given $1 \leq p, q < \infty$, we will say that the restriction estimate $R(p \mapsto q)$ holds if

$$(1) \quad \|\widehat{fd\sigma}\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{S}^{n-1})},$$

for all smooth functions f on $\mathbb{S}^{n-1}$. Here $fd\sigma$ is a measure supported on the sphere, with the Fourier transform given by

$$\widehat{fd\sigma}(x) = \int_{\mathbb{S}^{n-1}} f(\xi) e(-\xi \cdot x) d\sigma(\xi).$$

We write $e(z) = e^{2\pi iz}$. The following conjecture took shape in the late 1960s. It was the brainchild of Elias Stein, who also produced the first results in its direction.

Conjecture 2.2 (Restriction conjecture for $\mathbb{S}^{n-1}$). *The estimate $R(p \mapsto q)$ holds if and only if*

$$(2) \quad \frac{n-1}{p'} \geq \frac{n+1}{q}$$

and

$$(3) \quad q > \frac{2n}{n-1}.$$

This conjecture is expected to hold with the sphere replaced by any hypersurface with nonzero Gaussian curvature. The “only if” part is known, as it easily follows by testing with simple examples of functions f . It is also not hard to show that the restriction conjecture implies the Kakeya conjecture. This may come as a surprise, as the former is of oscillatory nature, while the latter has purely measure theoretic nature. The oscillations are however eliminated efficiently (this is called *square root cancellation*) by testing (1) with a randomized function f

$$f(\xi) = \sum_{\tau} \pm 1_{\tau}(\xi),$$

that assigns values 1 or -1 to each cap τ of size $\sim \delta$ in a partition of $\mathbb{S}^{n-1}$.

Prior to Bourgain's work, the Kakeya conjecture had been verified in $\mathbb{R}^2$. When $n \geq 3$, Kakeya sets in $\mathbb{R}^n$ were known to have dimension at least $\frac{n+1}{2}$. The full restriction conjecture was also known in $\mathbb{R}^2$. In higher dimensions, the range $q \geq \frac{2(n+1)}{n-1}$ was covered by the work of Stein and Tomas from the early 1970s. Around the same time, similar results were proved by Strichartz for the paraboloid, and would later have a significant impact in dispersive PDEs. This remained the state of the art for almost two decades.

Bourgain's arrival on the scene was like thunder, with the two landmark papers [23] and [22] completely revolutionizing the field. The first paper introduced a whole new toolbox in Fourier restriction. Here are a few examples.

First, he proved that Kakeya sets in $\mathbb{R}^3$ have dimension at least $\frac{7}{3}$. His argument revolves around the idea of organizing the tubes into clusters called *bushes*. This marked the birth of a combinatorial approach to the Kakeya problem, which would later lead, through the combined efforts of various mathematicians, to uncovering similar structures with increasing levels of sophistication.

In the same paper, Bourgain managed for the first time to break the Stein–Tomas exponent barrier $\frac{2(n+1)}{n-1}$ for the restriction problem. To achieve this, he introduced a few innovations that would become instrumental for most of the subsequent developments in the field. First, he demonstrated how one can use the intersection pattern of tubes to gain information about restriction estimates. Crucial to his analysis is the use of exponential sums, which allow for a discrete reformulation of the restriction conjecture. In a mind-boggling combination of different techniques, Bourgain also brings to bear local restriction estimates, reverse square function inequalities, and a commanding use of the uncertainty principle, according to which the magnitude of functions with spectrum inside a ball of radius r is essentially constant on balls of radius $\frac{1}{r}$.

The companion paper [22] investigates the related problem about the boundedness of Hörmander operators, that we define below. To start, we denote by B_n the unit ball centered at the origin in $\mathbb{R}^n$. Let

$$\phi(x, \xi) : B_n \times B_{n-1} \rightarrow \mathbb{R}$$

be a smooth function and let $a(x, \xi)$ be a smooth bump function supported on $B_n \times B_{n-1}$. Consider the family of Hörmander operators $T^N = T^{N, \phi}$ for $N \geq 1$

$$T^N f(x) = \int_{B_{n-1}} f(\xi) e(N\phi(\frac{x}{N}, \xi)) a(\frac{x}{N}, \xi) d\xi.$$

One special case of interest is when ϕ is linear in x ,

$$\phi(x, \xi) = x \cdot \Psi(\xi) = x_1 \xi_1 + \cdots + x_{n-1} \xi_{n-1} + x_n \psi(\xi).$$

Note that if in addition we take $a \approx 1$, the above expression simplifies to

$$T^N f(x) \approx 1_{|x| \leq N} \int_{|\xi| \leq 1} f(\xi) e(x \cdot \Psi(\xi)) d\xi.$$

For such linear phases ϕ , there is a unique underlying hypersurface Ψ , given by $\Psi(\xi) = \partial_x \phi(x, \xi)$. A uniform estimate (with respect to N) for the operators T^N is in fact tantamount to a restriction estimate relative to this hypersurface. In the general case, the hypersurface $\partial_x \phi(x, \xi)$ is allowed to vary with x .

For linear phases, it is expected that the boundedness properties of T^N remain the same, subject to the requirement that Ψ has everywhere nonzero Gaussian curvature. In other words, the sphere can be replaced with any such hypersurface in the restriction conjecture, Conjecture 2.2.

There is a similar nondegeneracy condition for the general phase functions ϕ that we do not bother to define. Hörmander conjectured in [29] that the following holds true and also gave a proof for it in the case $n = 2$.

Conjecture 2.3. *Assume that ϕ is nondegenerate. For $q > \frac{2n}{n-1}$ and uniformly over $N \geq 1$, we have*

$$\|T^N f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{\infty}.$$

It came as a great surprise when Bourgain disproved this conjecture rather dramatically. His work exposes a fundamental difference between nondegenerate and positive definite phases, as well as between odd and even dimensions. At the heart of his counterexamples are certain tubes which, due to the nonlinearity of the phase, are now curved, rather than straight. These tubes, unlike their straight counterparts introduced earlier in the narrative, can pack themselves in a smaller area. Curved Kakeya sets can have smaller Hausdorff dimension than the ambient space!

Roughly speaking, in the odd-dimensional case one may arrange for $T^N f$ to be concentrated in a thin neighbourhood of a low degree algebraic variety of dimension $\frac{n+1}{2}$. On the other hand, in the even-dimensional case, $\frac{n+1}{2}$ is not an integer, and the best one can hope to do is $\frac{n+2}{2}$.

The paper [22] opened a new line of investigation in restriction theory that would later stimulate the use of algebraic methods in the field. Most notably, the sharp (and correct) version of Hörmander's conjecture for positive definite phases was recently proved by Guth, Hickman, and Iliopoulou [28], using the polynomial method.

3. DISCRETE FOURIER RESTRICTION

The measure $f d\sigma$ in (1) is absolutely continuous with respect to $d\sigma$. We might therefore view (1) as a *continuous* restriction estimate. Right after writing his aforementioned papers [23] and [22], Bourgain shifted his attention to a discrete analogue, where the measures $f d\sigma$ are weighted sums of point masses on manifolds,

such as the sphere and the paraboloid. The special case when these points are in $\mathbb{Z}^n$ bears additional significance, since the Fourier transforms of such measures are periodic exponential sums. The new problem can be formulated, somewhat vaguely, as follows.

Question 3.1 (Discrete restriction). For each $N \geq 1$, let $S_N \subset \mathbb{Z}^n$ be a finite collection of lattice points with cardinality satisfying $\log |S_N| \sim \log N$. Find the largest $p \geq 2$ such that for each $\epsilon > 0$ and each $a_s \in \mathbb{C}$,

$$(4) \quad \left\| \sum_{s \in S_N} a_s e(s \cdot x) \right\|_{L^p(\mathbb{T}^n)} \lesssim_\epsilon N^\epsilon \|a_s\|_{l^2(S_N)}.$$

This question did not appear out of the blue on Bourgain's radar. Only a few years earlier, he had proved his celebrated result on $\Lambda(p)$ -sets of integers. Essentially, these are (collections of) sets S_N for which (4) holds with N^ϵ replaced with a constant independent of N . In [24] Bourgain brilliantly proved the existence of $\Lambda(p)$ -sets using probabilistic constructions. See the companion paper [34] for a sketch of his argument.

In 1993, Bourgain published three remarkable papers [21], [20], and [19] on the topic of discrete Fourier restriction. While [21] went somewhat unnoticed, the impact of [20] and [19] was immediate. These two stand currently as his most cited papers.

In [21] he considers the discrete restriction problem for sets S_N representing the lattice points on the sphere $N\mathbb{S}^{n-1}$ of radius N . The associated exponential sums in this case are eigenfunctions of the Laplacian on $\mathbb{T}^n$. This problem lies at the intersection of at least three areas of mathematics: Fourier analysis, number theory, and spectral theory. Let us say a few words about the latter.

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 2$, and let Δ_g be the associated Laplace–Beltrami operator. We denote by λ^2 a typical eigenvalue of $-\Delta_g$, and by e_λ a corresponding eigenfunction

$$-\Delta_g e_\lambda = \lambda^2 e_\lambda.$$

For $p \geq 2$ we define

$$\mu(p) = \max\left(\frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{p}\right), n\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}\right).$$

Sogge proved in [33] that

$$\|e_\lambda\|_{L^p(M)} \lesssim \lambda^{\mu(p)} \|e_\lambda\|_{L^2(M)}.$$

This inequality is saturated when M is the round sphere, but a better estimate is expected to hold when M has nonpositive curvature.

And indeed, in [21] Bourgain improved the exponent $\mu(p)$ for the flat torus $M = \mathbb{T}^n$. His proof adapts the Stein–Tomas argument from the continuous Fourier restriction world. To run this argument, he derives kernel estimates using the theory of Kloosterman sums. This is a typical situation where he puts his previously acquired expertise to work in completely new contexts. Through his prior work in ergodic theory he has already exposed the successful marriage between the Fourier transform and the circle method. And his dramatic inroads from 1991 into the Fourier restriction problem have taught him the value of the Stein–Tomas argument.

The paper [20] analyzes the discrete restriction problem for the paraboloid. The methods are similar to those from the case of the sphere, but the motivation is different. The exponential sums in this case are solutions to the Schrödinger equation with spatial domain $\mathbb{T}^n$. The results are in fact analogous to those that Strichartz proved two decades earlier in the case when the spatial domain is $\mathbb{R}^n$. While of independent intrinsic interest, these restriction problems have a wide range of applicability in the regularity theory of dispersive PDEs. The methods introduced in [20] (and expanded in [19] to also address the Korteweg–de Vries equation) would dominate the mathematical landscape for two decades, until the advent of decoupling.

4. KAKEYA CONJECTURE, NUMBER THEORY, AND ADDITIVE COMBINATORICS

Bourgain is largely responsible for uncovering arithmetic structure in Kakeya sets and for introducing new lines of attack on Conjecture 2.1 along this venue.

The following (slight modification of Montgomery’s) conjecture is discussed in [18]. This is a statement about large values of Dirichlet sums.

Conjecture 4.1. *Let $T \leq N^2$ and let $a_n \in \mathbb{C}$ be arbitrary complex coefficients with modulus less than 1. We consider the associated Dirichlet series*

$$D(t) = \sum_{n=1}^N a_n n^{it}.$$

For each 1-separated set $\mathcal{T}$ in $[0, T]$ and each $\epsilon > 0$, we have (uniformly over a_n)

$$\sum_{t \in \mathcal{T}} |D(t)|^2 \lesssim_{\epsilon} N^{1+\epsilon} (N + |\mathcal{T}|).$$

To put this conjecture into perspective, it is known to imply a strong upper bound on the size of the gaps between consecutive primes, essentially the same as what would follow assuming the Riemann hypothesis

$$p_{n+1} - p_n \lesssim_{\epsilon} n^{\frac{1}{2}+\epsilon}.$$

The best result to date (see [1]) has $\frac{21}{40}$ in place of the exponent $\frac{1}{2}$.

In [18] Bourgain proved the striking result that Conjecture 4.1 implies the Kakeya conjecture, Conjecture 2.1. The interested reader may find a nice presentation of a slightly stronger statement in [35, Section 11.4]. In Bourgain’s approach, the link between the two conjectures is provided by the following conjecture about arithmetic progressions.

Conjecture 4.2 (Arithmetic Kakeya conjecture (mild form)). *Let k, N be positive integers. Write $F_k(N)$ for the size of the smallest set of integers containing, for each $d \in \{1, \dots, N\}$, a k -term arithmetic progression with common difference d . Then for each $\eta > 0$,*

$$\lim_{N \rightarrow \infty} \frac{\log F_{N^\eta}(N)}{\log N} \geq 1.$$

Essentially, Bourgain proved that Conjecture 4.1 $\implies$ Conjecture 4.2 $\implies$ Conjecture 2.1. A detailed account of different versions of the arithmetic Kakeya conjecture appears in [27].

The connection between the Kakeya problem and additive number theory was explored further in [16]. This work illustrates perfectly Bourgain’s awareness of

recent developments in other fields and his ability to import ideas and techniques to new, seemingly unrelated contexts. By using a slicing argument similar to the one in [18], Bourgain has observed that any hypothetical Kakeya set in $\mathbb{R}^n$ of small enough dimension would violate the following principle, which we state somewhat loosely as follows.

Restricted sums-differences principle. Assume A, B are subsets of a torsion-free abelian group, and let $G \subset A \times B$ be such that

$$|A|, |B|, |\{a + b : (a, b) \in G\}| \leq N.$$

Then there is $\alpha < 2$ (independent of A, B, G) such that

$$(5) \quad |\{a - b : (a, b) \in G\}| \leq N^\alpha.$$

In [16] Bourgain has verified this principle with $\alpha = \frac{25}{13}$, and used it to prove that Kakeya sets in $\mathbb{R}^n$ have dimension $\geq \frac{13}{25}n + \frac{12}{25}$. Here is a sketch of his argument. Consider a Kakeya set of dimension d . This amounts to the existence of roughly δ^{-n+1} $(\delta, 1)$ -tubes with δ -separated directions, that cover a small enough area. We may arrange for these tubes to lie between the horizontal hyperplanes $\mathcal{H}_1$ and $\mathcal{H}_2$, given by $\{x_n = 0\}$ and $\{x_n = 1\}$, respectively. Bourgain introduces a third hyperplane $\mathcal{H}_3$, given by $\{x_n = \frac{1}{2}\}$.

We call A, B, C the intersections between $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ and the central axes of these tubes, respectively. The dimension d of our Kakeya set forces the cardinalities of the lower dimensional sets A, B , and C to be essentially smaller than $N = \delta^{-(d-1)}$. To make this accurate, we would need to work with slight perturbations of A, B, C , but we brush this minor technical point under the rug. Let G be the collection of all pairs $(a, b) \in A \times B$, such that a and b represent the two ends of one of our tubes. Note that $\{a + b : (a, b) \in G\} = 2C$, so A, B, G satisfy the requirements in the sums-differences principle. Thus

$$|\{a - b : (a, b) \in G\}| \leq N^{\frac{25}{13}} = \delta^{-(d-1)\frac{25}{13}}.$$

On the other hand, since the differences $a - b$ with $(a, b) \in G$ represent directions of distinct tubes, it must be that $|\{a - b : (a, b) \in G\}| \sim \delta^{-n+1}$. Combining the last two estimates leads to the desired estimate $d \geq \frac{13}{25}n + \frac{12}{25}$.

The impact of [16] can hardly be overestimated. By connecting them to the sums-differences problem, Bourgain showed that Kakeya sets encode deep additive structure. Katz and Tao have later refined Bourgain's approach in [30] by incorporating more hyperplanes. What emerges is that if, subject to as many linear constraints as needed, one managed to lower the value of α in (5) arbitrarily close to 1, this would completely solve the Kakeya conjecture!

5. THE BOURGAIN-GUTH INDUCTION ON SCALES: PAVING THE WAY TO DECOUPLING

After a few years, Bourgain revisited the continuous restriction problem in his paper [14] with Guth. This will prove to be another groundbreaking work, that will lead to a plethora of less anticipated, far-reaching developments in many areas of mathematics.

One of the ideas that became influential with the turn of the millennium was the use of bilinear restriction estimates in order to prove their linear counterparts.

At the heart of this argument lies a certain induction on scales that dwells on a Whitney-type decomposition of the frequency domain. However, this mechanism is no longer efficient if one is willing to use multilinear estimates of higher order instead of bilinear ones. Such multilinear restriction inequalities became available in 2006, through the work on Bennett, Carbery, and Tao [2]. But for a few years, nobody knew how to exploit them.

The 2011 paper [14] introduces a fundamentally new way of running the induction on scales that replaces the Whitney-type decomposition with a uniform one. The crucial aspect of this new approach is that it allows us to successfully input the Bennett–Carbery–Tao inequality, leading to new linear restriction estimates even in three dimensions.

Soon after this, Bourgain put the new method to test for other related problems. One of them goes back to the old work of L. Carleson. Let $e^{it\Delta}f$ be the solution of the free Schrödinger equation

$$\begin{cases} iu_t - \Delta u = 0, & (x, t) \in \mathbb{R}^n \times \mathbb{R} \\ u(x, 0) = f(x), & x \in \mathbb{R}^n. \end{cases}$$

The question is to identify the smallest Sobolev exponent $\alpha(n)$, such that whenever $f \in H^\alpha(\mathbb{R}^n)$ with $\alpha > \alpha(n)$, we have almost everywhere convergence of the solution to its initial data

$$\lim_{t \rightarrow 0} e^{it\Delta} f(x) = f(x), \text{ for a.e. } x.$$

Solving this problem amounts to controlling a certain maximal function, with a flavor similar to a restriction estimate.

When $n = 1$ the problem was solved by Carleson and by Dahlberg and Kenig, but the case $n \geq 2$ turned out to be significantly harder. In [12] Bourgain used the new induction on scales to improve the upper bounds for $\alpha(n)$ in the case $n \geq 3$. A few years later, in [4], he constructed a simple, yet clever example which provided the correct lower bound on $\alpha(n)$. This example turned out to be very inspirational. Within two years, the upper bound matching Bourgain’s lower bound would also be proved. The papers [26] and [25] completely solved Carleson’s problem in all dimensions.

Bourgain’s aforementioned result on the Schrödinger maximal function came as part of a well-established framework; it was a follow-up to his earlier work on the problem. The next set of applications of the Bourgain–Guth induction on scales that we will describe are the ones originating in [13]. To many of us, this paper seemed to come out of nowhere. It would later become transparent that [13] marked the beginning of a “dream theory”, one that Jean has fantasized about many years before the right tools became available to formalize it. This will be discussed at length in the next section.

6. DECOUPLING

We close with a survey of Bourgain’s work from his last few years. As we will see, this is well connected with his earlier work, and can in many ways be seen as a culmination and completion of a life-long program.

To give the reader a better idea about the broad context of the central problem discussed in this section, we start with a rather general formulation. Given a smooth function $F : \mathbb{R}^n \rightarrow \mathbb{C}$ and a positive measure set $S \subset \mathbb{R}^n$, we define the Fourier

projection of F onto the space $L^2(S)$ as

$$\mathcal{P}_S F(x) = \int_S \widehat{F}(\xi) e(x \cdot \xi) d\xi.$$

Definition 6.1 (Abstract l^2 decoupling). Let $p \geq 2$ and let $\mathcal{S}$ be a finite family consisting of pairwise disjoint sets $S_i \subset \mathbb{R}^n$.

Let $\text{Dec}(\mathcal{S}, p)$ be the smallest constant for which the inequality

$$(6) \quad \|F\|_{L^p(\mathbb{R}^n)} \leq \text{Dec}(\mathcal{S}, p) \left(\sum_i \|\mathcal{P}_{S_i} F\|_{L^p(\mathbb{R}^n)}^2 \right)^{1/2}$$

holds for each $F : \mathbb{R}^n \rightarrow \mathbb{C}$ with Fourier transform supported on $\bigcup S_i$.

Such an inequality will be referred to as *decoupling*. While—due to purely L^2 orthogonality reasons—it is always the case that $\text{Dec}(\mathcal{S}, 2) = 1$, the main interest lies in understanding the decoupling constants $\text{Dec}(\mathcal{S}, p)$ for $p > 2$.

Since

$$F = \sum_i \mathcal{P}_{S_i} F,$$

the triangle inequality in L^p combined with Hölder's inequality leads to a first estimate

$$\|F\|_{L^p(\mathbb{R}^n)} \leq |\mathcal{S}|^{1/2} \left(\sum_i \|\mathcal{P}_{S_i} F\|_{L^p(\mathbb{R}^n)}^2 \right)^{1/2}.$$

This rather trivial inequality holds true for arbitrary collections of sets S_i , but one can do better for specific examples. Let us assume, for instance, that the “geometry” of the sets is such that it forces square-root cancellation

$$|F| \approx \left(\sum_i |\mathcal{P}_{S_i} F|^2 \right)^{1/2},$$

at least in some average sense. A classical example of this nature is provided by the Littlewood–Paley theory, where the sets S_i are dyadic boxes partitioning the frequency domain. If this is the case, we are led to the stronger inequality

$$\|F\|_{L^p(\mathbb{R}^n)} \lesssim \left(\sum_i \|\mathcal{P}_{S_i} F\|_{L^p(\mathbb{R}^n)}^2 \right)^{1/2}.$$

This may be viewed as a manifestation of “ L^p orthogonality”. If it holds for some $p = p_c > 2$, interpolation with $p = 2$ shows that it also holds in the range $2 \leq p \leq p_c$.

We now turn our attention to the main case of interest for us, when the sets S_i partition the thin neighborhood of a curved manifold. For most applications, it will suffice to show that the decoupling constants grow mildly with the cardinality of the family $\mathcal{S}$, rather than being independent of $|\mathcal{S}|$.

Problem 6.2 (l^2 decoupling for manifolds). Let $\mathcal{M}$ be a d -dimensional manifold in $\mathbb{R}^n$, parameterized by a smooth function $\psi : U(\subset \mathbb{R}^d) \rightarrow \mathbb{R}^{n-d}$,

$$\mathcal{M} = \mathcal{M}^\psi = \{(\xi, \psi(\xi)) : \xi \in U\}.$$

For each $\delta \in (0, 1)$, let $\Theta_{\mathcal{M}}(\delta)$ be a partition of the δ -neighborhood of $\mathcal{M}$ into almost rectangular boxes θ of thickness δ .

Find the range of $p \geq 2$ such that

$$(7) \quad \text{Dec}(\Theta_{\mathcal{M}}(\delta), p) \lesssim_\epsilon \delta^{-\epsilon} \text{ (as } \delta \rightarrow 0\text{)}.$$

To my knowledge, the first to consider a question in this vein was Thomas Wolff. In [36] he proved a decoupling for the cone in $\mathbb{R}^3$, and used it to derive a host of applications in geometric measure theory and PDEs, most notably some new local smoothing estimates for the wave equation.

A decade later, in [13], Bourgain used decoupling to introduce a new host of applications, to the theory of exponential sums. This can be achieved by specializing—via a limiting procedure—the test function F in (6) to a weighted sum of Dirac deltas. The principle can be summarized as follows.

Theorem 6.3 (l^2 decoupling implies square root cancelation for exponential sums). *Let $\mathcal{M}$ and $\Theta_{\mathcal{M}}(\delta)$ be as in Problem 6.2. For each $\theta \in \Theta_{\mathcal{M}}(\delta)$, let $\xi_{\theta} \in \mathcal{M} \cap \theta$.*

Assume that $\text{Dec}(\Theta_{\mathcal{M}}(\delta), p) \lesssim_{\epsilon} \delta^{-\epsilon}$. Then for each ball $B_R \subset \mathbb{R}^n$ with radius $R \geq \delta^{-1}$, we have

$$\left(\frac{1}{|B_R|} \int_{B_R} \left| \sum_{\theta} e(\xi_{\theta} \cdot x) \right|^p dx \right)^{\frac{1}{p}} \lesssim_{\epsilon} \delta^{-\epsilon} |\Theta_{\mathcal{M}}(\delta)|^{\frac{1}{2}}.$$

In the same paper, Bourgain made significant progress on the l^2 decoupling for hypersurfaces in $\mathbb{R}^n$ with positive principal curvatures, proving the sharp inequality in the partial range $2 \leq p \leq \frac{2n}{n-1}$. His argument takes advantage of the two major developments that happened in the field since the work of Wolff: the multilinear restriction inequality of Bennett, Carbery, and Tao and the Bourgain–Guth induction on scales.

Here is what I found most intriguing about [13] when I first read it. The methods were purely Fourier analytic, yet the main result was strong enough to easily imply progress on various problems that were previously attacked using a considerable infusion of number theory. Also, as if this was not outrageous enough, there was no serious indication that Bourgain’s approach could not be further refined to cover the full expected range $2 \leq p \leq \frac{2(n+1)}{n-1}$ of decoupling estimates. If this were indeed proved to be true, it would in particular lead to an unexpected, number theory–free solution to the question about Strichartz estimates for tori, discussed earlier in this note. Too good to be true, right?

About a year later, I started to toy with Jean’s decoupling, in order to improve his eigenfunction estimate for the Laplacian on tori (see the discussion in Section 3). This was a problem dear to Jean’s heart, and our roads have inevitably intersected. Our best efforts in [8] combined incidence geometry with the Siegel mass formula—two otherwise very sharp methods—to produce new results in four and five dimensions. On the other hand, it became apparent that a decoupling inequality for the sphere in the full range would be strong enough to surpass our results. This gave us an additional motivation to pursue the question of optimal decouplings for hypersurfaces.

Within one year, our efforts panned out: in [5] we came up with a complete solution to the problem for the sphere and the paraboloid in all dimensions. Our argument adds to the one in [13] a certain bootstrapping component, that takes advantage of the multilinear restriction inequality at many scales. In the same paper, we also solved the corresponding problem for the cone. The key observation here is that the cone can be locally approximated with parabolic cylinders, and the decoupling theory of these manifolds is identical to that of the paraboloids.

The impact of our paper [5] was two-fold. On the one hand, it completed the investigation on periodic Strichartz estimates initiated by Bourgain in [20]. We state the result for reader's convenience.

Theorem 6.4 (Strichartz estimates on tori). *Let u be the solution of the Schrödinger equation on $\mathbb{T}^{n-1}$, with initial data ϕ*

$$\begin{cases} 2\pi i u_t(x, t) = \Delta u(x, t), & (x, t) \in \mathbb{T}^{n-1} \times \mathbb{R} \\ u(x, 0) = \phi(x), & x \in \mathbb{T}^{n-1}. \end{cases}$$

Assume that $\text{supp}(\hat{\phi}) \subset [-N, N]^{n-1}$. Then for each $\epsilon > 0$ we have the following estimate (sharp, apart from N^ϵ losses)

$$(8) \quad \|u\|_{L^p(\mathbb{T}^{n-1} \times [0, 1])} \lesssim_\epsilon \begin{cases} N^\epsilon \|\phi\|_2, & 2 \leq p \leq \frac{2(n+1)}{n-1} \\ N^{\frac{n-1}{2} - \frac{n+1}{p} + \epsilon} \|\phi\|_2, & p \geq \frac{2(n+1)}{n-1}. \end{cases}$$

On the other hand, our paper initiated a long-term program of investigating the decoupling phenomenon for other manifolds, with the potential for new applications. Curves looked particularly appealing, as there is a plethora of conjectured exponential sum estimates associated with them. Jean's broad expertise came in handy again. In [3] he followed a strategy developed by Huxley, to set a new world record on the growth of the Riemann zeta on the critical line. The key new ingredient was a decoupling inequality for a certain curve in $\mathbb{R}^4$.

Curves with torsion proved to be more complex than hypersurfaces. Bourgain's aforementioned result in [3] was about decoupling the curve into arcs of a particular length. If one considers longer arcs, it turns out that the sharp range of estimates, the methods of proof, as well as the spectrum of applications will vary. In particular, the arcs of maximum length started to capture our attention. It soon became apparent that we have a strategy for attacking one of the most important problems in analytic number theory. After the preliminary work [6], we joined forces with Larry Guth in [7] and proved a sharp decoupling into "long arcs" for the moment curve

$$\{(t, t^2, \dots, t^n) : t \in [0, 1]\}.$$

Our proof uses the multilinear Kakeya inequalities not just for tubes, but for neighborhoods of affine spaces of arbitrary codimension.

Using the principle stated in Theorem 6.3, we derived the following result.

Corollary 6.5. *For $p \geq 2$ and $a_k \in \mathbb{C}$ we have*

$$\left\| \sum_{k=1}^N a_k e(x_1 k + \dots + x_n k^n) \right\|_{L^p([0, 1]^n)} \lesssim_\epsilon N^\epsilon \left(1 + N^{\frac{1}{2} - \frac{n(n+1)}{2p}}\right) \|a_k\|_{l^2}.$$

The case when all a_k are equal to 1 was known as the main conjecture in Vinogradov's mean value theorem. The case $n = 3$ had been proved earlier by T. Wooley using a number theoretical approach. I will not discuss any of the (many) consequences of this result, just an equivalent reformulation in the language of Diophantine systems.

For positive integers s, n, N , let $J_{s,n}(N)$ be the number of solutions $(k_1, \dots, k_{2s}) \in \{1, \dots, N\}^{2s}$ of the system

$$\begin{cases} k_1 + \dots + k_s = k_{s+1} + \dots + k_{2s} \\ (k_1)^2 + \dots + (k_s)^2 = (k_{s+1})^2 + \dots + (k_{2s})^2 \\ \dots\dots\dots \\ (k_1)^n + \dots + (k_s)^n = (k_{s+1})^n + \dots + (k_{2s})^n. \end{cases}$$

It is clear that there are $\sim N^s$ “trivial” solutions, those with $\{k_1, \dots, k_s\} = \{k_{s+1}, \dots, k_{2s}\}$. Corollary 6.5 shows in fact that if $1 \leq s \leq \frac{n(n+1)}{2}$, there aren’t “too many” nontrivial solutions. More precisely

$$J_{s,n}(N) \lesssim_\epsilon N^{s+\epsilon}.$$

I think it is fair to say that in his last few years, decoupling has become Jean’s favorite toy. He has written a significant number of papers, as well as a few short, more or less complete unpublished notes where he derives all sorts of consequences. His work in [9] and [10] with Watt is notable in this regard. They combine number theory with a refinement of the cone decoupling to improve the known estimates for the Gauss circle problem.

I will close this note with a quick reflection on the work described here. Of course, we all wish that Jean had lived longer and had proved many more wonderful theorems. At the same time, we need to acknowledge how much he has changed mathematics, in particular the circle of ideas related to restriction theory. With the theorems described in this last section, it can be argued that his life-long efforts have found some definitive form. Over the last three decades, he instinctively developed tools such as induction on scales and wave packet analysis, and applied them repeatedly to enrich the interplay between oscillatory problems and geometric measure theory. What could be more satisfactory at the end of one’s career, than to see this long quest culminate with the proof of fundamental results in so many areas of mathematics?

ABOUT THE AUTHOR

Ciprian Demeter is professor of mathematics at Indiana University, Bloomington. He specializes in harmonic analysis.

REFERENCES

- [1] R. C. Baker, G. Harman, and J. Pintz, *The difference between consecutive primes. II*, Proc. London Math. Soc. (3) **83** (2001), no. 3, 532–562, DOI 10.1112/plms/83.3.532. MR1851081
- [2] J. Bennett, A. Carbery, and T. Tao, *On the multilinear restriction and Kakeya conjectures*, Acta Math. **196** (2006), no. 2, 261–302, DOI 10.1007/s11511-006-0006-4. MR2275834
- [3] J. Bourgain, *Decoupling, exponential sums and the Riemann zeta function*, J. Amer. Math. Soc. **30** (2017), no. 1, 205–224, DOI 10.1090/jams/860. MR3556291
- [4] J. Bourgain, *A note on the Schrödinger maximal function*, J. Anal. Math. **130** (2016), 393–396, DOI 10.1007/s11854-016-0042-8. MR3574661
- [5] J. Bourgain and C. Demeter, *The proof of the l^2 decoupling conjecture*, Ann. of Math. (2) **182** (2015), no. 1, 351–389, DOI 10.4007/annals.2015.182.1.9. MR3374964
- [6] J. Bourgain and C. Demeter, *Mean value estimates for Weyl sums in two dimensions*, J. Lond. Math. Soc. (2) **94** (2016), no. 3, 814–838, DOI 10.1112/jlms/jdw063. MR3614930
- [7] J. Bourgain, C. Demeter, and L. Guth, *Proof of the main conjecture in Vinogradov’s mean value theorem for degrees higher than three*, Ann. of Math. (2) **184** (2016), no. 2, 633–682, DOI 10.4007/annals.2016.184.2.7. MR3548534

- [8] J. Bourgain and C. Demeter, *New bounds for the discrete Fourier restriction to the sphere in 4D and 5D*, Int. Math. Res. Not. IMRN **11** (2015), 3150–3184, DOI 10.1093/imrn/rnu036. MR3373047
- [9] J. Bourgain and N. Watt, *Decoupling for perturbed cones and the mean square of $|\zeta(\frac{1}{2} + it)|$* , Int. Math. Res. Not. IMRN **17** (2018), 5219–5296, DOI 10.1093/imrn/rnx009. MR3862121
- [10] J. Bourgain and N. Watt, *Mean square of zeta function, Gauss circle problem and divisor problem revisited*, arXiv:1709.04340 (2017).
- [11] J. Bourgain, *On oscillatory integral operators in higher dimensions*, Advances in analysis: the legacy of Elias M. Stein, Princeton Math. Ser., vol. 50, Princeton Univ. Press, Princeton, NJ, 2014, pp. 47–62. MR3329846
- [12] J. Bourgain, *On the Schrödinger maximal function in higher dimension*, Tr. Mat. Inst. Steklova **280** (2013), no. Ortogonal'nye Ryady, Teoriya Priblizhenii i Smezhnye Voprosy, 53–66, DOI 10.1134/s0081543813010045; English transl., Proc. Steklov Inst. Math. **280** (2013), no. 1, 46–60. MR3241836
- [13] J. Bourgain, *Moment inequalities for trigonometric polynomials with spectrum in curved hypersurfaces*, Israel J. Math. **193** (2013), no. 1, 441–458, DOI 10.1007/s11856-012-0077-1. MR3038558
- [14] J. Bourgain and L. Guth, *Bounds on oscillatory integral operators based on multilinear estimates*, Geom. Funct. Anal. **21** (2011), no. 6, 1239–1295, DOI 10.1007/s00039-011-0140-9. MR2860188
- [15] J. Bourgain, *Harmonic analysis and combinatorics: how much may they contribute to each other?*, Mathematics: frontiers and perspectives, Amer. Math. Soc., Providence, RI, 2000, pp. 13–32, DOI 10.1007/bf02791532. MR1754764
- [16] J. Bourgain, *On the dimension of Kakeya sets and related maximal inequalities*, Geom. Funct. Anal. **9** (1999), no. 2, 256–282, DOI 10.1007/s000390050087. MR1692486
- [17] J. Bourgain, *Analysis results and problems related to lattice points on surfaces*, Harmonic analysis and nonlinear differential equations (Riverside, CA, 1995), Contemp. Math., vol. 208, Amer. Math. Soc., Providence, RI, 1997, pp. 85–109, DOI 10.1090/conm/208/02736. MR1467003
- [18] J. Bourgain, *On the distribution of Dirichlet sums*, J. Anal. Math. **60** (1993), 21–32, DOI 10.1007/BF03341964. MR1253227
- [19] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II. The KdV-equation*, Geom. Funct. Anal. **3** (1993), no. 3, 209–262, DOI 10.1007/BF01895688. MR1215780
- [20] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations*, Geom. Funct. Anal. **3** (1993), no. 2, 107–156, DOI 10.1007/BF01896020. MR1209299
- [21] J. Bourgain, *Eigenfunction bounds for the Laplacian on the n -torus*, Internat. Math. Res. Notices **3** (1993), 61–66, DOI 10.1155/S1073792893000066. MR1208826
- [22] J. Bourgain, *L^p -estimates for oscillatory integrals in several variables*, Geom. Funct. Anal. **1** (1991), no. 4, 321–374, DOI 10.1007/BF01895639. MR1132294
- [23] J. Bourgain, *Besicovitch type maximal operators and applications to Fourier analysis*, Geom. Funct. Anal. **1** (1991), no. 2, 147–187, DOI 10.1007/BF01896376. MR1097257
- [24] J. Bourgain, *Bounded orthogonal systems and the $\Lambda(p)$ -set problem*, Acta Math. **162** (1989), no. 3-4, 227–245, DOI 10.1007/BF02392838. MR989397
- [25] X. Du and R. Zhang, *Sharp L^2 estimates of the Schrödinger maximal function in higher dimensions*, Ann. of Math. (2) **189** (2019), no. 3, 837–861, DOI 10.4007/annals.2019.189.3.4. MR3961084
- [26] X. Du, L. Guth, and X. Li, *A sharp Schrödinger maximal estimate in $\mathbb{R}^2$* , Ann. of Math. (2) **186** (2017), no. 2, 607–640, DOI 10.4007/annals.2017.186.2.5. MR3702674
- [27] B. Green and I. Z. Ruzsa, *On the arithmetic Kakeya conjecture of Katz and Tao*, Period. Math. Hungar. **78** (2019), no. 2, 135–151, DOI 10.1007/s10998-018-0270-z. MR3950735
- [28] L. Guth, J. Hickman, and M. Iliopoulou, *Sharp estimates for oscillatory integral operators via polynomial partitioning*, Acta Math. **223** (2019), no. 2, 251–376, DOI 10.4310/acta.2019.v223.n2.a2. MR4047925
- [29] L. Hörmander, *Oscillatory integrals and multipliers on FL^p* , Ark. Mat. **11** (1973), 1–11, DOI 10.1007/BF02388505. MR340924

- [30] N. H. Katz and T. Tao, *Bounds on arithmetic projections, and applications to the Keakeya conjecture*, Math. Res. Lett. **6** (1999), no. 5-6, 625–630, DOI 10.4310/MRL.1999.v6.n6.a3. MR1739220
- [31] C. Kenig, *On the work of Jean Bourgain in nonlinear dispersive equations*, Bull. Amer. Math. Soc., **58** (2021), no. 1 (to appear).
- [32] I. Laba, *From harmonic analysis to arithmetic combinatorics*, Bull. Amer. Math. Soc. (N.S.) **45** (2008), no. 1, 77–115, DOI 10.1090/S0273-0979-07-01189-5. MR2358378
- [33] C. D. Sogge, *Concerning the L^p norm of spectral clusters for second-order elliptic operators on compact manifolds*, J. Funct. Anal. **77** (1988), no. 1, 123–138, DOI 10.1016/0022-1236(88)90081-X. MR930395
- [34] T. Tao, *Exploring the toolkit of Jean Bourgain*, Bull. Amer. Math. Soc., **58** (2021), no. 1 (to appear).
- [35] T. H. Wolff, *Lectures on harmonic analysis*, University Lecture Series, vol. 29, American Mathematical Society, Providence, RI, 2003. With a foreword by Charles Fefferman and a preface by Izabella Laba; Edited by Laba and Carol Shubin. MR2003254
- [36] T. Wolff, *Local smoothing type estimates on L^p for large p* , Geom. Funct. Anal. **10** (2000), no. 5, 1237–1288, DOI 10.1007/PL00001652. MR1800068

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON INDIANA
Email address: demeterc@indiana.edu