

Mathematical theory of scattering resonances, by S. Dyatlov and M. Zworski, Graduate Studies in Mathematics, Vol. 200, American Mathematical Society, Providence, RI, 2019, xi+634 pp., ISBN 978-1-4704-4366-5

The behavior of waves on a compact Riemannian manifold is governed by the eigenvalues and eigenfunctions of the Laplacian. In this setting, a solution u of the wave equation has an expansion of the form

$$u(t, x) = \sum_{j=0}^{\infty} (a_j e^{it\lambda_j} + b_j e^{-it\lambda_j}) \phi_j(x),$$

where $-\lambda_j^2$ are the eigenvalues of the Laplacian with corresponding eigenfunctions ϕ_j , and the coefficients a_j and b_j are determined by the initial data. (The equality above always holds in the distributional sense, and it holds in stronger senses if the solution is more regular.) In other words, in the compact setting, the discrete data of the eigenvalues determine the frequencies at which waves can oscillate.

In the noncompact setting, however, the set of eigenvalues is typically not rich enough to describe the behavior of waves. *Scattering resonances* provide one replacement for this data; just as the eigenvalues of the Laplacian on a compact manifold can be viewed as poles of the resolvent $(-\Delta - E)^{-1}$, scattering resonances can be defined as poles of a meromorphic continuation of the resolvent.

As a concrete illustration, if V is a compactly supported potential on \mathbb{R}^3 , the resolvent $(-\Delta + V - E)^{-1}$ is a family of bounded operators on L^2 ; aside from possibly finitely many poles corresponding to negative eigenvalues, the family is holomorphic for $E \in \mathbb{C} \setminus [0, \infty)$. Passing to the double cover of $\mathbb{C} \setminus [0, \infty)$ corresponds to taking $E = \lambda^2$. The resolvent

$$R_V(\lambda) = (-\Delta + V - \lambda^2)^{-1}$$

defines a family of bounded operators on $L^2(\mathbb{R}^3)$ which is meromorphic in the upper half-plane. Considered as an operator from compactly supported L^2 functions to functions that are locally in L^2 , this family meromorphically continues to the entire complex plane. The scattering resonances are precisely the poles of this family in the lower half-plane.

The study of resonances is motivated by their ability to explain phenomena found in physics, quantum chemistry, and acoustics. One physical view of the above discussion is that we expect a closed system (such as one modeled by a compact Riemannian manifold) to be described entirely by the eigenvalues and eigenfunctions of an underlying Hamiltonian. In an open setting, such as the one described by potential scattering on \mathbb{R}^3 , the underlying Hamiltonian often has a purely continuous spectrum, and we do not expect to find positive energy eigenstates. On the other hand, one does expect to find states that persist for some time before tunneling to infinity: these are described by resonances and resonant states.

The analogy with eigenvalues is not limited to the discreteness of the resonances: solutions of the wave equation with potential on \mathbb{R}^3 enjoy a resonance wave expansion. More precisely, if u is an appropriately regular solution of

$$\partial_t^2 u - \Delta u + Vu = 0$$

on \mathbb{R}^3 and σ_j are the poles of the resolvent above (so including both resonances and possibly finitely many eigenvalues), then a contour-deformation argument shows that for each compact set $K \subset \mathbb{R}^3$ and constant C , there are functions a_j so that

$$u = \sum_{\operatorname{Im} \sigma_j > -A} e^{-i\sigma_j t} a_j(x) + r(t, x),$$

where r decays exponentially in t at the rate $\exp(-At)$. In this case we see that the real part of the resonance encodes a characteristic frequency of oscillation (as was the case with eigenvalues) and the imaginary part encodes a characteristic rate of decay. The resonances closest to the real axis are “physical” in that they decay slowly and so take longer to tunnel to infinity.

In addition to existence theorems and resonance wave expansions, the analogy with eigenvalues can be further fleshed out, though the statements on the resonance side are often considerably more difficult to establish. Two notable examples include resonance counting estimates (such as the upper bound [10] in analogy with the Weyl law for eigenvalues) and trace formulae (such as the one due to Melrose [8] in analogy with the Selberg trace formula). Both examples present new challenges in the setting of resonances. Unlike in the case of eigenvalues, tight lower bounds for the resonance counting function are known in only a few settings; similarly, the available trace formulae require considerably more machinery than their counterparts for eigenvalues.

A final aspect to the analogy between eigenvalues and resonances arises from the correspondence principle in classical mechanics, and it concerns the relationship between their distribution and the underlying Hamiltonian dynamics on the cotangent bundle (or, more precisely, the cosphere bundle). For resonances, the underlying classical phase space is noncompact, so the presence or absence of trapped trajectories (i.e., those trajectories that remain confined to a compact set) is a primary determinant of the distribution of resonances. We typically expect trapping to produce resonances near the real axis, while the absence of trapped trajectories should give resonance-free regions near \mathbb{R} . If the underlying classical system has no trapped trajectories, the resonance wave expansion above holds for all A , while if the trapped set is “thin” enough and the dynamics nearby are hyperbolic, the resonance wave expansions typically only hold only up to a threshold value determined by the surrounding dynamics [2].

The book [5] under review is a masterly exposition and the definitive work on this subject for the foreseeable future. It details the above themes in several contexts, including one-dimensional scattering theory (where more explicit formulas are available), compactly supported potential scattering in odd-dimensional Euclidean space, black box scattering in Euclidean space, and scattering on asymptotically hyperbolic spaces.

A particular strength is the book’s treatment of the meromorphic continuation of the resolvent on asymptotically hyperbolic spaces through propagation estimates. Originally due to Vasy [9], this approach has spurred a flurry of activity in general relativity [6, 7] and dynamical systems [3, 4]. The book provides an accessible treatment of this technique and application to other domains in its fifth chapter. This approach is distinct from the one taken in Borthwick’s book [1], and it provides a complementary perspective for those primarily interested in hyperbolic manifolds.

The book also provides a detailed account of the influence of trapping in the semiclassical limit, i.e., in the relationship between the asymptotic distribution of

resonances (as the real part of the resonance goes to infinity) and the behavior of the underlying classical dynamical system. There are extensive discussions of the presence and size of resonance-free regions (related to the rate of decay expected for solutions of wave equations) as well as of the effect that trapped trajectories have on the distribution of resonances.

The appendices of the book are another significant strength: the authors include sections on spectral theory, Fredholm theory, complex analysis, and semiclassical analysis to make the book essentially self-contained. A careful reading of these appendices provides a solid introduction to the basic techniques and results required for graduate students to begin research in the field. Of particular note is the appendix covering semiclassical analysis: in addition to serving as an excellent text for an introductory course in microlocal or semiclassical analysis, it includes an exposition of positive commutator estimates and propagation of singularities that is perhaps the clearest currently in the literature.

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