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Geometric relativity, by Dan A. Lee, Graduate Studies in Mathematics, Vol. 201, American Mathematical Society, Providence, RI, 2019, xii+361 pp., ISBN 978-1-4704-5081-6

1. When general relativity met geometric analysis

Let us begin with a brief historic account on how the theory of general relativity eventually met the branch of mathematics called geometric analysis.

Einstein's theory of general relativity is largely built upon a Lorentzian manifold, called *spacetime*. Despite its geometric framework, general relativity had been for a long time viewed as a branch of physics. Almost a half century later after Einstein published his first paper on general relativity, in 1973 a two-week American Mathematical Society conference on differential geometry held at Stanford University had included a session of general relativity. In this conference, the theoretical physicist R. Geroch and mathematicians J. Kazdan and F. Warner separately posted problems from seemingly very different motivations. One problem is a fundamental problem about positivity of energy/mass that is of interest to the physics community, while the other is motivated by a broader intellectual curiosity about connections between topology and metrics of positive scalar curvature. Nevertheless, both problems arrive at the same conjecture:

Geroch–Kazdan–Warner Conjecture. *A three-dimensional torus cannot carry a Riemannian metric of positive scalar curvature.*

In 1979 Richard Schoen and Shing-Tung Yau proved this conjecture [5], using ideas and innovative techniques interconnecting topology, minimal surface theory, and nonlinear partial differential equations. They made fundamental contributions and pioneered the discipline of mathematics called *geometric analysis*. Very soon they extended the arguments to the Riemannian positive mass theorem [6] and, with more new techniques, to the full positive energy conjecture [7].[1] Those ideas introduced by Schoen and Yau demonstrate amazing interconnections between analysis, geometry, topology, and physics. Since then, general relativity has been a strong driving force for the modern development of geometric analysis and itself eventually becomes a vibrant branch of mathematics.

2. A slice of spacetime

A spacetime $(\mathbf{N}, \mathbf{g})$ is a four-dimensional Lorentzian manifold, where the metric $\mathbf{g}$ is not positive definite but has one negative eigenvalue corresponding to the *time* direction. The *Einstein equation* describes an explicit relation between the spacetime curvatures and matter content:

$$\mathrm{Ric}(\mathbf{g}) - \tfrac{1}{2} R(\mathbf{g})\mathbf{g} = \text{matter model.}$$

[1] We refer the reader to the expository article by M. Eichmair and the author [2, pp. 1352-1355] on Schoen and Yau's proof of the Geroch–Kazdan–Warner conjecture and its connections to the positive mass theorems.

As simple as the Einstein equation may look at first glance, those "null" directions along which the metric $\mathbf{g}$ is degenerate lead to very distinct phenomena. Those directions are physically significant as they are the directions where light travels. The simplest example of spacetime is the Minkowski spacetime

$$\mathbf{g} = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2.$$

Minkowski spacetime is *flat* because all of its curvature tensors are zero. An isolated gravitational system is modeled by spacetime that asymptotically approaches Minkowski spacetime. We call such spacetime *asymptotically flat*.

An important physical conserved quantity of the spacetime called the *ADM mass* (named after physicists Arnowitt, Deser, and Misner) can be defined for an asymptotically flat spacetime and should measure the total mass content. The positive mass theorem, in loose terms, says that the ADM mass of a "physically reasonable" spacetime is always positive, except Minkowski spacetime which has zero ADM mass.

The degeneracy in the null directions causes difficulties in studying spacetime with great generality. A successful systematic approach is to study just one "slice" of the spacetime and to retrieve some useful spacetime information from the slice. Let M be a hypersurface in $(\mathbf{N}, \mathbf{g})$. As already accustomed to geometers since the 18th century, the geometry of the ambient space $(\mathbf{N}, \mathbf{g})$ can be largely read off from the induced metric g on M and the induced second fundamental form k. Such a triple (M, g, k) is called an *initial data set*; see Figure 1. Alternatively, from a

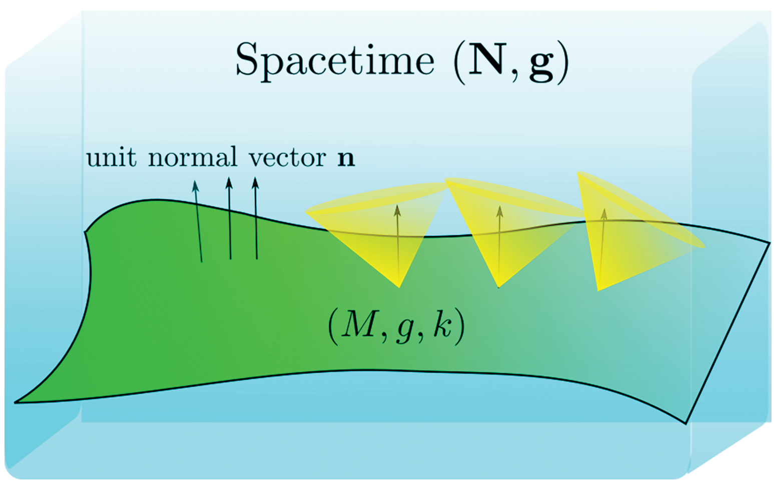

FIGURE 1. A null cone indicates the directions along which the spacetime metric $\mathbf{g}$ is zero or negative. An initial data set (M, g, k) is a triple of a hypersurface M, the induced Riemannian metric g (the tangential part of $\mathbf{g}$), and the induced second fundamental form k (the tangential part of the covariant derivative of the unit normal $\mathbf{n}$). An important special case, the so-called Riemannian case, assumes $k \equiv 0$, and in this case the dominated energy condition is reduced to the condition that the Riemannian manifold (M, g) has nonnegative scalar curvature.

Riemannian geometer's aspect, one can think that an initial data set (M, g, k) is just a Riemannian manifold (M, g) with added constraints on a symmetric 2-tensor k.

The advantage to working with (M, g, k) is that we can start to use a large "toolbox" from Riemannian geometry. A very important special case of $k \equiv 0$ is called the *Riemannian case*. In this case, the problems arising in general relativity can be described purely in terms of differential geometry. The *dominant energy condition* from physics becomes the geometric condition that (M, g) has nonnegative scalar curvature.

Our intellectual desire to understand the large-scale structure of the universe has motivated countless questions to be answered. These motivations from general relativity have been a strong driving force for the modern development in geometry and analysis.

3. Minimal surfaces and black holes

Perhaps the most intriguing topic in Riemannian geometry that appears (unexpectedly!) in general relativity is the topic of minimal surfaces. Minimal surface theory originated from the curiosity about soap bubbles—a household object that we surely love to play with. On the other hand, the theory of gravity is to describe large-scale, celestial objects, such as stars, galaxies, and black holes. Over the past few decades, remarkable progress has led to the astonishing realization that black holes are governed by the same mathematical principles that describe minimal surfaces. In fact, in some idealized situations, such as the Riemannian case defined earlier, the "boundary" of a black hole *is* modeled by a minimal surface.

Here we illustrate how the large toolbox of Riemannian geometry helps us to understand general relativity. We will discuss some of the essential ideas of Hawking's black hole topology theorem, which says that the two-dimensional cross-section of the boundary of a black hole must be topologically a 2-sphere. Those ideas also appear in Schoen and Yau's proof of the Geroch–Kazdan–Warner conjecture.

Let (M, g) be a three-dimensional Riemannian manifold. For a two-sided compact surface without boundary $\Sigma \subset M$, let ν be a unit normal vector on Σ. We define the *second fundamental form* A as the tangential part of the covariant derivative $\nabla \nu$ on Σ. The *mean curvature* of Σ, denoted H, is the scalar function on Σ obtained by taking the trace of A.

Let $\Phi_t : \Sigma \to M$ be a smooth one-parameter family of immersions with a parameter $t \in (-\epsilon, \epsilon)$. Suppose that $\Sigma = \Phi_0(\Sigma)$. By tracking the trajectory of each point of Σ with change of t, the velocity of Φ_t is expressed by the *deformation vector* X. Namely,

$$\left.\frac{\partial \Phi_t}{\partial t}\right|_{t=0} = X.$$

Suppose X is normal to Σ and we write $X = \eta\nu$. Denote by $\Sigma_t := \Phi_t(\Sigma)$. The areas of Σ_t, $\operatorname{area}(\Sigma_t)$, is a scalar function in t. Computing the first derivative of the area function gives

$$\left.\frac{d}{dt}\operatorname{area}(\Sigma_t)\right|_{t=0} = \int_\Sigma H\eta \, d\mu.$$

We say that Σ is a *minimal surface* if its mean curvature $H \equiv 0$; i.e., Σ is a critical point of the area function. If Σ is a minimal surface, computing the second derivative of the area functional gives

$$\frac{d^2}{dt^2}\text{area}(\Sigma_t)\bigg|_{t=0} = \int_\Sigma -\left[\Delta_\Sigma \eta + (\text{Ric}(\nu,\nu) + |A|^2)\eta\right]\eta\, d\mu,$$

where Δ_Σ is the Laplace–Beltrami operator on Σ and Ric is the Ricci curvature of g. A minimal surface Σ is said to be *stable* if the second derivative of the area function is nonnegative for all such deformation vectors X. Equivalently, we can say that a minimal surface Σ is stable if there is a real number $\lambda_1 \geq 0$ and a positive function η on Σ such that

$$L\eta := -\left[\Delta_\Sigma \eta + (\text{Ric}(\nu,\nu) + |A|^2)\eta\right] = \lambda_1 \eta,$$

that is, the first eigenvalue of the operator L is nonnegative.

Now, suppose that (M, g) has scalar curvature $R_g \geq 0$ and that Σ is a stable minimal surface. By letting $u = \log \eta$, we can rewrite the above equation $L\eta \geq 0$ as

$$\Delta_\Sigma u + |\nabla u|^2 + (\text{Ric}(\nu,\nu) + |A|^2) \leq 0.$$

The Ricci curvature term $\text{Ric}(\nu,\nu)$ relates the Gauss curvature of Σ, denoted R_Σ, by the Gauss equation, so we get $\text{Ric}(\nu,\nu) + |A|^2 = \frac{1}{2}(R_g - R_\Sigma + |A|^2)$, where we have also used $H = 0$. The above inequality then takes the following form:

$$\Delta_\Sigma u + |\nabla u|^2 + \tfrac{1}{2}(R_g - R_\Sigma + |A|^2) \leq 0.$$

We integrate the above inequality, invoke the Gauss–Bonnet theorem, and drop nonnegative terms to conclude that

$$0 \geq \int_\Sigma |\nabla u|^2 + \tfrac{1}{2}(R_g - R_\Sigma + |A|^2) \geq -2\pi\chi(\Sigma),$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ. This shows that the Euler characteristic of Σ is either negative or zero, which implies that topologically Σ is either a sphere or a torus. In fact, if Σ is a torus (i.e., the borderline case that $\chi(\Sigma) = 0$), we have strong consequences that $R_g, \text{Ric}(\nu,\nu), A$ must all vanish along Σ.

As it was mentioned earlier, this argument has originally appeared in Hawking's and in Schoen and Yau's work. With an extra argument by perturbing the surfaces in the spacetime, Hawking was able to exclude the borderline case and thus to conclude the black hole topology theorem. The ideas displayed in this argument have driven many more techniques and led to amazing applications in the study of minimal surfaces, scalar curvature, and the positive mass theorems.

4. The book

What we have described in the previous section is just a snapshot of many profound and intriguing ideas that interconnect several branches of mathematics and physics. The book *Geometric relativity* will guide the reader through a tour of the beautiful geometric, analytical, topological wonderland that was developed to answer the positive mass theorem and related questions. Dan Lee has made fundamental contributions in this research area; see, e.g., [1, 3, 4]. He is also a very skilled writer who did an exceptional job in this book of explaining intuitive concepts while providing the right amount of technical details.

The book consists of two main parts. In the first part, the book focuses on the Riemannian case. In this setting, most of the results can be formulated in terms of questions in Riemannian geometry. Those results motivated by general relativity also advance our understanding of connections between scalar curvature and minimal hypersurfaces. The author has selected a wide range of topics that include scalar curvature rigidity, the Riemannian positive mass theorem, the Riemannian Penrose inequality, and quasi-local masses of Bartnik and of Brown and York. It is a unique feature of this book that Schoen and Yau's minimal surface approach and Witten's spin approach to the Riemannian positive mass are both presented in detail.

In the second part, the book is aimed at discussing more recent developments on initial data sets, in particular the full positive mass theorem, also known as the *spacetime positive mass theorem*. Studying geometry of initial data sets is of physical significance as it would advance our understanding of spacetime geometry. On the other hand, this is a topic that very few textbooks in differential geometry have touched upon. The starting chapter of the second part, Chapter 7, provides a concise introduction to spacetime geometry, trapped surfaces, and the famous Penrose incompleteness theorem. After those, the reader will find detailed proofs to the positivity part of the spacetime positive mass theorem, including density theorems for initial data sets that are useful in other problems.

Despite the extended breadth of this book, several research topics that would fit well in the category of "geometric relativity" are not discussed in it. Some of those topics include the gluing construction of initial data sets, asymptotically hyperbolic manifolds, and the corresponding hyperbolic positive mass theorems.

It should also be noted that the book is not intended to be completely self-contained. The book may be used as a textbook for a graduate topic course after a semester course on differential geometry (even better with another semester course on partial differential equations), but proofs of several results presented in this book are only sketched, left as exercises with hints, or are not present at all in some cases. As the author has expressed in the introduction of the book "our goal is less to give a complete proof than to give the reader a guide for how to understand those proofs." The more advanced reader who is seeking more thorough and in-depth understanding can find the original research papers which are clearly referred in the book.

To my knowledge, this is the only textbook aimed at the graduate level that includes most current research topics in mathematical relativity on the spacetime positive mass theorem and marginally outer trapped surfaces. It may be also the only textbook that compares side-by-side both the minimal surface approach and the spinor approach to the positive mass theorems. The book is a useful resource for graduate students and researchers who want to enter this active research area that is continuing to thrive.

References

[1] Hubert L. Bray and Dan A. Lee, *On the Riemannian Penrose inequality in dimensions less than eight*, Duke Math. J. **148** (2009), no. 1, 81–106, DOI 10.1215/00127094-2009-020. MR2515101

[2] Hubert L. Bray, William P. Minicozzi II, Michael Eichmair, Lan-Hsuan Huang, Shing-Tung Yau, Karen Uhlenbeck, Rob Kusner, Fernando Codá Marques, Chikako Mese, and Ailana Fraser, *The mathematics of Richard Schoen*, Notices Amer. Math. Soc. **65** (2018), no. 11, 1349–1376. MR3822985

[3] Michael Eichmair, Lan-Hsuan Huang, Dan A. Lee, and Richard Schoen, *The spacetime positive mass theorem in dimensions less than eight*, J. Eur. Math. Soc. (JEMS) **18** (2016), no. 1, 83–121, DOI 10.4171/JEMS/584. MR3438380

[4] Lan-Hsuan Huang and Dan A. Lee, *Equality in the spacetime positive mass theorem*, Comm. Math. Phys. **376** (2020), no. 3, 2379–2407, DOI 10.1007/s00220-019-03619-w. MR4104553
[5] R. Schoen and Shing Tung Yau, *Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature*, Ann. of Math. (2) **110** (1979), no. 1, 127–142, DOI 10.2307/1971247. MR541332
[6] Richard Schoen and Shing Tung Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65** (1979), no. 1, 45–76. MR526976
[7] Richard Schoen and Shing Tung Yau, *Proof of the positive mass theorem. II*, Comm. Math. Phys. **79** (1981), no. 2, 231–260. MR612249

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