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SELECTED MATHEMATICAL REVIEWS

related to papers in the previous section on the work of
MICHAEL ATIYAH

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Atiyah, M. F.; Singer, I. M.

The index of elliptic operators on compact manifolds.

Bulletin of the American Mathematical Society **69** (1963), 422–433.

The authors here describe their solution to the index problem for elliptic operators on closed manifolds. Their result may also be thought of as a beautiful and far-reaching generalization of Hirzebruch's Riemann-Roch theorem—both in statement and in the spirit of the proof.

The authors formulate the index problem in the following general setting. Let E and F be vector-bundles over the compact manifold X, where everything is C^∞ throughout. It then makes sense to speak of a differential operator D from E to F, and such an operator induces a linear map $D\colon \Gamma(E) \to \Gamma(F)$ from the sections of E to those of F. When D is elliptic, both the kernel and the cokernel of D are finite-dimensional, and the difference of these dimensions is by definition the index of D. Alternately, one has the equality index $(D) = \sum(-1)^i \dim H^i(X;K)$, where K is the kernel sheaf of D, as follows from rudimentary sheaf theory, and the local "onto" property of elliptic operators: $H^0(X;K) \simeq \ker D$, $H^1(X;K) \simeq \operatorname{coker} D$, $H^i = 0$, $i \geq 2$.

The "index problem" is to give a description of this integer in terms of the topological data implicit in the elliptic operator. To describe these one needs the following interpretation of the highest-order terms in D. Let $T(X)$ be the cotangent bundle of X, and let $T_0(X)$ be the subset of non-zero vectors in $T(X)$. The projection $T(X) \to X$ will be denoted by π. With this understood, the highest-order terms of D are seen to define a definite homomorphism $\sigma(D)\colon \pi^*E \to \pi^*F$ of the pulled-back bundles on $T(X)$. Further, D is elliptic if and only if this homomorphism, called the symbol of D, is an isomorphism on $T_0(X)$. This isomorphism—or rather its stable homotopy class $[\sigma(D)]$—is to be thought of as the topological "twist" of the elliptic operator D.

One may relate $[\sigma(D)]$ in various ways to more standard topological objects. Maybe the simplest construction is the following one. Let $B(X)$ and $S(X)$, respectively, stand for the unit ball and unit sphere bundle of $T(X)$ endowed with some fixed Riemannian structure. Let $W(X) = B(X) \cup_{S(X)} B(X)$ be the manifold obtained by glueing two copies of $B(X)$ together along their boundary, i.e., $W(X)$ is the doubled manifold constructed from $B(X)$. Now one uses $\sigma(D)$ to construct a bundle $E \cup_{\sigma(D)} F$ on $W(X)$ by taking π^*E on one copy of $B(X)$, π^*F on the other, and glueing them together over $S(X)$ by means of the isomorphism $\sigma(D)$. (Alternatively and really equivalently, one may use $\sigma(D)$ to construct a difference element in $K(B(X), S(X))$ and that is the point of view taken in the paper under review.)

A formula quite equivalent to the index formula of the paper now is of the form $\operatorname{index}(D) = \int_{W(X)} \operatorname{ch}(E \cup_{\sigma(D)} F) \bigwedge \pi^*A(X)$. Here ch denotes the Chern character and $A(X)$ is a differential form on X which we will not specify here, but which is an explicit polynomial in the characteristic classes of X.

The index formula easily yields the results of the pioneers in this field such as Agranovič, Dynin, Gel′fand, Seeley, Vol′pert, etc. The formula is also seen to generalize the Riemann-Roch theorem of Hirzebruch to (not necessarily) algebraic complex manifolds and bundles. (To this one need only follow up our earlier expression for index (D) as an Euler characteristic.)

The proof the index formula, as well as its various consequences in special cases, is clearly summarized in the note. For the sake of analysts we remark only that the general setting of operators on vector-bundles—as opposed to systems—is essential for the proof.

R. Bott

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Atiyah, M. F.; Singer, I. M.

The index of elliptic operators. I.

Annals of Mathematics. Second Series **87** (1968), 484–530.

The earlier proof of the Atiyah-Singer index theorem as given in the book by R. Palais [*Seminar on the Atiyah-Singer index theorem*, Ann. of Math. Studies, No. 57, Princeton Univ. Press, Princeton, N.J., 1965; MR0198494] used cobordism theory and was in this respect modelled on the proof of the Riemann-Roch theorem due to the reviewer [*Neue topologische Methoden in der algebraischen Geometrie*, Ergeb. Math. Grenzgeb., Heft 9, Springer, Berlin, 1956; MR0082174; second edition, 1962; MR0137706; third edition in English, Springer, New York, 1966; MR0202713]. This proof did not lend itself to certain generalizations (for example, in the equivariant case) because the corresponding cobordism theories are not known. The index theorem of the paper under review includes the equivariant case: Suppose a compact Lie group G operates differentiably on a compact differentiable manifold X such that the action is compatible with a linear elliptic problem on X. Then the index of this elliptic problem is an element of the representation ring $R(G)$. In the simplest case the elliptic problem is given by an elliptic linear differential operator $D\colon C^\infty(E) \to C^\infty(F)$, where E, F are complex vector bundles over X. The group G operates on the kernel and on the cokernel of D. Thus we have two finite-dimensional representations of G. The difference of the two representations as elements of $R(G)$ is the index of the elliptic problem. The symbol of such an elliptic problem is an element of $K(BX, SX)$ if one forgets about the G-action. (For the notation, see the review of the book of Palais [loc. cit.].) If a G-action is given, then equivariant K-theory must be used and the symbol is an element of $K_G(BX, SX)$. The authors define K and K_G for a locally compact space as the corresponding reduced groups of the one-point-compactification of the space. With this understanding we have $K_G(BX, SX) = K_G(TX)$, where TX is the total space of the covariant tangent bundle. Using "enough operators" (pseudo-differential operators) and homotopy properties which ensure that the index depends only on the symbol, the authors define the analytical index a-ind: $K_G(TX) \to R(G)$. This is a homomorphism of $R(G)$-modules and has by its very definition the property that the index of an elliptic problem equals a-ind of its symbol. The definition of an $R(G)$-homomorphism t-ind: $K_G(TX) \to R(G)$ in topological terms is given in § 3. The "index theorem" is the main theorem 6.7 of the paper and asserts that a-ind and the topological index t-ind coincide. For the definition of t-ind and for the investigation of properties of a-ind and t-ind, the following construction is fundamental. Suppose X, Y are G-manifolds with $X \subset Y$ and X compact. Then $TX \subset TY$ and the normal bundle W of $TX \subset TY$ equals $N \oplus N$ lifted to TX, where N is the normal bundle of X in Y. Since $N \oplus N$ carries a complex structure, so also does W. The de Rham complex of the exterior powers of W can be tensored with any complex of vector bundles over TX lifted to W which has compact support in TX. The result is a complex of vector bundles over W with compact support, because the de Rham complex is canonically trivialized outside a zero-section of W. In this way we get a homomorphism $K_G(TX) \to K_G(W)$. Since W may be regarded as a tubular neighborhood of TX in TY and since the one-point-compactification of TY maps onto the one-point-compactification of W, we have a homomorphism $K_G(W) \to K_G(TY)$. The composition is an $R(G)$-homomorphism $i_!\colon K_G(TX) \to K_G(TY)$, where $i\colon X \to Y$ denotes the embedding. (Of course, many details are omitted in this review. Constructions using the alternating sum of the exterior powers occur in several earlier papers of Atiyah and others.)

To define t-ind for a compact differentiable G-manifold X we embed X in a real representation space E of G. This is always possible [R. Palais, J. Math. Mech. **6** (1957), 673–678; MR0092927]. Let i be the embedding. We have $i_!\colon K_G(TX) \to K_G(TE)$. Under the embedding j of the origin P in E we have $j_!\colon K_G(TP) = K_G(P) = R(G) \to K_G(TE)$. $j_!$ is an isomorphism. This is a special case of the equivariant form of the Bott periodicity theorem [the first author, *K-theory*, Benjamin, New York, 1967; MR0224083; Russian translation, Izdat. "Mir", Moscow, 1967; MR0224084; Quart. J. Math. Oxford Ser. (2) **19** (1968), 113–140; MR0228000].

t-ind is defined by $j_! \circ$ (t-ind) $= i_!$. The authors show that the definition is independent of the choice of the embedding. t-ind is the identity of $R(G)$ if (A1) X is a point. The diagram

(A2)
$$\begin{tikzcd} K^G(TX) \arrow[rr, "i_!"] \arrow[dr, "t-\mathrm{ind}"'] & & K_G(TY) \arrow[dl, "t-\mathrm{ind}"] \\ & R(G) & \end{tikzcd}$$

commutes for any inclusion $i\colon X \to Y$ with X, Y compact G-manifolds. An index function ind is given if we have for every compact differentiable manifold X an $R(G)$-homomorphism $K_G(TX) \overset{\mathrm{ind}}{\to} R(G)$. If such an index function ind satisfies (A1) and (A2), then ind = t-ind (Proposition 4.1). For the analytical index, (A1) is trivial. To prove the main theorem, axiom (A2) has to be checked for the analytical index. This is not easy because for an operator D on X with symbol $\gamma(D)$ we have to construct an operator on Y with symbol $i_!\gamma(D)$ and show that the index of this new operator equals the index of D. This construction is the essential analytical part of the proof of the main theorem. "Once this has been done, we can take Y to be a sphere, and the general index theorem is reduced to the case of operators on the sphere. For these the problem is easily solved." At this point one recognizes that the whole proof has "in spirit, at least" much in common with Grothendieck's proof of the Riemann-Roch theorem. Since it is difficult to verify (A2) directly, it is shown that certain axioms, (B1), (B2$''$) and (B3), imply (A2) for any index function. In § 8 the axioms (B1) and (B2$''$) are proved for the analytical index; in § 9 the axiom (B3) is proved. A special case of (B3) is the behaviour of the index function if one takes the Cartesian product of two G-manifolds with elliptic problems. The analytical index behaves multiplicatively in this case. (B3) generalizes this multiplicative property to differentiable fibre bundles. In this case the bundle of "indices along the fibres" may not be trivial over the base and enters essentially in the formulation (B1) is an excision axiom: Let U be a (non-compact) G-manifold and $j\colon U \to X$, $j'\colon U \to X'$ two open G-embeddings into compact G-manifolds X, X'. Then the following diagram commutes.

$$\begin{tikzcd}
& K_G(TX) \arrow[dr, "\mathrm{ind}"] & \\
K^G(TU) \arrow[ur, "j^*"] \arrow[dr, "j'^*"'] & & R(G) \\
& K_G(GX') \arrow[ur, "\mathrm{ind}"'] &
\end{tikzcd}$$

Observe that the one-point-compactifications of TX and TX' map onto the one-point compactification of TU. j^* and j'^* are induced by these maps.

The excision property of the analytical index was observed by R. T. Seeley [Trans. Amer. Math. Soc. **117** (1965), 167–204; MR0173174]. The axiom (B2$''$) is a normalisation axiom for certain operators on S^1 and S^2. Information on operators on other spheres follows by using excision and the multiplicative property. The idea of the proof for the essential property (A2) of the analytical index is sketched by the authors as follows (§ 1): let $i\colon X \to Y$ be an inclusion of compact manifolds (we forget the G-action). Let U be a tubular neighborhood of X and Z its double. Then the excision property (B1) shows that $\mathrm{ind}\, i_!A = \mathrm{ind}\, k_!A$, where $A \in K(TX)$ and $k\colon X \to Z$ is the inclusion in the double. Z is fibred over X by spheres. The multiplicative property and the information on spheres gives the desired result $\mathrm{ind}\, i_!A = \mathrm{ind}\, A$.

This paper contains an impressive amount of analysis. The theory of pseudo-differential operators (Hörmander, Kohn-Nirenberg, Seeley) is essential to have "enough operators" to realize all elements of $K_G(TX)$ as symbols and to carry through all constructions.

The analytical index was "calculated" in this paper by topological terms (topological index defined by K-theory). In the following Parts II [#5244] and III [#5245] this topological index will be interpreted in two steps. In Part II the topological index is expressed in terms of fixed-point sets of G. This is done in K-theory. It leads to a general "Lefschetz fixed-point theorem" where the fixed point set of an element of G is a disjoint union of submanifolds. For isolated fixed points this is contained in the fixed point theorem of the first author and R. Bott [Ann. of Math. (2) **86** (1967), 374–407; MR0212836], a theorem on differentiable maps g with isolated simple fixed points, where g need not be invertible. In Part III the result is reformulated in cohomological terms. If G is the identity, this gives the index theorem in its well-known cohomological form [R. Palais, loc. cit.]. In general, it gives the cohomological form of the fixed-point theorem.

F. Hirzebruch

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Atiyah, M. F.; Bott, R.

The Yang-Mills equations over Riemann surfaces.

Philosophical Transactions of the Royal Society of London. Series A. Mathematical and Physical Sciences **308** (1983), *no.* 1505, 523–615.

If P is a principal bundle with group G over a compact oriented Riemannian manifold M, the Yang-Mills functional L is defined on the space $\mathcal{A}$ of all connections in P, by the formula $L(A) = \|F(A)\|^2$, where $F(A)$ is the curvature form of A. The Euler-Lagrange equations of L are known as the Yang-Mills equations. A connection A such that $F(A)$ satisfies these equations (i.e. a critical point of L) is known as a Yang-Mills connection. The study of these equations, particularly when $\dim M = 4$, is an active area of current research in mathematical physics with which both authors have been closely associated. In this article the case where $\dim M = 2$ is examined in detail; although this is "trivial" from the physicists' point of view, it leads to some very interesting mathematics. The manifold M has the structure of a Riemann surface, so one expects the complex structure to play a role. Indeed, if $G = \mathrm{U}_n$, $\mathcal{A}$ can be identified with the space of holomorphic structures on the Hermitian vector bundle E associated to P (the $(0,1)$-part of the covariant derivative defines a $\bar{\partial}$-operator and hence an almost complex structure, which is integrable for dimensional reasons). Those holomorphic structures corresponding to the minimum points (i.e. stable extrema) of L turn out to be related in a simple way to the stable holomorphic structures of E (in the sense of Mumford). The theme of the paper is to apply Morse theory to L, in order to obtain the cohomology of the moduli space of such holomorphic structures. Unfortunately, a direct attack is fraught with difficulties. For example, the space $\mathcal{A}$ is contractible, so nontrivial results are achieved only if the action of the gauge group $\mathcal{G}$ (the group of automorphisms of P) is taken into account; L is equivariant with respect to this action. As $\mathcal{G}$ does not act freely on $\mathcal{A}$, it is necessary to consider the homotopy quotient $\mathcal{A}//\mathcal{G}$. The fact that $\mathcal{A}$ is infinite-dimensional gives rise to analytical problems with the Morse theory, and to avoid these the authors utilize a stratification of $\mathcal{A}$ occurring naturally in algebraic geometry, which behaves as a "gradient flow stratification" for L. A crucial property of this stratification is that the "lowest stratum" corresponding to the minimum points of L contains essentially all the information needed to construct the other strata; since the topology of the space $\mathcal{A}//\mathcal{G}$ is just that of the classifying space $B\mathcal{G}$ and is known, Morse theory now leads to an inductive formula for the cohomology of the lowest stratum and hence of the moduli space of stable holomorphic structures. The Betti numbers of the moduli space had been obtained recently by Desale and Ramanan by entirely different methods, using the Weil conjectures, but additional results are proved here concerning the multiplicative structure and the absence of torsion. It will be seen, therefore, that the paper begins with certain differential equations of mathematical physics, traverses areas of differential geometry, algebraic topology, and algebraic geometry, and finally connects up with important results in number theory. It goes without saying that this review can only be an inadequate sketch of a large project; recommended summaries are the authors' own readable and extensive introduction, and other articles [the authors, Geometry and analysis, 11–20, Indian Acad. Sci., Bangalore, 1980; MR0592249; Bott, The Chern symposium 1979 (Berkeley, Calif., 1979), 11–22, Springer, New York, 1980; MR0609555]. Some highlights of the paper will now be indicated.

The first two sections contain a review of Morse theory, introducing a version needed for the functional L on $\mathcal{A}//\mathcal{G}$. Sections 3 and 4 give a detailed treatment of the Yang-Mills functional and the Yang-Mills equations. The basic material concerning (Yang-Mills) connections in P and (stable) holomorphic structures of E is contained in Sections 5 to 8. If E admits a flat connection A, this is certainly a critical point of L, and it is induced from a connection in the standard flat bundle $\tilde{M} \to M$ by a unitary representation of $\pi_1 M$ (the holonomy representation). Generalizing, it is shown that for any Yang-Mills connection A, the 0-form $*F(A)$ is given essentially by a fixed element $X \in \mathfrak{u}_n$, and that the connection is induced via a projective unitary representation ρ of $\pi_1 M$. If X is in the centre of $\mathfrak{u}_n$, A is said to be central, and it gives a minimum point of L. A general Yang-Mills connection A breaks up into a sum of central connections in the summands of some direct sum decomposition of E, corresponding to the decomposition of ρ into irreducible summands. This is an important principle: the general critical point (for E) may be expressed in terms of minima (for subbundles of E). Those Yang-Mills connections for which the bundle decomposition of E is of "type μ" form a subset $\mathcal{N}_\mu \subseteq \mathcal{A}$.

These results are then interpreted in terms of holomorphic structures; $\mathcal{A}$ will now be identified with the space $\mathcal{C}$ of $\bar{\partial}$-operators. The group $\mathcal{G}^c$ of smooth automorphisms of E acts on $\mathcal{C}$, the orbits being equivalence classes of holomorphic structures ($\mathcal{G}^c$ may be identified with the complexification of the gauge group $\mathcal{G}$). Harder and Narasimhan associated to a holomorphic structure a canonical flag of holomorphic subbundles of E, so one may write $\mathcal{C} = \bigcup_\mu \mathcal{C}_\mu$, where $\mathcal{C}_\mu$ (a $\mathcal{G}^c$-equivariant subset) denotes those holomorphic structures for which the flag has type μ. Generically the flag is trivial (i.e. $\{0\} \subseteq E$). In this case E is semistable and the corresponding stratum is denoted $\mathcal{C}_{\mathrm{ss}}$. From the point of view of equivariant cohomology, $\mathcal{C}_\mu$ is equivalent to a product $\mathcal{C}_{\mathrm{ss}}^{(1)} \times \mathcal{C}_{\mathrm{ss}}^{(2)} \times \cdots \times \mathcal{C}_{\mathrm{ss}}^{(k)}$, where E has a decomposition $E = E_1 \oplus E_2 \oplus \cdots \oplus E_k$ of type μ and where $\mathcal{C}_{\mathrm{ss}}^{(i)}$ denotes the semistable holomorphic structures for E_i. The relation with the previous paragraph is provided by the result of Narasimhan and Seshadri that stable holomorphic structures are precisely those which arise from irreducible projective unitary representations of $\pi_1 M$ [see also S. K. Donaldson, Differential Geom. 18 (1983), no. 2, 269–277; MR0710055]. In the most favourable case, where the rank and degree of E are coprime, one has (a) the subset of stable holomorphic structures $\mathcal{C}_{\mathrm{s}}$ is equal to $\mathcal{C}_{\mathrm{ss}}$, (b) the representation of $\pi_1 M$ corresponding to $X \in \mathfrak{u}_n$ is irreducible if and only if X is central, and (c) $\mathcal{C}_{\mathrm{s}}/\mathcal{G}^c$ and $\mathcal{N}_{\mathrm{s}}/\mathcal{G}$ are isomorphic compact complex manifolds. Moreover, if $\mathcal{A}_\mu$ is the subset of $\mathcal{A}$ corresponding to $\mathcal{C}_\mu \subseteq \mathcal{C}$, then $\mathcal{N}_\mu$ is the subset of $\mathcal{A}_\mu$ on which L takes its minimum value. This suggests that the stratification $\mathcal{A} = \bigcup_\mu \mathcal{A}_\mu$ is the Morse stratification of $\mathcal{A}$ with respect to L. Rather than go into the analysis needed to prove this, the authors observe that to get cohomological information from a Morse function, one only needs to know the stratification. Thus, they suppress L from now on and work entirely with the stratification $\mathcal{C} = \bigcup_\mu \mathcal{C}_\mu$.

In Section 9 it is shown, using an idea first introduced by Bott and Samelson, that the stratification has the appropriate topological properties (it is "equivariantly perfect"), from which the inductive formula for the cohomology of the moduli space $\mathcal{C}_{\mathrm{s}}/\mathcal{G}^c$ (in the case where the rank and the degree of E are coprime) is obtained. The authors remark that the ensuing calculations turn out to be astronomical.

It is instructive, however, to consider the very simple case where $M = S^2$ and $G = \mathrm{SU}_n$ (then E has rank n and degree 0). Each $\mathcal{C}_\mu$ consists of a single $\mathcal{G}^c$-orbit, since the classification of holomorphic structures here is discrete: E splits as a sum of holomorphic line bundles. The critical set $\mathcal{N}_\mu$ consists of a single $\mathcal{G}$-orbit. The gauge group $\mathcal{G}$ has a normal subgroup $\mathcal{G}_0$ consisting of those automorphisms which are the identity over a fixed base point of M; $\mathcal{G}_0$ acts freely on $\mathcal{A}$ and $\mathcal{G}/\mathcal{G}_0 \cong G$ ($= \mathrm{SU}_n$ here). Dividing by this group rather than by $\mathcal{G}$, one finds that $\mathcal{A}/\mathcal{G}_0$ is topologically the loop space $\Omega\,\mathrm{SU}_n$, and each $\mathcal{N}_\mu/\mathcal{G}_0$ consists of an equivalence class of homomorphisms $S^1 \to \mathrm{SU}_n$ (i.e. a conjugacy class of geodesics in SU_n). This brings to mind the Morse-theoretic analysis of the energy functional on $\Omega\,\mathrm{SU}_n$ carried out by Bott and H. Samelson [Amer. J. Math. 80 (1958), 964–1029; MR0105694; correction; MR0170351], and indeed the index of an element of $\mathcal{N}_\mu/\mathcal{G}_0$, calculated from the second variation formula for L in Section 5, is shown to agree with the index of the corresponding geodesic.

Section 10 extends some of the results to groups G other than U_n, using the adjoint bundle of P instead of E. A special feature of the functional L on A is that the same critical points are obtained for any functional of the form $A \mapsto \int_M \varphi(*F(A))$, where φ is an invariant convex function on $\mathfrak{g}$ (L is given by $\varphi(x) = \mathrm{tr}(x^*x)$). In fact, this situation is an infinite-dimensional example of a phenomenon noted by Mumford and Sternberg for a reductive group G acting on a Kahler manifold X. To such an action one associates a moment map $X \to \mathfrak{g}^* \cong \mathfrak{g}$, which may then be composed with an invariant convex function on $\mathfrak{g}$. It turns out that there is a connection between stability of a critical point of such a function and stability of the point in the sense of algebraic geometry. These aspects have been studied further by Guillemin and Sternberg, and by Kirwan [see F. C. Kirwan , Cohomology of quotients in symplectic and algebraic geometry, Princeton Univ. Press, Princeton, N.J., 1984]. Section 11 compares the number-theoretic approach of Desale and Ramanan, and Harder and Narasimhan, with the differential geometric one developed here. The remaining Sections 12–15 contain technical results and background information referred to earlier in the article.

Martin A. Guest

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Hodge, W. V. D.; Atiyah, M. F.

Integrals of the second kind on an algebraic variety.

Annals of Mathematics. Second Series **62** (1955), 56–91.

This paper, whose main results were announced earlier [C.R. Acad. Sci. Paris **239** (1954), 1333–1335; MR0068869], is concerned with a study of integrals of the second kind on an algebraic variety, using as main tool the theory of stacks (or sheaves or faisceaux). Chapter I deals with simple and double integrals of the second kind on an algebraic variety of arbitrary dimension. The definition is expressed in terms of stacks, but, in the case of double integrals of the second kind, is shown to be equivalent to one of Picard-Lefschetz. The main theorem states that the number of independent 2-forms of the second kind is $R_2 - \varrho$, where R_2 is the second Betti number and ϱ is the Picard number, the maximum number of inequivalent divisors. Emphasis is laid on geometric interpretations of stack-theoretic results and on relations with classical theory.

Chapter II begins with a summary of the Leray-Cartan spectral theory and is devoted to the general theory of meromorphic q-forms. Let V denote a non-singular irreducible algebraic variety of dimension m over the complex field, and let W be an algebraic subvariety of dimension $m-1$. Denote by $\Omega^q({}^*W)$ the stack of germs of q-forms holomorphic in $V-W$ and having polar singularities on W. Let $\Omega({}^*W)=\sum_q \Omega^q({}^*W)$, and let $\Omega(*)$ be the direct limit of $\Omega({}^*W)$, as W runs over the subvarieties of V. Similarly, denote by $\tilde{\Omega}^q(W)$ the stack of germs of complex-valued q-forms of class C^∞ on $V-W$, and having arbitrary singularities on W, and define $\tilde{\Omega}(W)$, $\tilde{\Omega}(*)$ in an analogous manner. The authors view as their main problem the study of the relationship between the analytic stack $\tilde{\Omega}({}^*W)$ and the stack $\tilde{\Omega}(W)$, the latter depending on the geometry of V relative to its subvarieties. By supposing W to be ample and taking a fundamental stack $F=\sum_p F^p$, where F^p is the stack of germs of C^∞ p-forms, the authors introduce two spectral sequences ${E_r}^{p,q}(W)$ and ${\tilde{E}_r}{}^{p,q}(W)$. The term ${E_\infty}^{p},{}^{q}(W)$ gives a filtration of the group $G^{p+q}(W)$, the residue class group of all meromorphic $(p+q)$-forms on V with no singularities except those on W, modulo the derived forms. Similarly, ${\tilde{E}_\infty}{}^{p,q}(W)$ gives a filtration of $H^{p+q}(W)\cong H^{p+q}(V-W,C)$, the $(p+q)$-th cohomology group of $V-W$ over the complex field. If W is moreover simple, ${E_r}^{pq}(W)$ and ${\tilde{E}_r}{}^{pq}(W)$ are isomorphic for $r\geq 2$. From this one derives

$$\dim {G_q}^q(W)=\dim {E_{q+1}}^{q,0}(W)=\dim\left(w^*H^q(V,C)\right),\ q\geq 1,$$

where $w\colon V-W\to V$ is the injection map. The study of the general case will depend on the conjecture on the resolution of singularities of algebraic varieties by birational transformations. The authors derive several results based on this conjecture. These include an identification of the author's definition of forms of second kind and the definition in terms of residues. Finally, a filtration of $G^{q*}=\lim_W G^q(W)$ is suggested, which is birationally invariant and hence should lead to a birationally invariant spectral sequence.

S. Chern

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Atiyah, Michael; Maldacena, Juan; Vafa, Cumrun

An M-theory flop as a large N duality.

Journal of Mathematical Physics **42** (2001), *no.* 7, 3209–3220.

Dualities in string theory sometimes involve geometric transitions from a geometric background with D-branes (and therefore involving strings with boundaries), to a different background with no D-branes in which only closed strings propagate. An example of this duality is the AdS/CFT conjecture of Maldacena. Another example is the transition considered by R. Gopakumar and C. Vafa in the context of topological strings [Adv. Theor. Math. Phys. **3** (1999), no. 5, 1415–1443; MR1796682] and extended by Vafa to the full type II theory [J. Math. Phys. **42** (2001), no. 7, 2798–2817; MR1840317]. In this duality, one considers type IIA theory on the background $T^*S^3\times\mathbf{R}^4$ with N D6-branes wrapping $S^3\times\mathbf{R}^4$. After the geometric transition, the background is the resolved conifold with no D-branes, but with N units of flux through the S^2. The purpose of this influential paper is to understand this duality by lifting the whole construction to M-theory on a seven-manifold of G_2 holonomy.

The construction goes as follows. Consider the seven-manifold defined as a hypersurface in $\mathbf{C}^4$ by

$$(|z_1|^2 + |z_2|^2) - (|z_3|^2 + |z_4|^2) = V.$$

This is nothing but the spin bundle over S^3, with the topology of $\mathbf{R}^4 \times S^3$. Depending on the sign of V, the three-sphere is identified with the locus $z_3 = z_4 = 0$, or with the locus $z_1 = z_2 = 0$. This means that, as V changes sign and goes through $V = 0$, we have a flop transition similar to the familiar one in the Calabi-Yau case. In M-theory, the parameter V gets complexified (like the Kähler parameters in type II theory) and the true parameter is $V_M = V + iC$, where C is the flux of the 3-form field of M-theory through S^3. Physically, the theories before and after the flop are supposed to be equivalent. This equivalence is preserved if we mod the theory by a group action of g on the manifold, so we have schematically

$$Q_G[V_M] = Q_{G'}[-V_M]$$

where Q is M-theory on the corresponding background, the subscript G means that we modded out by the corresponding group, and G' is the group that acts on the flopped theory and that is obtained from G by exchanging z_1, z_2 with z_3, z_4.

This simple observation has deep consequences, as shown in this paper. First of all, for an appropriate choice of the action G which leads to an A_{N-1} singularity on the geometry, the left-hand side of the duality describes $\mathcal{N} = 1$ super Yang-Mills theory on $\mathbf{R}^4$ at low energies, with gauge group $\mathrm{SU}(N)$. On the other side of the duality, we obtain a theory with N vacua, and no singularity. This is precisely the description of the $\mathcal{N} = 1$ super Yang-Mills theory in the infrared, which in this language is purely geometric.

Furthermore, one can go to type IIA theory by choosing a circular eleventh dimension of the geometry. Doing this in the appropriate way, the left-hand side of the duality gives an $\mathbf{R}^3$ fibration of S^3 with a singularity at the origin. This is precisely type IIA theory on T^*S^3 with N D6-branes wrapping S^3. The right-hand side gives type IIA theory on the resolved conifold, with N units of flux. Therefore, the geometric transition in type IIA theory, which involves two very different backgrounds, turns out to be simply a flop in M-theory.

Another paper which discusses a similar scenario and obtains similar results is by B. S. Acharya ["On realising $N = 1$ super Yang-Mills in M-theory", preprint, http://arXiv.org/abs/hep-th/0011089]. Further issues raised by this paper have been explored in [M. Atiyah and E. Witten, "M-theory dynamics on a manifold of G_2 holonomy", preprint, http://arXiv.org/abs/hep-th/0107177; A. Brandhuber et al., Nuclear Phys. B **611** (2001), no. 1-3, 179–204; MR1857379].

Marcos Mariño

From MathSciNet, July 2021

MR0206940 (34 #6756) 55.30; 57.30

Atiyah, M. F.

K-theory and reality.

The Quarterly Journal of Mathematics. Oxford. Second Series **17** (1966), 367–386.

In this paper, the author introduces a new K-theory. Let X be a topological space X provided with a homeomorphism $\tau\colon X \to X$ such that $\tau^2 = 1$. The first example of such a space is the set of complex points on a real algebraic variety;

because of this analogy with algebraic geometry, the author calls such an X a "real space". The sort of vector-bundle which one has to consider over such an X is a complex vector-bundle E provided with a conjugate-linear self-map τ' over the map $\tau\colon X \to X$ and such that $(\tau')^2 = 1$. The author calls such an E a "real vector bundle". In terms of such bundles, one defines the Grothendieck group $KR(X)$; this is a ring.

The author's objective in introducing the ring $KR(X)$ is twofold. (i) It is needed for the theory of real elliptic operators; in fact, it is shown in § 5 that the symbol $\sigma(P)$ of a real elliptic operator lies in the group $KR(B(X), S(X))$, where $B(X)$ and $S(X)$ are appropriate ball and sphere bundles. (ii) It is advantageous for topology, since it leads to an understanding of the relation between different K-theories. To support this, the author first shows in § 2 how to modify an earlier proof [the author and R. Bott, Acta Math. **112** (1964), 229–247; MR0178470] so as to obtain a periodicity theorem for KR. Next, in § 3 he shows how this leads to an elegant proof of the periodicity theorem for KO-theory. On the way one encounters various K-theories already known, and exact sequences between them.

In more detail, let $R^{p,q}$ be R^{p+q} with an involution which reverses the first p coordinates and preserves the last q coordinates; let $B^{p,q}$ and $S^{p,q}$ be the unit ball and sphere in $R^{p,q}$. Using these for suspension instead of the usual sphere, one obtains bigraded groups

$$KR^{p,q}(X,Y) = KR^{\sim}(X \times B^{p,q}/X \times S^{p,q} \cup Y \times B^{p,q}).$$

The periodicity theorem shows that $KR^{p,q}(X,Y) = KR^{p+1,q+1}(X,Y)$, so one can write KR^{p-q} for $KR^{p,q}$. Next, one can consider KR-theory with coefficients in Y, given on X by $KR^*(X \times Y)$. It is proved that KR-theory with coefficients in $S^{p,0}$ has period 2 if $p = 1, 4$ if $p = 2$, and 8 if $p = 4$. If $p = 1$ the theory in question is ordinary complex K-theory. If $p = 2$ the theory in question is self-conjugate K-theory. From the case $p = 4$ the author deduces an isomorphism $KR(X) \cong KR^{-8}(X)$; if the involution τ on X is trivial, this reduces to the usual periodicity theorem for the real case. This requires a lemma, which is proved in § 4 with the aid of Clifford algebras.

{The reviewer is conscious that the paper contains points of interest not mentioned above; he pleads that this is a paper of 19 pages which cannot be summarised adequately in less than 20, and urges topologists to read it.}

J. F. Adams

From MathSciNet, July 2021

MR0167985 (29 #5250) 57.30; 55.30

Atiyah, M. F.; Bott, R.; Shapiro, A.

Clifford modules.

Topology. An International Journal of Mathematics **3** (1964), *no.* suppl, suppl. 1, 3–38.

According to the authors, "the purpose of the paper is to undertake a detailed investigation of the role of Clifford algebras and spinors in the KO-theory of real vector bundles. On the one hand the use of Clifford algebras throws considerable light on the periodicity theorem for the stable orthogonal group. On the other hand the use of spinors seems essential in some of the finer points of the KO-theory which centre round the Thom isomorphism".

Part I is entirely algebraic, and studies Clifford algebras and spinor groups. The material is essentially known, but its presentation is improved. In particular, the authors emphasise the grading (over Z_2) of the Clifford algebra. They can thus write formulae which involve signs given by the standard "anticommutative law" of algebraic topology; in this way the algebra becomes simpler and more natural. A feature of this approach is that the spinor group with two pathwise-components, which is a double covering of $\mathrm{O}(k)$, arises as naturally as the spinor group with one pathwise-component, which is a double covering of $\mathrm{SO}(k)$. § 4 gives the structure of the Clifford algebras, and §§ 5, 6 discuss their representation theory.

In Part II the authors give a complete treatment of the "difference bundle construction" in K-theory. This includes a Grothendieck-type construction for the relative groups $\mathrm{K}(X,Y)$ (9.1) and a discussion of the products in these groups (10.3, 10.4).

In Part III, §§ 11, 12, the authors set up the Thom isomorphism φ for real K-theory. Their construction of φ is clearly good; in particular, if φ is applied in a Whitney sum bundle, it satisfies a product formula. [Since it is one of the main objects of the paper to prove this formula, it is remarkable that the formula is not formally stated; the reader is left to deduce it from 11.3.] However, the authors seem less satisfied with their proof that φ is an isomorphism. In fact, the crucial step (11.5) amounts to case-by-case checking, using the results of R. Bott [Bull. Soc. Math. France **87** (1959), 293–310; MR0126281], that φ has the correct effect when the base-space is a point.

An alternative construction of the Thom isomorphism has been given in R. Bott [Bull. Amer. Math. Soc. **68** (1962), 395–400; MR0152995]. This approach is convenient for computing the effect of representations, but does not lead to the product formula. It is therefore desirable to prove that the two constructions coincide. This is done in §§ 13, 14, by studying the sphere as a coset space of the spinor group.

{The reviewer remarks that the work of D. W. Anderson on $\mathrm{KO}(BG)$ apparently yields a third approach to the Thom isomorphism for real K-theory.}

Having shown in the course of the work that Clifford algebras are related to the Bott periodicity theorem, the paper ends (§ 15) by showing that they are related to certain questions about vector bundles over stunted projective spaces and about vector-fields on spheres.

Although the main interest of the paper lies in real K-theory, it is a feature of the method that the real and complex cases can be treated in parallel throughout.

J. F. Adams

From MathSciNet, July 2021

MR0139181 (25 #2617) 57.30; 14.52

Atiyah, M. F.; Hirzebruch, F.

Vector bundles and homogeneous spaces.

Proc. Sympos. Pure Math., Vol. III, 7–38, *American Mathematical Society, Providence, R.I.*, 1961.

This paper summarizes some of the authors' researches on the K-theory, with special emphasis on the K-groups of a homogeneous space.

The authors start by defining the functors K^{-n} $(n \geq 0)$ on the category of pairs of finite CW complexes. This is done by setting $K^{-n}(X,Y)$ equal to the homotopy classes of basepoint-preserving maps of the nth suspension of X/Y into $\mathbf{Z} \times B_U$, where B_U is the universal base-space of the infinite unitary group. (When Y is vacuous, X/Y is defined as the disjoint union of X with a point * which plays the role of basepoint.) They then interpret the periodicity format $\Omega^2 B_U \simeq B_U$ as a canonical isomorphism $K^{-n}(X,Y) \xrightarrow{\simeq} K^{-(n+2)}(X,Y)$, and thereby extend the definition of K^n to all integers. Now they observe with the aid of the Puppe sequence that the resulting functors $\{K^n\}$ satisfy all the axioms of a cohomology theory save the dimension axiom. They also define a graded ring structure for the functor $K^{(n)}$ and are finally led to the "abbreviated" functor $(X,Y) \to K^*(X,Y) = K^0(X,Y) + K^{-1}(X,Y)$ from pairs (X,Y) to $\mathbf{Z}_2$-graded rings.

The study of this functor is now based on the following three of its properties: (1) If p is a point, then $K^n(p) = \mathbf{Z}$ (n even), $K^n(p) = 0$ (n odd). (This is a restatement of the corresponding formula for $\pi_n(\mathbf{Z} \times B_U)$.) (2) The usual Chern-character extends to give a natural transformation of cohomology theories $K^*(X,Y) \rightsquigarrow H^*(X,Y;\mathbf{Q})$ which is a ring homomorphism. (Here the salient fact is that the adjoint of the periodicity map, i.e., the map $j\colon S^2 \times B_U \to B_U$ takes the universal character $\underline{ch} \in H^{**}(B_U;\mathbf{Q})$ into $X \otimes \underline{ch}$ where X generates $H^2(S^2)$.) (3) There is a spectral sequence with E_2-term $H^*(X,Y)$ which converges to a graded group associated to $K^*(X,Y)$. Further, the differential operators in this sequence raise dimension by an odd number. (This theorem may be interpreted as the proper generalization of the Eilenberg-Steenrod uniqueness theorem; a spectral sequence of the type $H^*(X,Y;K^*(p)) \Rightarrow K^*(X,Y)$ exists whenever K^* satisfies all the axioms of Eilenberg-Steenrod, save possibly the dimension axiom.)

As an example of the power of this approach we cite the following immediate corollary of (3): If $H^*(X,\mathbf{Z})$ is free of torsion, then $K^*(X)$ is (unnaturally) isomorphic to $H^*(X)$, $\mathbf{Z}_2$-graded by the even and odd dimensional parts.

The more delicate results announced in this paper depend on the "differentiable Riemann-Roch theorem" of the author [Bull. Amer. Math. Soc. **65** (1959), 276–281; MR0110106]. With the aid of both these tools the authors are able to make considerable progress in their program to prove the K-analogues of the theorems about the ordinary cohomology of homogeneous spaces and the classifying spaces of compact Lie groups.

R. Bott

From MathSciNet, July 2021

MR0934202 (89k:53067) 53C80; 32L10, 53C05, 58E15, 58F07, 58G30, 81E13

Atiyah, Michael; Hitchin, Nigel

The geometry and dynamics of magnetic monopoles. (English)

M. B. Porter Lectures.

Princeton University Press, Princeton, NJ,, 1988, viii+134 pp. pp., \$25.00,
ISBN 0-691-08480-7

Let $A = \sum_{\mu=0}^{3} A_\mu dx_\mu$ be a connection one-form on a principal G-bundle ($G = \mathrm{SU}(2)$) over Minkowski space and φ a section of a vector bundle associated with the G-bundle by a representation, i.e., A_μ and φ are Lie algebra-valued functions on $\mathbf{R}^4$. The Yang-Mills-Higgs equations have the following form: (i) $D_A F = 0$; $D_A{*F} = -[\varphi, D_A\varphi]$; $D_A{*D_A\varphi} = 2\lambda\varphi(|\varphi|^2 - 1)$, where $F = dA + A \wedge A$, $D_A = d + A$, and λ is a constant.

The purpose of this book is to investigate soliton-like solutions of these equations, called magnetic monopoles. The finiteness of the energy gives the following behavior at infinity for solutions of (i): $|F| = O(r^{-2})$; (ii) $|D_A\varphi| = O(r^{-2})$, $|\varphi| = 1 - k/2r + O(r^{-2})$, where $r^2 = x_1^2 + x_2^2 + x_3^2$, k is the degree of the map $|\varphi|^{-1}\varphi\colon S_R^2 \to S^2$, where S_R^2 is a sphere of large radius R and S^2 is the unit sphere in the Lie algebra. The integer k is called the magnetic charge of the monopole solution.

The existence of static solutions of (i), (ii) with $\lambda \neq 0$ and $k = 1$ was first proved by 't Hooft and Polyakov in 1974. In 1975 Bogomol′nyĭ and Prasad-Sommerfield calculated the limit of the 't Hooft-Polyakov solution as $\lambda \to 0$ and noted that this limit not only satisfies (i) and (ii) with $\lambda = 0$ and $k = 1$, but also gives the minimum of the energy $\int_{\mathbf{R}^3}(F,F)+(D_A\varphi, D_A\varphi)$ for fields A, φ satisfying conditions (ii). The corresponding variational equations on $\mathbf{R}^3$ are the Bogomol′nyĭ equations $F = *D\varphi$. In 1977–78 Manton showed the existence of static solutions of the Bogomol′nyĭ equations with magnetic charge $k > 1$ (multimonopoles) and showed that Bogomol′nyĭ equations are equivalent to self-dual Yang-Mills equations $F = *F$ in Euclidean $\mathbf{R}^4$ when the connection A has the form $A = \varphi dx_0 + A_1 dx_1 + A_2 dx_2 + A_3 dx_3$. So the Penrose twistor theory (results of Atiyah, Ward, Hitchin, Drinfel′d, Manin) is applied to these equations. Between 1979 and 1985 all multimonopole static solutions of the Bogomol′nyĭ equations were found and described in a series of remarkable works (Weinberg, 1979, Jaffe, Taubes, 1980, Ward, 1980, 1981, Prasad, Rossi, 1981, Corrigan, Goddard, 1981, Forgács, Horváth, Palla, 1981, Hitchin, 1982, 1983, Nahm, 1982, Taubes, 1983, 1985, Hurtubise, 1983, 1985, Donaldson, 1984).

This book gives an excellent exposition of all the main methods for construction of k-monopoles. Let us consider the space $\mathcal{A}_k$ of all solutions (A, φ) of Bogomol′nyĭ equations with magnetic charge k. The group $\mathcal{G}$ of gauge transformations acts on $\mathcal{A}_k$ to produce the parameter space of all static k-monopoles $M_k = \mathcal{A}_k/\mathcal{G}$. It is a $(4k-1)$-dimensional manifold. A tangent vector $\dot{c}$ of M_k may be represented by a vector $\dot{A}, \dot{\varphi}$ in $\mathcal{A}$ which is orthogonal to the gauge group orbit through (A, φ), which means that $D_A^*\dot{A} + [\varphi, \dot{\varphi}] = 0$; then $h(\dot{c}, \dot{c}) = \int_{\mathbf{R}^3}(\dot{A}, \dot{A}) + (\dot{\varphi}, \dot{\varphi})$ defines a Riemannian metric on M_k. Manton (1982) has argued that the geodesic flow on M_k with respect to $h(\dot{c}, \dot{c})$ is the low-energy approximation to the true evolution of dynamic monopoles.

The book under review contains several deep results in the direction of the realisation of Manton's program.

Very briefly, the main contents of the book are the following: (1) The translation group of $\mathbf{R}^3$ acts freely on M_k and so one can introduce a reduced monopole space M_k^0 of dimension $4k-4$. The space M_k^0 has a Donaldson parametrization by rational scattering functions (Chapter 2). (2) The metric $h(\dot{c}, \dot{c})$ is finite and complete (Chapter 3). (3) The manifold M_k with this metric is hyper-Kähler (Chapter 4). (4) M_k with the hyper-Kähler metric is equivalent to its twistor space Z_k and this twistor space is found (Chapters 5, 6). (5) The space M_2^0 has an explicit description and this is used for finding the conformal structure of M_2^0. The metric on M_2^0 is an SO(3)-invariant anti-self-dual Einstein metric (Chapters 7, 8, 9). (6) An explicit form for the metric on M_2^0 can be found (Chapters 10, 11). (7) The geometry of geodesics in M_2^0 becomes understood; asymptotic expansions for the behavior of two monopoles near the "collision states" are found (Chapters 12, 13, 14).

The book also contains detailed background material, intriguing comparisons with the KdV equations and many interesting unsolved problems.

G. M. Khenkin

From MathSciNet, July 2021

MR0089473; 19,681d 53.3X

Bott, Raoul

Homogeneous vector bundles.

Annals of Mathematics. Second Series **66** (1957), 203–248.

Let M be a compact connected Lie group, G the complexification of M, U a closed complex subgroup of G such that $X = G/U = M/V$ (where $V = U \cap M$) is compact and simply-connected. Then G is a complex analytic principal U-bundle over X, and hence any complex analytic representation $\tilde{\varphi}$ of U on a vector space E induces a complex analytic vector bundle $\mathbf{E}$ on X. Let $\mathcal{S}\mathbf{E}$ denote the sheaf of germs of complex analytic cross-sections of $\mathbf{E}$, and let $H^*(X, \mathcal{S}\mathbf{E})$ denote the cohomology of X with coefficients in $\mathcal{S}\mathbf{E}$. Then G operates in a natural fashion on $H^*(X, \mathcal{S}\mathbf{E})$ giving a representation $\tilde{\Phi}$. The author's problem is to determine $\tilde{\Phi}$ explicitly in terms of $\tilde{\varphi}$. This is the analogue of a result of Frobenius on representations of finite groups.

Let φ, Φ denote the restrictions of $\tilde{\varphi}$, $\tilde{\Phi}$ to V, M respectively. Then it is sufficient to determine Φ. The author's main theorem (Th. I) asserts:

$$H^*(X, \mathcal{S}\mathbf{E}) = \sum W \otimes H^*(u, v, \mathrm{Hom}(W, E)),$$

where W ranges over all irreducible M-modules (representation spaces), u, v are the real Lie algebras of U, V respectively, and the terms on the right-hand side are relative cohomologies of Lie algebras. This theorem has a number of important corollaries, one of which is essentially a theorem of Borel-Weil (unpublished).

The spaces X defined above have been studied in detail by H. C. Wang [Amer. J. Math. **76** (1954), 1–32; MR0066011], who has shown that X is Kählerian if and only if its Euler characteristic is non-zero. Moreover, if X is Kählerian then it is actually a rational algebraic variety [A. Borel, Proc. Nat. Acad. Sci. U.S.A., **40** (1954), 1147–1151; MR0077878; M. Goto, Amer. J. Math. **76** (1954), 811–818; MR0066396]. In this case the author proves the following important theorem (Th. IV), which was conjectured by Borel and Hirzebruch: if φ is irreducible then Φ is irreducible or zero. In particular, therefore, $H^q(X, \mathcal{S}\mathbf{E}) \neq 0$ for at most one value of q. The particular value of q (if it exists) is explicitly computed from φ by a very simple formula involving the roots of M.

Th. I is proved by using chain-complexes of C^∞-differential forms on X with values in $\mathbf{E}$. The proof of Th. IV uses the following: (i) the Borel-Weil theorem, (ii) the Leray spectral sequence for fibre bundles, (iii) the analytic Künneth formula of Grothendieck [Mem. Amer. Math. Soc. no. **16** (1955); MR0075539], (iv) the theorem of Kodaira on the vanishing of certain higher dimensional cohomology groups [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 1268–1273; MR0066693]. Thus Th. IV depends on Th. I only so far as (i) is concerned, and so one could avoid using the cohomology of Lie algebras by relying instead on the essentially simpler proof of Borel-Weil.

M. F. Atiyah
From MathSciNet, July 2021

MR0766741 (86i:58050) 58F05; 14D25, 14L30
Kirwan, Frances Clare
Cohomology of quotients in symplectic and algebraic geometry. (English)
Mathematical Notes, 31.
Princeton University Press, *Princeton, NJ*, 1984, i+211 pp., \$17.50,
ISBN 0-691-08370-3

If X is a (nonsingular) projective algebraic variety acted on linearly by the complexification G of a compact Lie group K, the quotient X/G will not in general have good properties. One way to obtain a sensible quotient is to take the "symplectic reduction" $\mu_K^{-1}(0)/K$, where $\mu_K\colon X\to\mathfrak{k}^*$ is the moment map for the symplectic action of K. Another way is to consider the algebro-geometric quotient X/G, i.e. the variety associated to the subring of the coordinate ring of X which consists of G-invariant functions. Under reasonable conditions, these two quotients coincide: both are the ordinary quotient X^s/G, where X^s is a Zariski open subset of X (the set of "stable" elements). For example, if $G=\mathbf{C}^*$ acts by scalar multiplication on $X=\mathbf{C}^{n+1}$, one obtains a good quotient only by first deleting the origin in $\mathbf{C}^{n+1}$. The quotient is then $\mathbf{C}\mathrm{P}^n$, which is also the quotient of S^{2n+1} ($\leq\mathbf{C}^{n+1}$) by $K=S^1$. (This example is not projective, of course, but the theory still applies.) In this book the situation is explored in some detail, in order to calculate the cohomology of the quotient. Both the "symplectic" and "algebro-geometric" approaches are given, leading to very explicit formulae for the Betti numbers.

Part I, the symplectic approach, is based on an application of Morse theory to the real-valued function $\|\mu_K\|^2$ on X. The author has to extend the classical theory of Morse and Bott somewhat, and uses some recent ideas of Atiyah and Bott concerning equivariant Morse theory [M. F. Atiyah and R. Bott, Philos. Trans. Roy. Soc. London Ser. A **308** (1983), no. 1505, 523–615; MR0702806]. It is shown that the Morse inequalities hold, and that in fact they are equalities. Thus one has a relation between the cohomology of X and that of each of the stable manifolds S_β (where $X=\bigcup S_\beta$). Of primary interest is the stratum S_0 containing the set $\mu_K^{-1}(0)$ of minimum points. However, a significant feature of the decomposition is that each of the other strata is related to the stable manifold containing the minimum points of a function $\|\mu_H\|^2$ for some subgroup H of K. Thus it is possible to set up an inductive procedure to find the cohomology of $\mu_K^{-1}(0)/K$ (in terms of that of X and G). It is not necessary to assume that X is a projective variety; indeed the theory is developed here for any compact symplectic manifold acted on symplectically by K.

In Part II, the algebro-geometric approach, a stratification $\{S_\beta\}$ of X is defined purely algebraically, based on work of Kempf. It is then shown that this has the essential properties of the "Morse stratification" of Part I, so that one obtains the formulae for the cohomology of X^s/G. Whereas the previous approach is valid for a symplectic manifold, this approach generalizes to the case of a variety over any algebraically closed field. For such varieties, Betti numbers may be obtained via the Weyl conjectures, and this is discussed in Section 15.

The ramifications of this work extend well beyond the computation of Betti numbers. First, much inspiration was provided by Yang-Mills theory for connections in a principal bundle over a Riemann surface [Atiyah and Bott, op. cit.]. In that case, an infinite-dimensional Lie group acts symplectically on an infinite-dimensional manifold; the corresponding function $\|\mu\|^2$ is the Yang-Mills functional. The interaction between differential geometry and algebraic geometry in this example suggested that similar general principles might apply elsewhere. One general principle which features in Part I is the role of convexity in Lie groups. For example, an orbit M of the coadjoint action of a Lie group K has a natural symplectic structure, invariant under K, for which the moment map μ_K is just the given inclusion of M in the dual of the Lie algebra $\mathfrak{k}^*$. Any maximal toral subgroup T of K also acts symplectically, and the moment map μ_T is obtained by composing μ_K with the projection from $\mathfrak{k}^*$ to $\mathfrak{t}^*$. It is an old result of Kostant that $\mu_T(M)$ is a convex subset of $\mathfrak{t}^*$. This was extended recently to any symplectic action of a torus by Atiyah and by Guillemin and Sternberg, using Morse theory. The convex set is the hull of the common critical points for the functions $\mu_T^x\colon m \mapsto \mu_T(m)(x)$ as x varies in $\mathfrak{t}$. The Morse theory of the function μ_T^x is closely related to that of $\|\mu_K\|^2$, and in fact this convex set is used to define the indexing set for the stratification $\{S_\beta\}$ referred to above. Another important idea explored in the book is the connection between the previously unrelated concepts of stability in critical point theory (i.e. in dynamical systems) and stability in algebraic geometry: the minimum stratum S_0 turns out to be just the open set X^s. This was observed in the situation of the Yang-Mills equations on Riemann surfaces [Atiyah and Bott, op. cit.], where the minima (i.e. stable critical points of the Yang-Mills functional) correspond to holomorphic bundles over the Riemann surfaces which are stable (in the sense of Mumford).

The notions of convexity and stability were discussed independently by V. Guillemin and S. Sternberg [Invent. Math. **67** (1982), no. 3, 491–513; MR0664117; ibid. **77** (1984), no. 3, 533–546; MR0759258]. Finally we note that when the "reasonable conditions" of the first paragraph above are not satisfied, it is still possible to develop the algebraic approach. This has been done by the author [Ann. of Math. (2) **122** (1985), no. 1, 41–85].

Martin A. Guest

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