BOOK REVIEWS

The random matrix theory of the classical compact groups, by Elizabeth Meckes, Cambridge Tracts in Mathematics, Vol. 218, Cambridge University Press, Cambridge, 2019, xi+212 pp., ISBN 978-1-108-41952-9

1. RANDOM MATRICES: WHAT, WHY AND HOW

Consider one of the compact classical groups: the orthogonal, unitary, and symplectic groups ($O(n)$, $U(n)$ and $Sp(2n)$, respectively), as well as the subgroups $SO(n)$ and $SU(n)$. These come naturally equipped with an invariant Haar probability measure and a Lie group structure as submanifolds of Euclidean space. A random matrix from one of the classical groups is a random matrix distributed according to the corresponding Haar measure.

The modern random matrix theory (RMT) traces its origins to the fields of statistics (through the work of Wishart [16]) and of high energy physics (through the work of Wigner [15]). Random matrices also appeared, however implicitly, in Weyl's work on the representation theory of compact Lie groups, and in particular in his integration formula. As follow up of a famous encounter of Dyson and Montgomery, it also became clear that random matrices appear in the study of Riemann's zeta function, although, in spite of much progress, that connection still remains elusive.

Given a classical group (which, for the purpose of this review, will be taken to be $U(n)$ unless stated otherwise), there are several very natural questions. A sample follows.

(1) How does one sample a random matrix X_n?
(2) What is the joint distribution of the *eigenvalues* $\{e^{i\theta_k}\}_{k=1}^n$ of X_n? Define $L_n = n^{-1} \sum_{k=1}^n \delta_{\theta_k}$, the *empirical measure* of eigenvalues of X_n. Does L_n converge (as $n \to \infty$)? At what speed?
(3) Let $f : S^1 \to \mathbb{C}$ be a smooth function. What can be said about the fluctuations of $S_n(f) := \sum_{k=1}^n f(e^{i\theta_k})$, for n large? What about nonsmooth functions, such as $f_z(\theta) = \log|z - e^{i\theta}|$, $z \in S^1$?
(4) What is the limit (as $n \to \infty$) of the point process of eigenvalues, rescaled so that the expected spacing is 1?
(5) Take a $p \times p$ minor $X_{n,p}$ of X_n. What can be said about the joint distribution of the entries of $X_{n,p}$?

(In question (4) above, a *process* on S^1 is a random atomic measure; equivalently, it is a random configuration of points in S^1.)

There are many answers to the first question. (Meckes gives six, of which my favorite starts with a matrix of independent and identically distributed (i.i.d.) complex standard Gaussians, and applies the Gram-Schmidt algorithm. That one gets the Haar measure is a small miracle, and the algorithm can be adapted to other situations.)

The answer to the second question requires more work. First, Weyl's integration formula gives that the joint density of the eigenvalues is

$$(1.1) \qquad \frac{1}{n!(2\pi)^n} \prod_{1 \leq j < k \leq n} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{\ell=1}^{n} d\theta_\ell = \frac{1}{n!(2\pi)^n} |\Delta(\theta)|^2 \prod_{\ell=1}^{n} d\theta_\ell,$$

where $\Delta(\theta)$ is a Vandermonde determinant. Based on symmetry considerations, it is not hard to see that the expectation of L_n is the normalized Lebesgue measure $\mathfrak{m}$ on $[0, 2\pi]$. That L_n actually converges to $\mathfrak{m}$ (weakly, in probability) requires other tools, my favorite being the use of the theory of *concentration of measure*, which gives an exponential in n^2 rate of convergence. The precise constant in that rate is given by the theory of *large deviations*. Amazingly, Meckes manages to concisely develop both theories just enough so that both these statements can be proved.

The third question is where the miracles of the theory of random matrices really begin to show up. Indeed, if θ_k were i.i.d., then $\bar{S}_n(f) := S_n(f) - ES_n(f)$ would fluctuate at scale $\sqrt{n}$, and in fact $\bar{S}_n(f)/\sqrt{n}$ would converge in distribution to the Gaussian law. However, for random matrices, due to repulsion between eigenvalues, the variable $\bar{S}_n(f)$ itself, with no further rescaling, converges to a nondegenerate Gaussian (as long as f is continuously differentiable). One of the most beautiful proofs of that fact is based on the following result of Diaconis and Shahshahani. For a positive integers, consider the functions $f_a(z) = z^a$, and thus $S_n(f_a) = \text{Tr } X_n^a$. Then for $k < n/2a$, the $2k$-th moment of $|S_n(f_a)|$ coincides with the $2k$-th moment of (the modulus of) a complex Gaussian of variance k. Moreover, for different a's, the expectation of products of $|S_n(f_a)|^{2k_a}$s factors to products of moments, as long as the total degree is smaller than n. This beautiful fact (whose proof uses some representation theory of the compact Lie groups, which Meckes develops) can then be used, together with concentration of measure, to derive the central limit theorem (CLT) for $S_n(f)$, for general smooth test functions f. It can also be used to handle the characteristic polynomial of X_n, using the development of $\log|z - x|$ in power series, although this is much more delicate and beyond the scope of the book.

To answer the fourth question, one returns to (1.1) and notes that the joint density of eigenvalues can be written as a determinant

$$\det_{k,\ell=1}^{n} K_n(\theta_k, \theta_\ell),$$

for appropriate kernel K_n; this means that the process of eigenangles is a *determinantal* process with kernel K_n. One feature of such processes is that the number of points falling in an interval can be represented as a sum of independent Bernoulli variables, of parameters equal to the eigenvalues of the restriction of K_n to that interval. Another is that it is enough to understand the convergence of the kernel in order to deduce the convergence of the correlation functions of the associated point processes. Indeed, based on the theory of determinantal processes (which Meckes

develops, again precisely to the extent needed), the rescaled point process of eigen-angles is seen to converge to the sine process, i.e., the stationary determinantal process with kernel $\sin(x)/x$, the famous point process from the Dyson–Montgomery lore.

Finally, the fifth question requires more classical probabilistic analysis. The answer is that the joint law of the entries of $X_{n,p}$ is close, in the strong sense of total variation, to a product of (identical) Gaussians, if $p^2 = o(n)$, and differs asymptotically from such product (again, in variation distance) if $p^2/n > c$ for any fixed c. Related statements, due to Chatterjee and Meckes, hold for various projections of X_n. The proof goes through Stein's method, which is a method of deriving explicit bounds on the error in variation distance between a given distribution and the Gaussian law, and gives Meckes an excuse to introduce and develop yet another tool.

The last chapter of the book is devoted to a brief introduction to the relation with the Riemann zeta function ζ. Recall the Dyson–Montgomery conjecture that states that the empirical measure of (normalized) spacing between consecutive zeros of ζ should asymptotically coincide with the empirical measure of spacings between eigenvalues of X_n. Further, Selberg had proved a CLT for $\log \zeta(1/2 + it)/\sqrt{1/2 \log \log T}$, when t is sampled uniformly in $[T, 2T]$ and T is large, and Montgomery conjectured the form of moments of $|\zeta(1/2 + it)|$ under the same conditions. Keating and Snaith proved both the CLT and the moment conjecture, when $\zeta(1/2 + it)$ with t uniform in $[T, 2T]$ is replaced by the characteristic polynomial $\det(I - X_n)$ with X_n random from $\mathrm{U}(n)$ with $n = \log T$, and they used this to make various predictions concerning ζ. This turned out to be a rich area of research, currently very active (see comments below), and Meckes' chapter provides a proof of the Keating–Snaith results on the random matrices side, as well as a review of some of the conjectures and numerical work, and a partial guide to the rich literature concerning this topic.

2. FURTHER TOPICS AND PERSPECTIVE

It is maybe appropriate to put some of the previous discussion in perspective. The theory of random matrices, and in particular its part motivated by physics, developed the notion of matrix ensembles, parametrized by $\beta = 1, 2, 4$ and corresponding to ensemble of matrices invariant by conjugation with elements of $\mathrm{O}(n)$, $U(n)$, and $\mathrm{S}(n)$, respectively. In the joint distributions of eigenvalues of these ensembles, the Vandermonde determinant in (1.1) is raised to power β. In particular, the *circular orthogonal ensemble* (COE), corresponding to $\beta = 1$, does not lead a determinantal process, and the COE does *not* correspond to the law of eigenvalues of a random orthogonal matrix. The *circular unitary ensemble* (CUE) (corresponding to $\beta = 2$) does however coincide with the law of eigenvalues of a random matrix sampled from $U(n)$.

In fact, circular ensembles can be extended to any $\beta > 0$ by replacing in (1.1) the power of $|\Delta(\theta)|$ by β. These matrices possess a very nice link to the theory of orthogonal polynomials on the unit circle, which we will not discuss, referring instead to [12], [13] for a comprehensive introduction. In particular, by a change of basis that does not modify the spectrum, the CUE matrix can be put into a 5-diagonal form, and the randomness captured by the *Verblunsky coefficients* associated to the orthogonal polynomials corresponding to the spectral measure of

these matrices. That the Verblunsky coefficients for the CUE (and more generally, for the CβE) are independent was observed by Killip and Nenciu [9], exhibiting again a small miracle in RMT.

The logarithm of the characteristic polynomial appearing in question (3) and in Chapter 7 of Meckes' book, originally studied by Keating and Snaith, has led to rich and exciting recent developments, linked also to studies of the Riemann zeta function. Indeed, the process $W_n(z) = S_n(\log|z - \cdot|)$ possesses logarithmic (in n) variance, and in fact is (asymptotically) a Gaussian *logarithmically correlated field*. A conjecture of Fyodorov, Hiary, and Keating [6] predicts the fluctuation of its maximum and links it to the fluctuations of the maximum of the Riemann zeta function on short intervals. Both sides of this correspondence have seen rapid development in the last few years (see [2], [4] and references therein), although on both sides the conjecture is still open. It is interesting to note that on the RMT side, the representation in terms of Verblunsky coefficients discussed in the previous paragraph is crucial in the analysis of [4]. We mention in passing high order correlations results by Rudnick and Sarnak that apply also to more general L-functions, and a proof of (a modification of the) Dyson–Montgomery correspondence in the function field case by Katz and Sarnak [8].

Finally, a big chunk of modern random matrix theory deals with the topic of universality, that is the fact that many asymptotic quantities such as spacing distributions and laws of characteristic polynomials (as processes) do not depend much on the specific matrix model but rather only on the parameter β. Meckes' book does not touch upon this vast area (in particular, as noted above it deals exclusively with $\beta = 2$ and with a limited class of models). We recommend [5, 14] for modern perspectives on the universality questions.

3. Summary

Random matrix theory sits at the lucky intersection of many branches of mathematics, as the analysis of random matrices involves tools from algebra, combinatorics, spectral theory, analysis, probability, and operator algebras. There exist now several textbooks on random matrices, written from different perspectives [1, 3, 5, 7, 8, 11, 14]. The current book restricts its attention to random matrices from the classical compact groups, with emphasis on deriving the background material along the way, while not trying to be encyclopedic. The choice of material, which includes answers to the questions in Section 1 as well as chapters on geometric applications of RMT and on the link with the Riemann zeta function, reflects the author's interests and has a definite algebraic emphasis that differentiates it from other books. By necessity, many topics do not appear—besides the topics mentioned in Section 2, these include CLTs in mesoscopic scales, transport methods, links with free probability, and links with integrable probability. The book was written from a very personal perspective of an author who clearly likes the subject and shares her enthusiasm with the reader and who tells a story that is not well covered in book form in the existing literature. The writing is clear and flowing, and makes for a very accessible book that is a pleasure to read. My only minor quibble is that in the middle of some of the proofs, the proof interupts and a lemma or theorem from the literature is imported without proof; it may have been better to spell out upfront what would be imported in such way.

The book makes for a wonderful companion to a topics class on random matrices, and an instructor can easily use it either as a stand-alone text or as complementing other textbooks.

Elizabeth Meckes passed away recently at the young age of 40 [10]. This book is a tribute to her passion for mathematics and her ability to tell a compelling mathematical story.

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Ofer Zeitouni

Department of Mathematics
Weizmann Institute of Science
Rehovot, Israel;
Courant Institute
New York University
New York City, New York