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Extrinsic geometric flows, by Ben Andrews, Bennett Chow, Christine Guenther, and Mat Langford, Graduate Studies in Mathematics, Vol. 206, American Mathematical Society, Providence, RI, 2020, xxv+759 pp., ISBN 978-1-4704-5596-5

1. Extrinsic and intrinsic geometries

Because the title of the book may turn away people who are more interested in flows such as the Ricci flow, I would like to start this review with a little history about the intimate relation between so-called intrinsic and extrinsic geometry. The study of geometry goes back to an era that predated Euclid. The subject of differential geometry, after the invention of calculus, started in the hands of Euler, who pioneered the study of plane curves through definitions of arc-length and radius of curvature. Monge, Lagrange, Cauchy, Laplace, Liouville, Gauss, Bonnet, Minkowski, Liebmann, Hilbert, and many others worked to shape and spearhead the study of Riemannian geometry after the transformational *Habilitation* of Riemann. By the very nature and the evolutional history of geometry, intrinsic and extrinsic geometries of a space/manifold are intimately entangled. More recently, this has manifested in several important developments of geometry. For example in the earlier proof (cf. [29]) of Hopf's high dimensional generalization of Poincaré's index theorem, which identifies the sum of the indices at the nondegenerate singular points of a smooth vector field on a closed manifold with the Euler characteristic, one embeds the manifold into a high dimensional Euclidean space and utilizes the associated Gauss map. The first proof of high dimensional generalization of the Gauss–Bonnet theorem by Allendoerfer and Weil also made use of extrinsic geometry [2] before the intrinsic proof discovered later by Chern [14]. Before the celebrated theorem of Nash, which provides an isometric embedding of a Riemannian manifold as a submanifold in a Euclidean space, there have been a lot of developments by the school of A. D. Aleksandrov relating the intrinsic and extrinsic geometries concentrating on the Minkowski and Weyl problems (with more details below). An excellent account of these developments can be read in the Buffet style value (both monetary and mathematically) book of Busemann [12]. Such an intimate connection between the two geometries also exists in subjects such as complex algebraic geometry.

By proving an existence theorem for the Weyl problem, the study of convex geometry originating from Minkowski's work (e.g., proving the celebrated Aleksandrov–Fenchel inequality on the mixed volumes of convex bodies) gradually evolved into the study of the intrinsic geometry of convex surfaces. He proved that every metric of nonnegative curvature given on a two-dimensional sphere can be isometrically immersed into three-dimensional Euclidean space in a form of a closed convex surface [1]. The result also induces further questions:

(i) *Is this immersion unique up to rigid motion?*
(ii) *If the metric is given on the sphere, is it a regular one and of positive Gaussian curvature?*
(iii) *Is it true then that the surface with this metric is regular?*

A. V. Pogorelov, another distinguished geometer from the school of Aleksandrov, answered these questions by using synthetic geometric methods [33]. First, he developed geometric methods to obtain a priori estimates for solutions of Monge–Ampère equations. He then used these equations to solve geometric problems. Based on geometric reasons, he also constructed a generalized solution of a Monge–Ampère equation and then proved its regularity for the regular right-hand side of the equation. L. Nirenberg simultaneously published his celebrated work [31] on the Weyl and Minkowski problems approaching some of above mentioned questions from a more analytic/PDE perspective. The pioneering works of H. Lewy [27], L. Nirenberg [31], E. Calabi [13], A.V. Pogorelov [33, 34], as well as the celebrated work of J. Nash, laid the foundation of the field of geometric analysis through the establishment of fundamental a priori estimates and the study of the Monge–Ampère equation. The continuous influence of these works, as well as the gradient estimate technique dated at least back to Bernstein's work of the first two decades of 1900, can be seen in the later important contributions made by Agmon, Douglis, and Nirenberg; Aubin; Caffarelli, Nirenberg, and Spruck; Calabi; Chen, Donaldson, and Sun; Cheng and Yau; Cheeger; Donaldson; Evans; Gromov; Jost; Kodaira; Krylov and Safanov; Li and Yau; F. Lin; Morrey; Schoen; Simon; Siu; Taubes; Tian; Trudinger; Uhlenbeck; Yau; etc. The methods and ideas evolved further in the works of geometric flows pioneered and/or developed by Andrews; Angenent; Böhm and Wilking; Brakke; Brendle; Colding and Minicozzi; Daskalopoulos and Sesum; Ecker; Eells and Sampson; Evans and Spruck; Firey; Hamilton; Haslhoffer and Kleiner; Huisken; Ilmanen; Ladyzenskaja, Solonnikov, and Ural'ceva; Naber; Perelman; Struwe; X.-J. Wang; White; etc. (The lists, limited by author's ignorance, are not intended to be exhaustive nor do they include any development of geometric PDEs of hyperbolic/dispersive type. We apologize for any omissions.) In particular, the book under review focuses on the flow of hypersurfaces of the Euclidean space.

2. Geometric inequalities

Geometric flows study parabolic PDEs related to these geometric problems. Outside of the many various motivations and applications in applied sciences, such as image processing and reconstruction, there are additional mathematical motivations as well. The geometric inequalities, most of which are related to the field of convex geometry, are such motivations from geometry to study the flows instead of motivation from PDEs.

2.1. **Isoperimetric inequalities.**

The Dido problem, described in the famous *Aeneid* by Publius Vergilius Maro, asks us to maximize the land encircled with a bull's hide, a problem encountered by the exiled Phoenician princess. This perhaps was the first time the *isoperimetric problem* was recorded. The problem was then studied by mathematicians throughout history varying from the studies of the ancients—Aristotle, Archimedes, and Zenodorus—to more recent works by Steiner, Weierstrass, Schwarz, and many others. A detailed history can be found in Blaschke's book [8].

The mean curvature flow,

$$\frac{\partial \mathbf{X}(x,t)}{\partial t} = \mathbf{H}(x,t), \text{ where } \mathbf{H}(x,t) \text{ is the mean curvature of the embedding,} \tag{2.1}$$

originated to model systems such as cell, grain, and bubble growth. In the 1950s, von Neumann studied soap foams whose interface tends to have constant mean curvature, whereas Mullins' work describes coarsening in metals, whose interfaces are not generally of constant mean curvature. One motivation of the mean curvture flow is to obtain a dynamic approach towards a solution to the Plateau problem. However, there is a proof of the isoperimetric inequality in the plane by P. Topping [35] via the curve shorting flow, a very special case of the mean curvature flow. The isoperimetric ratio and its monotonicity play important roles in the study of related geometric flows. The first three chapters of the book under review cover more recent development of this idea by Andrews and P. Bryan [5].

2.2. **Convex geometry.** Many geometric inequalities arise in the modern study of geometry. The excellent book [11] does an amazing job in surveying the subject (of geometric inequalities) as its analogous precedents [24], [7] on inequalities in general. Convex geometry plays an essential role concerning many geometric inequalities.

One result of pivotal importance is the Aleksandrov–Fenchel inequality for convex bodies, which asserts that for bounded convex bodies $K_1, \ldots, K_n$ in $\mathbb{R}^n$, the *mixed volume* $V(K_1, \ldots, K_n)$ satisfies

$$V(K_1, \ldots, K_n) \geq V(K_1, K_1, K_3, \ldots, K_n) \cdot V(K_2, K_2, K_3, \ldots, K_n). \tag{2.2}$$

It is related to Brunn–Minkowski theory. Aleksandrov gave two proofs of it. One is via the method combinatorics and the other is via a PDE method. Later, a proof via the Hodge index theorem for complex surfaces was discovered via a theorem of Bernstein which connects the mixed volume with the number of roots of polynomials in n-complex variables.

In his PhD thesis, Andrews gave a proof via a geometric flow designed specifically for the Aleksandrov–Fenchel inequality. An exposition of this is included in Chapter 18 of the book under review. There is a generalization to nonconvex setting in [22].

There are many open problems concerning geometric inequalities of convex bodies, such as the famous 1945 problem of Aleksandrov conjecturing the sharp upper bound of the area by its diameter. For some problems the geometric flow could be a powerful tool to tackle them. There are also dynamic/PDE problems in convex geometry which concern the behavior of the flow itself.

W. Firey, a geometer who mainly studies convex bodies of Euclidean space, proposed several problems concerning convex bodies in $\mathbb{R}^3$ and $\mathbb{R}^n$. One of them is *Firey's conjecture* regarding the fate of the rolling (convex) stones, which was first resolved by Andrews [4] for $\mathbb{R}^3$.

Theorem 2.1 (Andrews)**.** *The fate* (*asymptotical shape*) *of a rolling convex stone in* $\mathbb{R}^3$ *is a round sphere.*

The book under review contains an exposition on this problem and a solution of its generalizations to higher dimensions (and some variations of the flow) in Chapters 15–17.

3. Geometric flows of hypersurfaces

One can study the flow of high codimension submanifolds even though that is not the topic of the book under review. The book focuses on the flow of a hypersurface in a Euclidean space. In fact its main focuses are mean curvature flow and Gauss curvature flow.

3.1. Mean curvature flow.

As mentioned previously, the mean curvature flow has been studied in material science to model things such as cell, grain, and bubble growth. Mullins may have been the first to write down the mean curvature flow equation in general when he tried to model the grain boundaries in annealing metals. Mullins also found some of the basic self-similar solutions, such as the translating solution now known as the "Grim Reaper".

3.1.1. *The formulation in terms of geometric measure theory.* In his 1978 PhD thesis under the supervision of Almgren, Brakke formulated the mean curvature flow via varifolds. An existence theorem (for all time) is proved for a surface moving by its mean curvature, via an approximation procedure that yields a one-parameter family of varifolds that satisfies the definition of motion by mean curvature. The excellent book [18] by Ecker is based on lectures of the author devoted to the regularity theory of Brakke from a more PDE-centric point of view.

3.1.2. *The PDE approach.* In [25], Huisken started the study of the flow by the mean curvature of a smooth convex hypersurface. The study and the method were inspired by Hamilton's celebrated work initiating the powerful Ricci flow. The main theorem is that any smooth compact convex hypersurface shall shrink to a point and as it shrinks it becomes more round, and namely, the rescaled hypersurface converges to a round sphere. The proof is done via establishing several a priori estimates. Unlike the work of Hamilton, which is mainly based on his estimates on the maximum principle, his work utilizes integral estimates via an iteration scheme of Stampacchia (some also credited to De Giorgi). These integral estimates also play an important role in the later convexity estimate of Huisken and Sinestrari. These have been covered in book form in [36].

There is an excellent survey article [17] by Colding, Minicozzi and Pedersen, where the reader can find some aspects of the subject missed by the book under review. In particular, the article covers the stability theorem and the uniqueness of the tangent flow result obtained by Colding and Minicozzi, and it provides information on possible topological applications of mean curvature flow.

3.2. Flows by a speed function which is of a homogenous degree 1 function of the principal curvatures.

Mean curvature flow is one of many flows of convex hypersurfaces by a speed function which is a homogenous degree 1 function of the principal curvatures (eigenvalues of the second fundamental form) satisfying certain structure conditions since $H = \frac{\lambda_1+\cdots+\lambda_n}{n}$. Such functions include the square root of the scalar curvature and nth root of the Gauss curvature (if the dimension of hypersurface is n). These were the subject of study of two papers of B. Chow adapting the techniques of [25]. In the PhD thesis of Andrews (part of it appeared in [3]), formulating the flow in terms of the support function, a great simplification was achieved on Huisken and Chow's results. Meanwhile, the method allows much broader choices of the speed functions. The study here was instead motivated mainly by a series of papers of Caffarelli, Nirenberg, and Spruck on fully nonlinear

PDEs. In fact, the structure conditions on the speed function were inspired by the above mentioned work as well. The main result proves the similar convergence to a sphere result for all flows of closed convex hyperfurfaces with speed function of homogenous degree 1 of the second fundamental form.

Besides the study of the flow of the hypersurfaces in the Euclidean space, the thesis of Andrews also includes many other very interesting developments. In fact, a very general differential Harnack estimate (not necessarily with degree 1 speed function, in fact including some anisotropic flows) was obtained, which includes the case of the mean curvature flow, the flow by the power of Gauss curvature and harmonic mean curvature. By using the support function, the proof is done with very few computations, unlike the previous works on the mean curvature flow and the Gauss curvature flow. This is another example of Sylvester's conclusion "... a general proposition should be easier to prove than any special case of it." Of course it is only possible after a new and more general method was invented. However such a more conceptual proof has not been found for the Ricci flow. In Andrews' thesis, a flow approach to the Aleksandrov–Fenchel inequality was also successfully carried out, and new entropy inequalities were obtained. Finally, by using a flow of a hypersurface inside a Riemannian manifold, a "dented" 1/4 pinching sphere theorem was also obtained.

3.3. **The level set approach.** The level set method is motivated by the mathematical study of phase transition. The related flow is called an interface controlled model. There are also motivations from crystal growth modeling and image processing. The level set method is to represent the hypersurface as the zero set of a smooth (at the least continuous) function. This certainly has advantages since the zero set of a smooth function can have singularities. Hence, the method naturally allows flowing through the singularities. The foundation of the level set method was laid down by Chen, Giga, and Goto and, independently, by Evans and Spruck; the numerical work of Osher and Sethian preceded both of these. It is also related to the work on first order equations by Evans and Souganidis. The level set method together with the concept of the viscosity solutions allows proper formulation of flow through singularities but with a uniqueness. This advantage over the varifold formulation of Brakke's is important for applications. It also allows the topological changes along the flow. For mean curvature flow, this allows much progress in the study of mean convex hypersurface flow. The flexibility in the formulation makes it possible to attack some geometric and topological problems. An application is to provide the first rigorous construction of the so-called type II singularities (namely the ones forming more slowly than predicted). The work of Evans and Spruck focuses on the mean curvature flow while Chen, Giga, and Goto also include more general cases. The book [21] gives good coverage of the development up to the date of its publication. The survey [17] also contains more recent developments in this direction.

3.4. **The Gauss curvature flow and the flow by a positive power of the Gauss curvature.** The Gauss curvature flow was used by Firey to model the process of tumbling stones on the seashore. The Gauss curvature of a n-dimensional hypersurface is a degree n polynomial of the principal curvatures. The success of the flow by the Gauss curvature remains mostly for the convex hypersurfaces. However, the method most often extends to the flow by some positive power of Gauss curvature. Varying the power, the equation becomes more degenerate as the power becomes bigger and more singular as the power gets smaller. Studying them together gives an example of nonlinear parabolic equations with varying degeneracy. If the hypersurface is n-dimensional, one particular power is $\frac{1}{n+2}$, since $K^{\frac{1}{n+2}}$ is invariant under affine transformations of determinant 1. This was applied by Alvarez, Guichard, Lions, and Morel to study image processing. A thorough study of this flow was done by Andrews. Another such special power is $\frac{1}{n}$, which was first studied by Chow. This special case is included in the later, much more general study of the homogeneous degree 1 case by Andrews mentioned above. The large time behavior of Gauss curvature flow and the flow by any power bigger than $\frac{1}{n+1}$ was studied in [23] and [6] with the classification of the limits, namely the shrinking solitons (which was finally achieved in [10]). The work of [23] and [6] relies on some geometric consideration in terms of entropy (also called dual quermassintegrals in convex geometry) of convex bodies and its monotonicity under the flow. The entropy functional of a convex body K is defined as

$$(3.1)\quad \mathcal{E}(K) = \sup_{x \in K^0} \mathcal{E}(K, x),$$
$$\text{with } \mathcal{E}(K,x) = \frac{1}{|\mathbb{S}^n|} \int_{\mathbb{S}^n} \log u_x(\theta)\, d\theta,\ K^0 \text{ being the interior of } K.$$

Here $u_x(\theta) = \sup_{z \in K} \langle \theta, z - x \rangle$ is the support function of K with respect to x. A special version of it has been studied before in the works of Firey and Andrews. It only involves the support function of the convex body, does not involve any derivatives of the support function, and is unlike most other entropy quantities in the study of geometric flow, such as Perelman's entropy. Hence, it plays a fundamental role in obtaining the needed C^0-estimates. The result owes itself to the study of convex geometry since the classical Blaschke–Santalo inequality supplies the nonnegativity of the entropy. Similar success was achieved for the mean curvature flow, particularly in the work of Huisken and of Colding and Minicozzi (cf. [17]). It would be a sensational result if a similar entropy quantity not involving the derivatives of the data (e.g., the metric for the Ricci flow) could be found for the Ricci flow.

By combining [23] and [10], the following high dimensional analogue of Andrews' result holds.

Theorem 3.1. *The fate* (*asymptotical shape*) *of the rolling convex stone in* $\mathbb{R}^{n+1}$ *is a round* $\mathbb{S}^n$.

4. The book

As a graduate textbook the book has a lot of content with 20 total chapters, much more than one could possibly cover in a single semester/quarter or even a year-long graduate course. The book provides some suggestions on a possible short course with focuses on various topics, such as Gauss curvature flow, curve shorting flow, etc. The most suitable moment to teach a course out of the book is after a year-long sequence of PDE and a quarter/semester course of differential geometry, particularly the theory of submanifolds. One could also do a topic course using the materials of the book on self-similar solutions (and ancient solutions if you are ready to dive into the subject), though with the risk that this topic could be a bit specialized for a broader audience. After several tries, an experienced instructor can also pick their own selections of topics from the book for a semester-long graduate course on the flows of hypersurfaces type. The book also serves as an excellent reference to the subject for graduate students and mathematicians interested in acquiring techniques and ideas about the flow of hypersurfaces. There are several related earlier books concerning the flow of hypersurfaces in Euclidean space such as [16, 18, 21, 36]. The book of [21] focuses on the level set approach, which is almost completely orthogonal to the book under review. There are tangential overlaps with this book [16, 18, 36]. The book [16] primarily concerns the flow of curves covering anisotropic flow, while Ecker's book [18] concerns the regularity theorem of Brakke for the weak solution of a mean curvature flow of a hypersurface. The book [36] concerns the convexity estimate and the cylindrical estimate of Huisken and Sinestrari for the mean curvature flow of mean convex hypersurfaces with techniques which predate some of the more recent developments. The book [36] however covers some applications in mathematical relativity which is outside the scope of the book under review. In most cases, when overlapping, the current book provides an updated approach to the theorems covered by the three books mentioned above. For example, Chapter 3 contains an alternate proof of Grayson's theorem developed by Bryan and Andrews. Chapters 9 and 12 contain streamlined and alternate proofs of the above mentioned convexity and cylindrical estimates built upon the noncollapsing estimate in terms of the inscribed radius of Andrews. The book also contains many developments after the publication of the above mentioned three books on the topics of solitons and ancient solutions with an extensive list of references. A theme emphasized by the book is the interplay between differential Harnack estimates and the monotonicity formulae of entropy like quantities. This focus was the topic of the earlier survey paper [30], which contained a substantial simplification of calculations in the proof of differential Harnack estimates, discovered by Andrews, for the family of hypersurface flows, including mean curvature flow, harmonic mean curvature flow, and the flow by positive powers of the Gauss curvature. The survey paper [30] also illustrates at the least three different interplays between the differential Harnack estimates and the monotonicity for Ricci flow, Kähler Ricci flow, Hamilton's principle applied to mean curvature flow, harmonic map flow, Yang–Mills flows, and many other situations including Perelman's entropy monotonicity.

The book contains an exposition in three chapters, Chapters 15–17, of the flow by the power of Gauss curvature. The Gauss curvature is incarnated into the Weyl and Minikowski problems. The study of Gauss curvature flow was motivated by the study of the convex geometry [19]. One can deduce results in convex geometry from the study of the flow as illustrated by the results of [15], [26], [6], etc. On the other hand the analysis of the nonlinear PDEs also directly influenced the study of the Gauss curvature-like flows. The entropy monotonicity involving only the C^0 data (mentioned previously) seems to be known so far only for the extrinsic flows, which play a crucial role in the proof of the convergence to a soliton part in the resolution to the Firey's conjecture. The influence of the convex geometry is also obvious. In the study of convex geometry, there are extended studies of Minkowski type problems [9, 28]. Further synthesis of the techniques from the convex geometry and nonlinear elliptic and parabolic PDEs can lead to resolutions of problems in both fields.

Though it is a bit presumptuous to say that the book enables readers to clear their shelves of many research papers on the subject of the hypersurface flows, it certainly makes the road less rocky (or a little more "royal") for those who want to grasp some key developments and ideas of the subject. The book also contains many open problems which are good sources for graduate students who want to get an idea of the current status of the field. Despite its length, since on many topics only the ideas and simple applications are illustrated in the book, the reader should still read the research papers of the subject for further developments in depth, original ideas, motivations, and backgrounds of the particular problem that she or he is interested in.

Given the level of content and its concentration on the specialized topic of hypersurface flows, it is unrealistic to expect that this book belongs to the category of "coffee table books" such as [20, 32] (which should be owned by any mathematical club of undergraduate students or average mathematicians and should be shared with friends, colleagues, and students—a gift for beginners and experts alike). Similar to *Mathematical omnibus* [20], the book also includes photos of various mathematicians contributing to the subject for the purpose of inspiration. Instead of the artistic illustrations lavished through [20], the mathematical presentation of the book under review contains many figures to convey the ideas and insights of otherwise "cold" estimates. The competence and enthusiasm of the authors certainly inspire optimism that the content and the presentation of the book shall prevail in the test of the time. I recommend it to any graduate student (mathematician) who has solid background in both differential geometry and partial differential equations, and who is serious about learning some exciting developments of the geometric flows of hypersurfaces in the last decade.

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