Invitation to partial differential equations, by Mikhail Shubin, Graduate Studies in Mathematics, Vol. 205, American Mathematical Society, Providence, RI, 2020, xvii+319 pp., ISBN 978-0-8218-3640-8; 978-1-4704-5697-9

1. Introduction

In the Preface of the book under review, Mikhail Aleksandrovich Shubin, the book's author, explains the importance and the central role of the theory of partial differential equations (PDEs):

> It will not be an overstatement to say that everything we see around us (as well as hear, touch, and so on) can be described by PDEs, or, in a slightly more restricted way, by equations of mathematical physics. Among the processes and phenomena whose adequate models are based on such equations, we encounter, for example, string vibrations, waves on the surface of water, sound, electromagnetic waves (in particular, light), propagation of heat and diffusion, gravitational attraction of planets and galaxies, behavior of electrons in atoms and molecules, and also the Big Bang that led to the creation of our Universe.
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> It is no wonder, therefore, that the theory of PDEs forms today a vast branch of mathematics and mathematical physics that uses methods of all the remaining parts of mathematics (from which the PDEs theory is inseparable). In turn, PDEs influence numerous parts of mathematics, physics, chemistry, biology, finance, computer science and other sciences.
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> A disappointing conclusion is that no book, to say nothing of a textbook, can be sufficiently complete in describing this area. The only possibility remains: to show the reader pictures at an exhibition — several typical methods and problems — chosen by the author's taste. In doing so, the author has to severely restrict the choice of material.

The book under review represents a remarkable effort in describing the theory of PDEs. It concentrates on the questions of existence and uniqueness of solutions of classical problems of mathematical physics and on the study of regularity and other qualitative properties of these solutions. This is carried out without neglecting to present concrete methods of solving or approximating such problems. As such, the book offers the reader an introduction to the subject and leads him or her all the way to a research level in this core area.

Mikhail Aleksandrovich Shubin, who passed away after a long illness on May 13, 2020, was an outstanding mathematician and a leading expert in the theory of partial and pseudo differential operators and in spectral theory. He has made extensive and deep contributions to the subject for many years. The book under review is based upon the lectures on PDEs that Shubin gave at Moscow State University and at Northeastern University. In fact, the author has considerably

extended his lecture notes. Later, after Shubin's death, the book has been edited by M. Braverman, R. McOwen, and P. Topalov.

The book is intended to serve as a concise textbook for an introductory course on PDEs. It is aimed at senior undergraduate and beginning graduate students, preferably with a basic background in real and functional analysis. Surprisingly, the book does not require previous knowledge of partial differential equations. As such, the book should be also of interest to experts in analysis, mathematical physicists, differential geometers, applied mathematicians, and theoretical scientists.

In fact, the book covers classical and modern topics in the theory of *linear* PDEs, offering a panoramic view ranging from abstract problems to concrete examples (unfortunately, the book does not deal with nonlinear equations). Although the main topics are more or less classical, the presentation and organization differ from those of most other books on the subject.

In the next two sections I discuss the content of the book with emphasis on the main original features of it with which I most heartily agree. Roughly speaking, the first seven chapters of the book are mainly devoted to the classical theory of PDEs, while the rest of the book presents the topics using the modern theory of linear PDEs.

2. THE BOOK'S FIRST PART

The first chapter begins with the fundamental definitions of characteristics, and of ellipticity and hyperbolicity. In addition, the author demonstrates how, by a change of variables, one transforms an operator to its canonical form. The second chapter is devoted to a classical treatment of the one-dimensional wave equation followed by an appendix on calculus of variations and the Euler–Lagrange equations. The topic of the theory of Sturm–Liouville operators occupies the third chapter, while in Chapters 4 and 5 the author introduces, in an elegant and concise way, distribution theory and the Fourier transform defined on an appropriate space of distributions. In particular, a classical Liouville theorem for entire solutions with polynomial growth of constant coefficients linear homogeneous PDEs and a removability theorem of isolated singularity are proved. Finally, the central notion of a fundamental solution is defined, and its properties are discussed in detail. Recall that a distribution $E(x, y)$ is called a fundamental solution of a linear differential operator operator $A(x, D)$ if E satisfies $A(x, D)E(x, y) = \delta(x - y)$, where δ is Dirac's δ-function supported at the origin. It turns out that one gets important qualitative information on solutions of the nonhomogeneous equation $Au = f$ if the fundamental solution of A is known or estimated.

Chapter 6 is devoted to the classical theory of harmonic functions including the existence and properties of the Dirichlet Green's function, and the classical example due to Hadamard illustrating the instability of the Cauchy problem for the Laplace equation. In Chapter 7 the author studies the heat equation using the Fourier transform for distributions and the existence of the heat equation's fundamental solution. The chapter culminates in a stabilization theorem and Tikhonov's uniqueness theorem concerning the corresponding Cauchy problem.

We conclude this part of our review with Shubin's elegant and clear proof of the aforementioned Liouville theorem to demonstrate the applicability and importance of distribution theory and the Fourier transform to PDEs. (Similar proofs can be found in other textbooks, for example, [1, 3].)

We will not explain the fundamental notion of a tempered distribution which appears in the theorem below, but we mention that a function $u \in C(\mathbb{R}^n)$ of polynomial growth (i.e., u satisfies $|u(x)| \leq C(1 + |x|)^N$ in $\mathbb{R}^n$ for some C and $N \in \mathbb{N}$) is a tempered distribution.

Liouville theorem in $\mathbb{R}^n$. *Suppose that $p(D) = \sum_{|\alpha| \leq m} a_\alpha D^{(\alpha)}$ is a constant coefficient linear differential operator in $\mathbb{R}^n$ satisfying $p(\xi) = 0$ for $\xi \in \mathbb{R}^n$ if and only if $\xi = 0$. If u is a tempered distribution in $\mathbb{R}^n$ satisfying $p(D)u = 0$ in $\mathbb{R}^n$, then u is a polynomial in $\mathbb{R}^n$.*

Proof. We may assume that $u \neq 0$. Denote by $\tilde{u}$ the Fourier transform of the tempered distribution u. In particular, $\tilde{u}$ is also a tempered distribution. Applying the Fourier transform on the equation $p(D)u = 0$, we get

$$p(\xi)\tilde{u}(\xi) = 0.$$

Since $p(\xi) \neq 0$ for $\xi \neq 0$, it follows that the support of the tempered distribution $\tilde{u}$ is equal to $\{0\}$. Indeed, if $\varphi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$, then $\varphi/p \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$, and consequently,

$$\langle \tilde{u}(\xi), \varphi(\xi) \rangle = \left\langle \tilde{u}(\xi), p(\xi) \frac{\varphi(\xi)}{p(\xi)} \right\rangle = \left\langle p(\xi)\tilde{u}(\xi), \frac{\varphi(\xi)}{p(\xi)} \right\rangle = 0.$$

Thus, $\operatorname{supp} \tilde{u} \subset \{0\}$. By the classical characterization of distributions supported at a point, it follows that

$$\tilde{u}(\xi) = \sum_{|\alpha| \leq N} b_\alpha \delta^{(\alpha)}(\xi), \qquad a_\alpha \in \mathbb{C},$$

where α is a multi-index and $\delta^{(\alpha)}$ is the α-partial distributional derivatives of Dirac's δ-function. Taking the inverse Fourier transform, we obtain

$$u(x) = \sum_{|\alpha| \leq N} c_\alpha x^\alpha, \qquad c_\alpha \in \mathbb{C},$$

as required. $\qquad\square$

Example. If u is an entire harmonic function in $\mathbb{R}^n$ and $|u(x)| \leq C(1 + |x|)^N$, then by the above Liouville theorem, u is a (harmonic) polynomial of degree less or equal to N. The same holds for entire solutions of the equation $\Delta^m u = 0$, where $m \in N$.

Remark. The above Liouville theorem can be extended to elliptic linear operators with periodic coefficients in $\mathbb{R}^n$ by replacing the Fourier transform by the Floquet transform; see [2] and references therein.

3. The book's second part

The second part of the book starts with Chapter 8 by introducing Sobolev spaces via the Fourier transform approach. In particular, the author proves the Sobolev embedding and trace theorems. These fundamental results lead to the unique solvability of the Dirichlet problem for the Laplacian by using the Dirichlet principle.

Chapter 9 is devoted to the spectral theory of the Dirichlet Laplacian in a bounded domain $\Omega \subset \mathbb{R}^n$. In particular, the author proves the analyticity of the Dirichlet eigenfunctions and the Glazman and Courant variational principles for the Dirichlet eigenvalues. In Chapter 10 the author returns to the wave equation, and

with the help of Hamiltonian formalism he obtains the uniqueness of the Cauchy problem. The existence theorem of this problem in $\mathbb{R}^3$ is not derived in the traditional way (i.e., with the aid of Darboux equations and spherical means), but by using the corresponding symplectic form which leads to the classical Kirchhoff formula and Huygens principle. The two dimensional case is then derived using the classical descent method.

In Chapter 11 the book discusses in detail jump properties of volume, single layer and double layer potentials of the Laplacian. Finally, Chapter 12 deals with the wave front sets and short wave asymptotics for hyperbolic equations. In particular, the author demonstrates that a surface of jumps for L^1-solutions of nonhomogeneous linear differential equations $Au = f$ with smooth coefficients and of degree m is a characteristic of A.

4. Appreciation

Let me now discuss the main features of this book. The book is reader friendly, the theorems are clearly stated, and their proofs are accurate and quite detailed. The book is also essentially self-contained, for the benefit of beginners. A number of interesting problems, typically at a moderate level, appear at the end of each chapter, some of which contain important and useful results. Moreover, answers or hints are given for all the problems at the end of the book. Attention is given to concise bibliographical and historical notes.

5. Conclusions

The book under review is very well written, and we should acknowledge the editors for their excellent editorial work. The book is an in-depth, up-to-date, modern, clear exposition of the advanced theory of PDEs. Moreover, the prerequisites assumed do not go much beyond a first course in analysis and functional analysis, and the proofs of the main theorems are entirely self-contained. Therefore, the book should be accessible to a large audience, including advanced undergraduate, graduate students, and researchers in the related fields of PDEs.

The book would have benefited from inclusion of several important subjects which are not covered. For instance, a comprehensive treatment of the Cauchy–Kovalevskaya theorem for the Cauchy problem with analytic initial conditions, Hans Lewy's example of a linear equation with no solutions, the method of characteristics for solving first-order PDEs, the Perron method for solving the Dirichlet problem for second-order elliptic equations, and the basic techniques of one-parameter semigroups for solving parabolic equations.

Regardless of these remarks, I strongly recommend this excellent book to every graduate student studying PDEs or related areas. Naturally, it will also be of interest to many scientists in related subjects.

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