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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
RICHARD SCHWARTZ

MR0809718 (87j:53053) 53C15; 32F25, 53C57, 57R15

Gromov, M.

Pseudo holomorphic curves in symplectic manifolds.

Inventiones Mathematicae **82** (1985), *no.* 2, 307–347.

The paper under review opens a new effective approach to fundamental problems of symplectic topology. Let (M, ω) be a symplectic manifold. An almost complex structure J on M is said by the author to be tamed by ω if $\omega(x, Jx) > 0$ for all nonzero tangent vectors. Almost complex structures tamed by the given symplectic form are sections of a fiber bundle with a contractible fiber. In particular they will always exist. The author's theory shows that manifolds with such structures have (like Kähler complex analytic manifolds) many globally defined (pseudo)holomorphic curves (or J-curves), which leads to many deep results in the geometry and the topology of contact and symplectic manifolds. The following theorems illustrate the character of numerous results of the paper. Let S^2 be the 2-sphere with the area form ω_1 with $\int_{S^2} \omega_1 = A_1$ and let V_2 be a closed manifold of dimension $2(n-1)$ with a symplectic form ω such that $\int_{S^2} \omega_2 = kA_1$ for every smoothly mapped sphere $S^2 \to V$ for some integer $k = k(S^2 \to V)$. Theorem (2.3.C): Let J be a C^∞-smooth almost complex structure on $V = S^2 \times V_2$ tamed by the symplectic form $\omega_1 \oplus \omega_2$. Then there exists a (possibly singular and nonunique) rational (i.e., diffeomorphic to S^2) J-curve $C = C_v \subset V$ which contains a given point $v \in V$ and which is homologous to the sphere $S^2 \times v_2 \subset V$, $v_2 \in V_2$. If $n = 2$ and V_2 is not diffeomorphic to S^2, or $k > 1$, then C is regular and unique. If V_2 is diffeomorphic to S^2 and $k = 1$ then there exists a connected regular J-curve C in V which represents the homology class $p[S^2] + q[V_2] \in H^2(V; \mathbf{Z})$ for arbitrary nonnegative integers p and q and which has genus$(C) = pq - p - q + 2$. In fact, these curves C form a smooth manifold $M = M_{pq}(J)$ of dimension $2(pq+p+q)$. Corollary (0.3.A): Consider a symplectic diffeomorphism of the open round ball $B(R) \subset \mathbf{R}^{2n}$ onto an open subset $V' \subset \mathbf{R}^{2n}$ which is contained in the ε-neighborhood of the symplectic subspace $\mathbf{R}^{2n-2} \subset \mathbf{R}^{2n}$. Then R satisfies the inequality $R \leq \varepsilon$.

The next result shows the uniqueness of symplectic structure on $\mathbf{R}^4$. Theorem (0.3.C): Let an open manifold (V, ω) be symplectically diffeomorphic to $(\mathbf{R}^4, \omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$ at infinity. If the Hurewicz homomorphism $\pi_2(V) \to H_2(V; \mathbf{R})$ vanishes, then (V, ω) is symplectically diffeomorphic to $(\mathbf{R}^4, \omega_0)$. Now consider $\mathbf{C}^n$ with standard complex and symplectic structures. Theorem (0.4.A_2): For an arbitrary closed C^∞ smooth Lagrange submanifold $W \subset \mathbf{C}^n$ there exists a nonconstant holomorphic map $f\ (D^2, \partial D^2) \to (\mathbf{C}^n, W)$. It follows that the relative class $[\omega_0] \in H^2(\mathbf{C}^n, W; \mathbf{R})$ is nonzero. Corollary (0.4.A_2'): There exists a symplectic structure ω on $\mathbf{R}^{2n}$ for all $n \geq 2$ which admits no symplectic embedding into $(\mathbf{R}^{2n} = \mathbf{C}^n, \omega_0)$.

The author successfully applies his theory to prove new fixed point theorems for exact symplectic diffeomorphisms and to get many deep results in contact topology and between them, e.g., D. Bennequin's theorem [*Troisième rencontre de géométrie* *du Schnepfenried, Vol. 1* (Schnepfenried, 1982), 87–161, Astérisque, 107-108, Soc. Math. France, 1983; MR0753131] and some of its higher-dimensional analogues. One of the main tools of the theory is the compactness theorem for the space of pseudoholomorphic curves. The author introduces the notion of weak convergence of pseudoholomorphic curves to a “cusp-curve” and proves the following theorem (1.5.B): Let V be a compact manifold with almost complex structure J and Riemannian metric μ. Let C_j be a sequence of closed J-curves of fixed genus in (V, J, μ). If the areas $\mathrm{Area}_\mu C_j$ are uniformly bounded then some subsequence weakly converges to a cusp-curve $\overline{C}$ in V.

Compare 1.5.B with Bishop’s compactness theorem for analytic sets [see E. Bishop, Michigan Math. J. **11** (1964), 289–304; MR0168801].

Yakov Eliashberg

From MathSciNet, October 2021

MR0840401 (87m:53003) 53A04; 35K05, 52A40, 58E99, 58G11

Gage, M.; Hamilton, R. S.

The heat equation shrinking convex plane curves.

Journal of Differential Geometry **23** (1986), *no.* 1, 69–96.

The authors give a complete proof of the “shrinking conjecture” for convex plane curves. The “shrinking process” refers to the generation of a family of smooth closed curves by flowing at each point in the direction of the curvature vector k, and at a rate equal to the magnitude of k. This process may be represented by the parabolic differential equation $\partial X/\partial t = \partial^2 X/\partial s^2$ for the position vector $X(s, t)$, where for each fixed t, the parameter s represents arc length along the corresponding curve, so that $\partial^2 X/\partial s^2$ is the curvature vector. The conjecture is that starting with a smooth Jordan curve, this process defines a one-parameter family of smooth Jordan curves C_t and that the curves C_t tend to a circle. Here it is proved that if the initial curve is convex then the curves C_t are all smooth convex curves which converge to a point and become circular in the sense that (a) the ratio of the inscribed radius to the circumscribed radius approaches 1, (b) the ratio of the maximum to the minimum curvature tends to 1, and (c) all derivatives of the curvature tend uniformly to zero.

The latter part of the theorem (the evolving circularity of the curves C_t) depends crucially on two earlier papers of Gage [Duke Math. J. **50** (1983), no. 4, 1225–1229; MR0726325; Invent. Math. **76** (1984), no. 2, 357–364; MR0742856], while the former part—the fact that the process may be continued without developing singularities—uses earlier methods and results of Hamilton [J. Differential Geom. **17** (1982), no. 2, 255–306; MR0664497]. Many of the results in the paper are proved in a more general setting, such as the local existence theorem for solutions, and the fact that if the curvature remains bounded, then all the curves evolving out of a Jordan curve will remain Jordan curves.

Reviewer’s remark: Matthew Grayson, in a preprint, has given an argument to show that an initially Jordan curve must evolve into a convex curve and hence eventually tend to a circle. Thus the general conjecture is settled by reducing it to the case studied here.

R. Osserman

From MathSciNet, October 2021

MR1049697 (91k:58042) 58F05; 53C15, 57R50, 58C10
McDuff, Dusa
The structure of rational and ruled symplectic 4-manifolds.
Journal of the American Mathematical Society **3** (1990), *no.* 3, 679–712.

In this important paper the following classification result is proved. Let (V,ω) be a compact symplectic 4-manifold which contains a symplectically embedded copy C of S^2 with nonnegative self-intersection number. Then: (i) By blowing down a finite number of disjoint exceptional curves (symplectically embedded spheres with self-intersection number -1) in $V-C$, one can obtain a pair (V,C) for which $V-C$ contains no exceptional curves (minimal pair). (ii) A minimal pair is symplectomorphic either to ($\mathbf{C}\mathrm{P}^2$, line or quadric), ω being the standard form, or to (S^2-bundle over a surface, fiber or section), ω being determined by its cohomology class.

This is a very powerful generalization of a result by M. L. Gromov [Invent. Math. **82** (1985), no. 2, 307–347; MR0809718] characterizing the standard $\mathbf{C}\mathrm{P}^2$. The proof uses Gromov's holomorphic curves theory [op. cit.], a crucial point being that in dimension 4 these curves can be shown to be embedded for homological reasons (this is proved in another paper of the author ["The local behaviour of holomorphic curves", J. Differential Geom., to appear]).

The notion of symplectic blow up (and down) was defined previously by the author [ibid. **20** (1984), no. 1, 267–277; MR0772133]. The question of uniqueness of these operations is a subtle point related to the unsolved problem of the connectedness of the space of symplectic embeddings of a disjoint union of balls. Nevertheless, the author is able to show that the category of these symplectic manifolds is closed under these operations as well as under perturbations of ω through noncohomologous symplectic forms.

Jean-Claude Sikorav
From MathSciNet, October 2021

MR1095236 55M20; 52A37
Griffiths, H. B.
The topology of square pegs in round holes.
Proceedings of the London Mathematical Society. Third Series **62** (1991), *no.* 3, 647–672.

In this long and interesting paper the author proves several results about which geometric configurations can be inscribed in other geometric objects. For example, in 1929 Shnirel′man proved that a suitably smooth Jordan curve in $\mathbf{R}^2$ has an inscribed square. Here the author generalises this to Jordan curves which are injective immersions of S^1 in $\mathbf{R}^n$ of class C^1 such that if $n=2$ then there is an inscribed rectangle whose ratio of sides is any given positive real number. Shnirel′man's result is further generalised to show that a skew "box" (but possibly not a cube) can be inscribed in an injective immersion of S^2 of class C^1 in $\mathbf{R}^3$. The final major result in the paper generalises a result of the reviewer which itself generalises a result of Dyson. Namely: Suppose $h\colon S^2\to\mathbf{R}$ is continuous and $0\le\lambda\le\mu\le 2$. Then S^2 contains a rectangle of side lengths λ and μ such that h is constant on the vertices of the rectangle.

Roger Fenn
From MathSciNet, October 2021

MR3303043 51M25l; 51M16

Schwartz, Richard Evan

Lengthening a tetrahedron.

Geometriae Dedicata **174** (2015), 121–144.

The motivation for this work was a question posed to the author by Daryl Cooper: Suppose that you lengthen all the sides of a tetrahedron by one unit. Is the result still a tetrahedron, and (if so) does the volume increase? The author formulates the problem as follows: Let $\{d_{ij} \mid i \neq j \in 1,2,3,4\}$ be a list of real numbers. This list is tetrahedral if there are four distinct points V_1, V_2, V_3, V_4 in $\mathbb{R}^3$ such that $d_{ij} = \|V_i - V_j\|$ for all i, j. The list $\{d_{ij}+1\}$ will be the unit lengthening of $\{d_{ij}\}$.

In fact (Theorem 1.1), the unit lengthening of a tetrahedral list is also a tetrahedral list. If Δ_0 is the original tetrahedron and Δ_1 is the new tetrahedron then one gets the following inequality for the volumes:

$$\frac{\mathrm{vol}(\Delta_1)}{\mathrm{vol}(\Delta_0)} \geq \left(1 + \frac{6}{\sum_{i<j} d_{ij}}\right)^3.$$

This inequality is sharp, with equality for regular tetrahedra.

In order to prove Theorem 1.1 the author uses the Cayley-Menger determinant and considers pseudo-tetrahedra. Let K_4 be a complete graph on four vertices. A pseudo-tetrahedron is a non-negative labelling of the edges of K_4 so that, going around any 3-cycle of K_4, the edges satisfy the triangle inequality. Let $D = \{d_{ij}\}$ be a pseudo-tetrahedron; the Cayley-Menger determinant is

$$f(D) = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 \\ 1 & d_{21}^2 & 0 & d_{23}^2 & d_{24}^2 \\ 1 & d_{31}^2 & d_{32}^2 & 0 & d_{34}^2 \\ 1 & d_{41}^2 & d_{42}^2 & d_{43}^2 & 0 \end{vmatrix}.$$

If D represents a tetrahedron T_D, then one obtains $f(D) = 288V^2$, where $V = \mathrm{vol}(T_D)$.

Denote by X the space of pseudo-tetrahedra, which is a polyhedral cone in $\mathbb{R}^6$ with coordinates d_{ij}. This cone is partitioned into 48 cones X_D (the "chambers") labelled by certain decorations of K_4, with each X_D linearly isomorphic to the positive orthant. Let $g = D_{(1,1,1,1,1,1)}f$ be the directional derivative. Theorem 1.1 is a consequence of Theorem 1.3 that states the following: Let C be a constant; then the function $g\sum_{i<j} d_{ij} - Cf$ is non-negative on X if and only if $C \in [16, 36]$. In order to prove these two theorems, the author considers a special type of pseudo-tetrahedra (section 2) and uses a computational method which he names the Method of Positive Dominance. This method is explained in section 3 and the details on how to implement it in Java are given in section 4. According to the author, this paper has, in fact, a companion computer program that can be downloaded from `www.math.brown.edu/~res/Java/CM2.tar`.

What about selectively lengthening some subset of the edges of the tetrahedron? The author deals with this problem in section 5 and proves the following result: Theorem 1.4. Let $\beta \subset K_4$ be a friendly subset (i.e. $K_4 \backslash \beta$ is not a union of edges all incident to the same vertex). Let $g = D_\beta f$ denote the directional derivative of f along β. There is a nonempty union X_β of chambers of X and constants $A_\alpha < B_\beta$, with $B_\beta > 0$, such that the function $g\sum_{i<j} d_{ij} - Cf$ is non-negative on X_β if and only if $C \in [A_\alpha, B_\beta]$. Moreover, every chamber of $X\backslash X_\beta$ contains a point where $f > 0$ and $g < 0$.

The author also considers the case when β is a 3-cycle in K_4. This subset is not friendly, hence Theorem 1.4 does not apply, but it is still possible to obtain the following result: Theorem 1.5. Let $g = D_\beta f$ denote the directional derivative of f along β; then the function $g\sum_{i<j} d_{ij} - Cf$ is non-negative on X_β if and only if $C = 8$. Moreover, every chamber of $X\backslash X_\beta$ contains a point where $f > 0$ and $g < 0$.

Combining Theorems 1.4 and 1.5, one obtains: Corollary 1.6. Suppose that β is either a 3-cycle or a friendly subset of K_4. For every tetrahedron X_β, the lengthening along β locally increases the volume. Moreover, every chamber of $X\backslash X_\beta$ contains a tetrahedron such that the lengthening along β decreases the volume.

This article ends with an appendix containing the proof of the following result: Theorem 1.2. In every dimension the unit lengthening of a simplicial list is again a simplicial list and the new simplex has volume larger than the original. (The notion of simplicial list is the natural extension of the notion of tetrahedral list to simplices of other dimensions.) The argument given for this result is independent from the rest of the paper. The author states that this result also follows from a theorem attributed to von Neumann, and for further details refers to Corollary 4.8 in [J. H. Wells and L. R. Williams, *Embeddings and extensions in analysis*, Springer, New York, 1975; MR0461107].

Ana Pereira do Vale

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MR3887658 37D50; 52C23

Schwartz, Richard Evan

The plaid model. (English)

Annals of Mathematics Studies, 198.
Princeton University Press, *Princeton, NJ*, 2019, xii+268 pp.,
ISBN 978-0-691-18138-7; 978-0-691-181370

The book under review establishes the basic properties of the *plaid model*.

“The purpose of this monograph is to study a construction, based on elementary geometry and number theory, which produces for each rational parameter (satisfying some parity conditions) a cube filled with polyhedral surfaces. When the surfaces are sliced in one direction, the resulting curves encode all the essential information about the so-called special outer billiards orbits with respect to kites. When the surfaces are sliced in two other directions, they encode all the essential information in a 1-parameter family of Truchet tile systems.”

W. P. Hooper developed the Truchet systems in [Invent. Math. **191** (2013), no. 2, 255–320; MR3010377] in connection with polygon exchange transformations and renormalization. See [J. André, “Les planches de pavages de Truchet”, to appear] for a sample of Truchet tile systems in the plane. See also the book of the author [*The octogonal PETs*, Math. Surveys Monogr., 197, Amer. Math. Soc., Providence, RI, 2014; MR3186232].

“The monograph establishes some of the basic properties of the plaid model: the connection to outer billiards and to Truchet tilings, the connection to polytope exchange transformations (PETs), and some results about the size and distribution of the polygons in the slices of the model."

The plaid model was conceived when the author studied the outer billiard on kites. For the kite K_A with vertices $(-1,0)$, $(0,1)$, $(0,-1)$ and $(A,0)$ the author in [J. Mod. Dyn. **1** (2007), no. 3, 371–424; MR2318496; *Outer billiards on kites*, Ann. of Math. Stud., 171, Princeton Univ. Press, Princeton, NJ, 2009; MR2562898] proved the following result.

Theorem. When A is irrational, the outer billiards on K_A has an unbounded special orbit.

This special orbit is unbounded, both forward and backward, but returns in a neighborhood of every vertex of the kite infinitely many times. The outer billiards has a unidimensional invariant set formed by horizontal lines $(x, 2n+1)$, $n \in \mathbb{Z}$, and every outer billiard orbit in the invariant set is either periodic or special, and the set of periodic orbits is open and dense in the invariant set.

"The key step in understanding the special orbits on K_A is to associate an embedded lattice polygonal path to each special orbit. This path encodes the symbolic dynamics associated to the second return map to the union $\mathbb{R} \times \{-1,1\}$ of lines. When $A = p/q$ is rational, it is possible to consider the union of all these lattice paths at once. I call this union the *arithmetic graph* and denote it by Γ_A. When pq is even, every component of Γ_A is an embedded lattice polygon."

The book is organized in five parts: the plaid model and its properties in Chapters 1 to 7, the plaid PET in Chapters 8 to 12, the graph PET in Chapters 13 to 16, the plaid-graph correspondence in Chapters 17 to 20 and the distribution of the orbits in Chapters 21 to 26.

Roughly speaking, to define the plaid model one considers six families of lines in the plane, horizontal and vertical with integer coordinates, four families of slant lines with inclinations $P = \pm 2p/(p+q)$ and $Q = \pm 2q/(p+q)$, and intersection with the vertical axis Y at integer coordinates. The intersections of the slant lines with the horizontal and vertical lines (grid) are called *light points*. The *oriented plaid model* takes as input the even rational parameter p/q and assigns either the empty set or a direct edge e_R to each unit integer square $R = [m, m+1] \times [n, n+1]$. "The empty set is assigned when there are 0 relevant light points in the boundary of R. Otherwise, the directed edge e_R connects the center of the sides of R which contain the relevant light points and crosses these sides in the same direction as the transverse directions." The *plaid polygon* is a connected component of the union of edges of the plaid model.

The book contains eight fundamental theorems. See the Introduction for an overview of the main results.

The several computer programs which illustrate most of the results of this book are available from the web site of the author. An important commentary of the author about his experimental research is the following. "I discovered all the results in this monograph using the program, and I have extensively checked my proofs against the output of the program. While this monograph mostly stands on its own, the reader will get much more out of it by using the program while reading. I would say that the program relates to the material here the way a cooked meal relates to a recipe. Throughout the text, I have indicated computer tie-ins which give instructions for operating the computer program so that it illustrates the relevant phenomena. I consider these computer tie-ins to be a vital component of the monograph."

This book will be very useful for those interested in experimental mathematics using computer programs, global aspects of dynamical systems, outer billiards, renormalization of PETs and combinatorial aspects of periodic and special orbits of outer billiards, such as orbits of a kite. I consider this to be a masterpiece of mathematics.

Ronaldo Alves Garcia
From MathSciNet, October 2021

MR3995015 53D12; 57N35, 57R17

Yoshiyasu, Toru

On Lagrangian embeddings of closed nonorientable 3-manifolds.

Algebraic & Geometric Topology **19** (2019), *no.* 4, 1619–1630.

K. Fukaya [in *Morse theoretic methods in nonlinear analysis and in symplectic topology*, 231–276, NATO Sci. Ser. II Math. Phys. Chem., 217, Springer, Dordrecht, 2006; MR2276953] classified the diffeomorphism type of Lagrangian embeddings of closed orientable connected prime 3-manifolds $\mathcal{L}$ into $\mathbb{R}^6$ (here always thought of as endowed with the standard symplectic form)—precisely,

$$\mathcal{L} \overset{\text{Lag}}{\to} \hookrightarrow \mathbb{R}^6 \iff \mathcal{L} \cong S^1 \times \Sigma_g,$$

where Σ_g is a genus g surface. Removing the prime condition, the embedding problem becomes much more flexible. Based on the works [E. Murphy, "Loose Legendrian embeddings in high dimensional contact manifolds", preprint, `arXiv:1201.2245`; Y. M. Eliashberg and E. Murphy, Geom. Funct. Anal. **23** (2013), no. 5, 1483–1514; MR3102911], T. Ekholm et al. showed in [Geom. Funct. Anal. **23** (2013), no. 6, 1772–1803; MR3132903] that, after connect summing with $S^1 \times S^2$, we can embed any closed orientable 3-manifold L as a Lagrangian in $\mathbb{R}^6$, i.e., $L \# S^1 \times S^2 \overset{\text{Lag}}{\to} \hookrightarrow \mathbb{R}^6$. One can reinterpret this result as saying that any compact orientable 3-manifold M with boundary diffeomorphic to S^2 can be *concatenated* with $K = S^1 \times S^2 \setminus B$, so that $M \cup_{\partial M \sim \partial K} K \overset{\text{Lag}}{\to} \hookrightarrow \mathbb{R}^6$, where B is an open ball.

In this paper, the author proves an analogous result for any compact orientable 3-manifold M with ∂M diffeomorphic to the two torus T^2, where the 'stabilising manifold' is $K = S^1 \times (N_0 \setminus D)$ given by the product of the circle S^1 with the complement of a disk D in the Klein bottle N_0, i.e., $\mathcal{L} := M \cup_{\partial M \sim \partial K} K \overset{\text{Lag}}{\to} \hookrightarrow \mathbb{R}^6$ as a Lagrangian. Moreover, $\mathcal{L}$ can be taken to have minimal Maslov number 1.

The proof idea is similar to the one in [T. Ekholm et al., op. cit.], where the construction is given by concatenating a Lagrangian filling with a Lagrangian cap. The existence of the latter is a consequence of [Y. M. Eliashberg and E. Murphy, op. cit.]. The author constructs a filling $K = S^1 \times (N_0 \setminus D)$ of a loose Legendrian T^2.

It is implied from the main result that any 3-manifold $\mathcal{L}$, containing $K = S^1 \times (N_0 \setminus D)$ as the submanifold, embeds into $\mathbb{R}^6$ as a Lagrangian. In particular, for any closed orientable 3-manifold L, the connected sum $\mathcal{L} = L \# S^1 \times N_g$ embeds as a Lagrangian in $\mathbb{R}^6$, where N_g is the non-orientable surface of Euler characteristic $g \geq 0$. In fact, the author focusses the proof on the particular case $\mathcal{L} = L \# S^1 \times N_{2g}$, as a consequence of his Lemma 2.5, and points out that the general case follows an analogous argument.

Renato Vianna

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