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HAROLD WIDOM'S WORK IN TOEPLITZ OPERATORS

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ABSTRACT. This is a survey of Harold Widom's work in Toeplitz operators, embracing his early results on the invertibility and spectral theory of Toeplitz operators, his investigations of the eigenvalue distribution of convolution operators, and his groundbreaking research into Toeplitz and Wiener–Hopf determinants.

1. The first half of the 20th century

In 1911, Otto Toeplitz studied doubly-infinite matrices of the form

$$(a_{j-k})_{j,k=-\infty}^{\infty}$$

and showed in particular that such a matrix generates a bounded operator on $\ell^2(\mathbf{Z})$ if and only if the simply-infinite matrix $(a_{j-k})_{j,k=1}^{\infty}$ induces a bounded operator on $\ell^2(\mathbf{N})$. The former are now called Laurent matrices, whereas the latter are since then referred to as (infinite) Toeplitz matrices. It turns out that, and this is implicit already in Toeplitz's article, the matrices define bounded operators on ℓ^2 if and only if there is a function a in L^∞ defined on the unit circle $\mathbf{T}$ such that the entries of the matrices are just the Fourier coefficients of a, that is,

$$a_k = \frac{1}{2\pi}\int_0^{2\pi} a(e^{i\theta})e^{-ik\theta}\,d\theta \quad (k \in \mathbf{Z}).$$

If such a function a exists, it is unique and is called the symbol of the corresponding Laurent or Toeplitz matrix, and these, as well as the operators they induce, are denoted by $L(a)$ and $T(a)$, respectively.

A finite $n \times n$ Toeplitz matrix may be regarded as a truncation of $T(a)$, and accordingly we write $T_n(a) := (a_{j-k})_{j,k=1}^{n}$. For such matrices, a pioneering result goes back to Gabor Szegő, who in 1915 established his celebrated first limit theorem, which states that if a is positive, then the quotient $\det T_n(a)/\det T_{n-1}(a)$ converges to $G(a) := \exp(\frac{1}{2\pi}\int_0^{2\pi}\log a(e^{i\theta})\,d\theta)$ as $n \to \infty$. This theorem implies that if a is real-valued, in which case the matrices $T_n(a)$ are all Hermitian, and if we denote by $\lambda_1(T_n(a)) \le \cdots \le \lambda_n(T_n(a))$ the eigenvalues of $T_n(a)$, then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\varphi(\lambda_j(T_n(a))) = \frac{1}{2\pi}\int_0^{2\pi}\varphi(a(e^{i\theta}))\,d\theta$$

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for every *test function* $\varphi \in C(\mathbf{R})$. This is a first-order asymptotic result for the collective eigenvalue distribution of Toeplitz matrices. In 1952, eventually motivated by Lars Onsager's formula for the spontaneous magnetization of the two-dimensional Ising model, Szegő improved the result to a second-order asymptotic formula, which is now called Szegő's strong limit theorem. We refer to the article [DIK3] for an exhaustive treatment of this story. Almost at the same time, in 1953, M. Kac, W. L. Murdock, and G. Szegő succeeded in describing the behavior of the extreme eigenvalues $\lambda_j(T_n(a))$ and $\lambda_{n-j}(T_n(a))$ (j fixed and $n \to \infty$).

Since $L(a)$ is unitarily equivalent to the operator of multiplication by a on $L^2(\mathbf{T})$, it follows that $L(a)$ is invertible if and only if a is invertible in $L^\infty(\mathbf{T})$. Invertibilty of $T(a)$ is a much more delicate issue and is a problem that has been studied by many authors since the appearance of Toeplitz operators up to the present. In 1929, A. Wintner solved the problem for triangular matrices $T(a)$, and in 1954, P. Hartman and A. Wintner showed that if a is real-valued ($\Leftrightarrow T(a)$ is Hermitian), then the spectrum of $T(a)$ is the convex hull of the essential range of a.

Toeplitz operators are closely related to a series of other operators, namely, the operators coming from the Riemann–Hilbert boundary value problem, singular integral operators, and Wiener–Hopf integral operators. This connection was not fully understood in those times, but nowadays we know that every result on an operator belonging to one of the last three classes is a result on Toeplitz operators. Many mathematicians, including F. Noether, J. Plemelj, S. G. Mikhlin, G. Fichera, and T. Carleman, studied singular integral operators with continuous coefficients and realized that, stated in terms of Toeplitz operators with continuous symbols, for $T(a)$ to be invertible it is sufficient that a have no zeros on $\mathbf{T}$ and that the winding number of a about the origin be zero. Finally, in 1952, Israel Gohberg, by an ingenious application of the Gelfand theory of Banach algebras, was able to prove that these two conditions are also necessary for $T(a)$ to be invertible.

In 1931, Norbert Wiener and Eberhard Hopf published their paper on what is now called Wiener–Hopf factorization. This factorization amounts to factoring $T(a)$ into a product of an upper and a lower triangular matrix. However, a complete understanding of that method was gained only in the works of F. D. Gakhov in 1949 and of I. Gohberg and Mark Krein in the 1950s.

2. Invertiblity, Fredholmness, and spectra

Harold Widom entered the Toeplitz operators scene with his 1959 paper [2] jointly with Alberto Calderón and Frank Spitzer. This paper deals with Toeplitz operators $T(a)$ generated by symbols a in the Wiener algebra, that is, by symbols a satisfying $\sum |a_n| < \infty$. The authors consider $T(a)$ as an operator on ℓ^∞ and on ℓ^2, and they show that in both contexts $T(a)$ is invertible if and only if a has no zeros on $\mathbf{T}$ and has winding number zero about the origin. The approach is based on the Wiener–Hopf factorization $a(t) = a_-(t)t^\kappa a_+(t)$, which gives the inverse operator $T^{-1}(a) = T(a_+^{-1})T(a_-^{-1})$ in the case of invertibility ($\Leftrightarrow \kappa = 0$) and the kernel and co-kernel dimensions of $T(a)$ for $\kappa \neq 0$. The paper was submitted in May 1958, and in a note added in proof, the authors remark that a substantial part of their results are also in a 1958 paper by M. Krein. However, one theorem of [2] was not in Krein's paper: it replaces the condition $\sum |a_n| < \infty$ by the sole requirement that $a \in L^\infty(\mathbf{T})$ and says that $T(a)$ is invertible on ℓ^2 whenever a is invertible in $L^\infty(\mathbf{T})$ and $a/|a| = \exp(i\,\widetilde{v})$ with $v \in L^\infty(\mathbf{T})$ and $\widetilde{v}$ denoting the conjugate function of v.

In his 1960 paper [3], which was submitted in August 1958, Widom then laid the foundations for the invertibility theory of Toeplitz operators on ℓ^2. The paper has four theorems. In Theorems II and III, unaware of the work of Wintner and of Hartman–Wintner, he rediscovered their invertibility criteria for triangular and Hermitian Toeplitz matrices. Theorem I was a real breakthrough. It states that for $T(a)$ to be invertible, it is necessary and sufficient that $a = a_- a_+$ with $a_-^{\pm 1} \in L^2_-(\mathbf{T})$, $a_+^{\pm 1} \in L^2_+(\mathbf{T})$ such that the operator $f \mapsto a_+^{-1} P a_-^{-1} f$ is bounded on $L^2(\mathbf{T})$. Here $L^2_\pm(\mathbf{T})$ are the usual Hardy spaces and P is the orthogonal projection of $L^2(\mathbf{T})$ onto $L^2_+(\mathbf{T})$. Note that $P = (I + S)/2$, where S is the Cauchy singular integral operator. It was a lucky tie of events that just at that time, in 1960, H. Helson and G. Szegő were able to characterize the weights w for which S is bounded on $L^2(\mathbf{T}, w)$. Combining his Theorem I and the Helson–Szegő theorem, Widom arrived at the conclusion that $T(a)$ is invertible if and only if

$$a/|a| = \exp(i(c + u + \widetilde{v})),$$

where c is a real constant, u and v are two real-valued functions in $L^\infty(\mathbf{T})$, and $\|u\|_\infty < \pi/2$. This beautiful result, which was published in 1960 by Widom in [5] and was rediscovered by Allen Devinatz in 1964, is referred to in the textbooks as the Widom–Devinatz theorem. We should mention that an essential generalization of Widom's Theorem I, namely, its extension to Toeplitz operators with matrix-valued symbols on the Hardy spaces $L^p_+(\mathbf{T})$ was independently discovered by Igor Simonenko in 1961.

Theorem IV of Widom's paper [3] was another milestone. It concerns the case where a is piecewise continuous with at most finitely many jumps. Consider the continuous and naturally oriented curve in the plane that arises from the essential range of a by filling in line segments between the endpoints $a(t-0)$ and $a(t+0)$ of each jump. Widom proved that $T(a)$ is invertible on ℓ^2 if and only if this curve does not contain the origin and has winding number zero about the origin. This was the very beginning of a long and fascinating story.

The first chapter of this story was written by none other than Widom himself in [6]. The ℓ^2 theory of Toeplitz operators bifurcates into the ℓ^p and L^p theories for $1 < p < \infty$. The latter two theories are based on completely different techniques although, and this is something of a mystery, in the case of piecewise continuous symbols the final results are almost identical. In [6], Widom studied Toeplitz operators $T(a)$ with piecewise continuous symbols a on the Hardy space $L^p_+(\mathbf{R})$ of the upper half-plane. These operators are defined by $f \mapsto P(af)$, where $P = (I + S)/2$ and S is the Cauchy singular integral operator on $L^p(\mathbf{R})$. (One could equally well work on $L^p_+(\mathbf{T})$, the differences being only technical and psychological.) Widom again arrived at the boundedness of $f \mapsto a_+^{-1} P a_-^{-1} f$ on $L^p_+(\mathbf{R})$, understood that this is a question about the weights w for which S is bounded on $L^p(\mathbf{R}, w)$, and showed that S is bounded if

$$w(x) = (1 + |x|)^\alpha \prod_{k=1}^{m} |x - x_k|^{\alpha_k}$$

with

$$-1/p < \alpha_k < 1/q \quad \text{and} \quad -1/p < \alpha + \sum_{k=1}^{m} \alpha_k < 1/q,$$

where $1/p+1/q=1$. Using this insight, he was able to prove that $T(a)$ is invertible on $L^p_+(\mathbf{R})$ if and only if a certain curve does not contain the origin and has winding number zero about the origin. This curve results from the essential range of a by filling in certain circular arcs $\mathcal{A}_p(a(x-0),a(x+0))$ depending on p between the endpoints of the jumps at $x\in\mathbf{R}$ and the arc $\mathcal{A}_q(a(+\infty),a(-\infty))$ for the jump at infinity. Here, for two distinct points $\alpha,\beta\in\mathbf{C}$ and a number $r\in(1,\infty)$, we denote by $\mathcal{A}_r(\alpha,\beta)$ the circular arc at the points of which the line segment $[\alpha,\beta]$ is seen at the angle $2\pi/\max\{r,s\}$, where $1/r+1/s=1$, and which lies on the right (resp., left) of the oriented line passing first α and then β if $1<r<2$ (resp., $2<r<\infty$). For $r=2$, $\mathcal{A}_r(\alpha,\beta)$ is simply the line segment $[\alpha,\beta]$. For example, if $a(x)=\operatorname{sign}x$, then we have two circular arcs $\mathcal{A}_p(-1,1)$ and $\mathcal{A}_q(1,-1)$, and since $\mathcal{A}_q(1,-1)=\mathcal{A}_p(-1,1)$, it follows that $T(\text{sign})$ is invertible if and only if $p\neq 2$. Widom also computed the kernel and co-kernel dimensions of the operators if the curve has nonzero winding number. Overall, paper [6] contained the full Fredholm theory of Toeplitz operators with piecewise continuous symbols on $L^p_+(\mathbf{R})$, including an index formula.

In different language, particular cases of the Fredholm results of [6] were already evident in papers by B. V. Hvedelidze since 1947. The characterization of the weights w for which S is bounded on $L^p(\Gamma,w)$ has a long history, starting with G. H. Hardy and J. E. Littlewood and culminating with work by R. Hunt, B. Muckenhoupt, R. Wheeden (1973), A. Calderón (1977), and G. David (1984). In the late 1960s and the 1970s, I. Gohberg and N. Krupnik introduced their local principle by means of which they could not only give a simpler proof of Widom's result but also consider Lyapunov curves Γ with power weights w, the case of matrix-valued symbols, and Banach algebras generated by Toeplitz operators with piecewise continuous symbols. In 1972, R. Duduchava settled matters for Toeplitz operators on ℓ^p. The theory reached a certain final stage only in the 1990s by work of I. Spitkovsky (general weights w) and Yu. I. Karlovich and the second author (general curves Γ and general weights w). In these more general situations, Harold Widom's circular arcs undergo a metamorphosis into horns, logarithmic spirals, logarithmic horns, and eventually into logarithmic leaves with a halo [BK].

The invertibility and Fredholm criteria for Toeplitz operators with analytic, real-valued, or piecewise continuous symbols imply a description of the spectrum and of the essential spectrum of the operators. (The essential spectrum of an operator T is the set of all complex λ for which $T-\lambda I$ is not Fredholm, that is, not invertible modulo compact operators.) In all known cases, the spectrum and essential spectrum turned out to be connected sets, and in 1963, Paul Halmos posed the question whether the spectrum of $T(a)$ is connected for every $a\in L^\infty(\mathbf{T})$. In [10], submitted in April 1963, Widom proved that the answer is *Yes* for the spectrum of Toeplitz operators on ℓ^2, and in his paper [12] of 1966, he performed the same feat for Toeplitz operators on $L^p_+(\mathbf{T})$. In 1972, Ronald Douglas established the connectedness of the essential spectrum of Toeplitz operators on ℓ^2, and only in 2009, A. Yu. Karlovich and I. Spitkovsky [KS] were able to prove that both the spectrum and the essential spectrum of Toeplitz operators are always connected on $L^p_+(\Gamma,w)$ for $1<p<\infty$ and general curves Γ and weights w.

We cannot conclude this section without mentioning that several basic results on Toeplitz operators, which nowadays appear on the first pages of the textbooks, were established just around 1960, and that tracing back to the sources of these results is a subtle matter. For example, the Brown–Halmos theorem, according to which the spectrum of $T(a)$ is a subset of the convex hull of the essential range of a, though explicitly published for the first time by P. Halmos and A. Brown in 1963, was known to at least Widom and I. B. Simonenko already in 1960. As for Widom, the theorem is in his article [11], which is based on lectures at the IAS in 1960. We also remark that in the very early 1960s, Simonenko [Sim1, Sim2] already had the results of [14] on locally sectorial symbols and the theorem that a Toeplitz operator is invertible if and only if it is Fredholm of index zero, which was published by Lewis Coburn in 1967 and has been known as Coburn's lemma since then. Those years were indeed turbulent times.

3. Extreme eigenvalues of convolution operators

Another topic of Harold Widom's early work is extreme eigenvalues of integral operators of the form

$$(C_\tau f)(x) = \int_{-\tau}^{\tau} k(x-y)f(y)\,dy, \quad x \in (-\tau, \tau),$$

considered on $L^2(-\tau,\tau)$. These operators are the continuous analogue of finite Toeplitz matrices. Since the kernel of the operator is translation invariant, we may change integration over $(-\tau,\tau)$ to integration over $(0,2\tau)$ and therefore think of C_τ as the compression to $L^2(0,2\tau)$ of the Wiener–Hopf operator with the kernel $k(x-y)$, in which integration goes from 0 to ∞. The symbol of such operators is the Fourier transform of the function k, $\hat{k}(\xi) := \int_{-\infty}^{\infty} k(x)e^{i\xi x}\,dx$. Of interest is the case in which the function k is real-valued and even and in $L^1(\mathbf{R})$. In that case C_τ is a compact Hermitian operator and we may label the eigenvalues as $\lambda_1(C_\tau) \ge \lambda_2(C_\tau) \ge \cdots$. As predicted by Kac, Murdock, and Szegő, who studied the extreme eigenvalues of Hermitian Toeplitz matrices, the asymptotic behavior of $\lambda_j(C_\tau)$ for fixed j and for $\tau \to \infty$ depends heavily on the behavior of the symbol $\hat{k}$ near its maximum. Suppose that the maximal value is 1 and that it is attained at $\xi = 0$ and only there. Under the assumption that $\hat{k}(\xi) = 1 - c|\xi|^\alpha + o(|\xi|^\alpha)$ as $\xi \to 0$ and that some more minor technical conditions are satisfied, Widom proved that

$$\lambda_j(C_\tau) = 1 - \frac{c}{\mu_{j,\alpha}}\,\frac{1}{\tau^\alpha} + o\left(\frac{1}{\tau^\alpha}\right) \quad \text{as} \quad \tau \to \infty,$$

where the $\mu_{j,\alpha}$ are certain constants. For $\alpha = 2$, this was done in his 1958 paper [1], where he even improved the $o(1/\tau^2)$ to $\nu_{j,\alpha}/\tau^3 + o(1/\tau^3)$. Papers [7] and [8] of 1961 are for general $\alpha \in (0,\infty)$. The constants $\mu_{j,\alpha}$ are shown to be the eigenvalues of a certain positive definite integral operator with some kernel $K_\alpha(x,y)$ on $L^2(-1,1)$. If $\alpha = 2k$ is an even natural number, then $K_\alpha(x,y)$ is Green's function of the differential operator $u \mapsto (-1)^k u^{(2k)}$ on $(-1,1)$ with the boundary conditions $u^{(\ell)}(-1) = u^{(\ell)}(1) = 0$ for $0 \le \ell \le k-1$.

To prove these results, Widom derives a formula for the determinants of banded Toeplitz matrices and some kind of an analogue of this formula for integral operators. These formulas are of interest by themselves and the starting point of yet another story. Subtracting λI and setting the resulting determinants zero, he gets the eigenvalues, and a clever approximation argument then yields the desired result. Widom's 1963 paper [9] is devoted to the extreme eigenvalues of the compressions of convolution operators on $L^2(\mathbf{R}^n)$ to $L^2(\tau\Omega)$ as $\tau \to \infty$.

Extreme eigenvalues of Hermitian Toeplitz matrices, whose symbol has a prescribed behavior near the maximum that is governed by a parameter $\alpha \in (0,\infty)$, were thoroughly studied by Seymor Parter in the 1960s. As Harold told us, there was an agreement between Parter and him that Parter should focus on the Toeplitz case while he would embark on the Wiener–Hopf case.

This is also the right place for another story. In May 2008, the second author received a (handwritten!) letter from Peter Dörfler with the question whether there are results on the large n behavior of the maximal singular value (= spectral norm) of the $(n+1)\times(n+1)$ triangular Toeplitz matrices,

$$T_n = (-1)^\nu \begin{pmatrix} 0 & \binom{0}{\nu-1} & \binom{1}{\nu-1} & \cdots & \binom{n-1}{\nu-1} \\ & 0 & \binom{0}{\nu-1} & \cdots & \binom{n-2}{\nu-1} \\ & & & \ddots & \vdots \\ & & & & \binom{0}{\nu-1} \\ & & & & 0 \end{pmatrix},$$

composed of binomial coefficients with an integer $\nu \ge 1$. The matrix T_n is the representation of the operator taking the νth derivative, $f \mapsto D^\nu f$, in the orthonormal basis of Laguerre polynomials in the space $\mathcal{P}_n$ of algebraic polynomials of degree at most n with the Laguerre norm given by $\|f\|^2 = \int_0^\infty |f(x)|^2 e^{-x}\,dx$. Thus, the norm $\|T_n\|$ is just the best constant for which the so-called Markov-type inequality $\|D^\nu f\| \le M\|f\|$ holds for all $f \in \mathcal{P}_n$.

This question reminded the second author of an ingenious trick used in Harold Widom's 1966 paper [13] (and employed independently also by Lawrence Shampine in [Sha]). Let us consider an $n\times n$ matrix $A_n = (a_{jk})_{j,k=0}^{n-1}$ and denote by H_n the integral operator on $L^2(0,1)$ with the piecewise constant kernel $h_n(x,y) = a_{[nx],[ny]}$, where $[\cdot]$ stands for the integral part. Widom and Shampine proved that $\|A_n\| = n\|H_n\|$, thus transferring consideration of A_n on the sequence $\{\mathbb{C}^n\}$ of increasing spaces to the consideration of a sequence $\{H_n\}$ of operators in one and the same space $L^2(0,1)$. If one could show that after appropriate scaling the operators H_n converge in the operator norm to some nonzero operator H, that is, $n^{-\mu}H_n \to H$ in norm, it would follow that $n^{-\mu}\|H_n\| \to \|H\|$ and hence $\|A_n\| \sim \|H\| n^{\mu+1}$.

To compute $\|T_n\|$, we may ignore the factor $(-1)^\nu$ and the diagonal of zeros. In the resulting $n\times n$ matrix, the j,k entry is equal to $\binom{k-j}{\nu-1}$ for $j<k$. Consequently, if $x<y$, then the kernel of the scaled integral operator $n^{-(\nu-1)}H_n$ is

$$\begin{aligned} \frac{1}{n^{\nu-1}} a_{[nx],[ny]} &= \frac{1}{n^{\nu-1}} \binom{[ny]-[nx]}{\nu-1} \\ &= \frac{1}{(\nu-1)!} \frac{[ny]-[nx]}{n} \frac{[ny]-[nx]-1}{n} \cdots \frac{[ny]-[nx]-\nu+2}{n}, \end{aligned}$$

which converges uniformly to $(y-x)^{\nu-1}/(\nu-1)!$ as $n\to\infty$ and thus yields the asymptotics $\|T_n\| = \|L_\nu\| n^\nu (1+o(1))$, where L_ν is the Volterra integral operator on $L^2(0,1)$ given by

$$(L_\nu f)(x) = \frac{1}{(\nu-1)!} \int_x^1 (y-x)^{\nu-1} f(y)\,dy.$$

Clearly, $L_\nu = L_1^\nu$ and $\|L_\nu\| = \|L_\nu^*\|$ with

$$(L_\nu^* f)(x) = \frac{1}{(\nu-1)!} \int_0^x (x-y)^{\nu-1} f(y)\, dy.$$

Note that $\|L_1\| = 2/\pi$. For more on the subject and, in particular, for more about pieces of the amazing story around the norms of the Volterra operators L_ν, we refer to [BD].

4. Eigenvalue distribution

Widom made several fundamental contributions to the collective eigenvalue distribution of truncated Toeplitz and Wiener–Hopf operators and their generalizations, such as pseudodifferential operators. In this section, we focus our attention on two of his papers on this topic.

In his 1980 paper [24] with Henry Landau, he investigated the positive definite operator given on $L^2(-\tau,\tau)$ by

$$(C_\tau f)(x) = \frac{\gamma}{2\pi i} \int_{-\tau}^{\tau} \frac{e^{-i\alpha(x-y)} - e^{-i\beta(x-y)}}{x-y} f(y)\, dy, \quad x \in (-\tau,\tau).$$

This operator is of crucial interest in random matrix theory and in laser theory. For example, as observed by H. Brunner, A. Iserles, and S. Nørsett [BIN], if $\gamma = \pi$, $\alpha = -2$, $\beta = 2$, in which case the operator is convolution by $\sin(2t)/t$, the eigenvalues of C_τ are the singular values of the famous Fox–Li operator. The symbol of C_τ is $\gamma\chi_{(\alpha,\beta)}$, and hence it has two jumps. No general result of the type of Szegő's strong limit theorem delivered a second-order trace formula in this situation. By an extremely ingenious argument, Landau and Widom nevertheless succeeded in establishing a second-order result for the eigenvalues, which confirmed a conjecture by D. Slepian of 1965. The result says that if φ is in $C^\infty(\mathbf{R})$ and $\varphi(0) = 0$, then

$$\sum_{j=1}^{\infty} \varphi(\lambda_j(C_\tau)) = \tau\, \frac{\varphi(\gamma)(\beta-\alpha)}{\pi} + \frac{\log(2\tau)}{\pi^2} \int_0^\gamma \frac{\gamma\varphi(x) - x\varphi(\gamma)}{x(\gamma-x)}\, dx + O(1).$$

The other paper we want to emphasize here is [28] of 1990. One is tempted to think that the eigenvalues of the $n \times n$ Toeplitz matrices $T_n(a)$ somehow mimic the spectrum of the infinite Toeplitz matrix $T(a)$ as $n \to \infty$. This is indeed the case if a is real-valued, but already in 1960, P. Schmidt and F. Spitzer showed that this is in general no longer true if a is a Laurent polynomial ($\Leftrightarrow T(a)$ is banded). On the other hand, it was known that if a is piecewise continuous with exactly one jump and this jump is not too large, then the spectrum of $T_n(a)$ converges to the essential range of a. So what could the overall picture be? In [28], Widom raised the brave conjecture that except in rare cases, the eigenvalues of $T_n(a)$ are, in a sense, asymptotically distributed as the values of a. Such a rare case takes place, for instance, if a extends analytically a little into the interior or the exterior of $\mathbf{T}$, which happens in particular if a is a Laurent polynomial. And Widom proved this conjecture for various classes of symbols. One of the results of [28] says that if a is continuous, the range $a(\mathbf{T})$ is a Jordan curve, a is C^1 with nonvanishing derivative on $\mathbf{T} \setminus \{1\}$ but not in C^1 on all of $\mathbf{T}$, then the eigenvalues asymptotically cluster along $a(\mathbf{T})$; see Figure 1. The proof is based on a thorough analysis of the determinants $\det(T_n(a) - \lambda I)$. In the case at hand, the function $a - \lambda$ is nonvanishing but has nonzero winding number

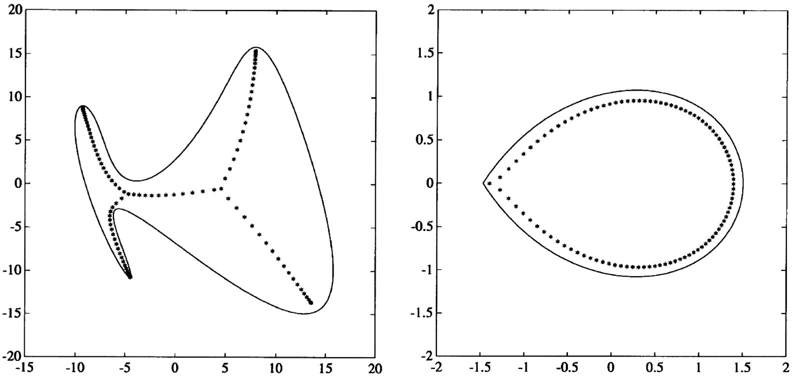

FIGURE 1. The left picture shows the range $a(\mathbf{T})$ of the Laurent polynomial

$$a(t) = -t^{-4} - (3+2i)t^{-3} + it^{-2} + t^{-1} + 10t + (3+i)t^2 + 4t^3 + it^4$$

and the 100 eigenvalues of $T_{100}(a)$. On the right we see the range $b(\mathbf{T})$ of $b(e^{i\theta}) = (\theta^4/100 + 1)e^{i\theta}$ (with $\theta \in [-\pi, \pi]$) and the 100 eigenvalues of $T_{100}(b)$. The eigenvalues of $T_n(a)$ are distributed as predicted by Schmidt and Spitzer. The function b is continuous but not C^1, so that, in accordance with Widom's result, the eigenvalues are canonically distributed.

about the origin, and getting asymptotic formulas for such determinants is one of the most difficult problems in the Toeplitz determinants business.

The ideas, methods, and results on extreme eigenvalues and eigenvalue distribution sketched in this and the previous sections foreshadowed some of the work on random matrix theory and gave hints of good things to come. We refer to [BBGM, DIK2] for recent developments concerning eigenvalues of large Toeplitz matrices and to [CDI] for the fascinating advances in the field of random matrices.

5. Toeplitz determinants with regular symbols

Widom made his debut at Toeplitz determinants with the 1960 paper [4] on Wiener–Hopf determinants. In 1954, Marc Kac proved a continuous analogue of Szegő's strong limit theorem, now known as the Akhiezer–Kac formula, and Widom extended this formula to the higher-dimensional case.

The revolutionary contributions of Widom to the topic are in his papers [17], [18], [19], which appeared from 1974 to 1976. Szegő's strong limit theorem says that, under certain assumptions, $\det T_n(a)/G(a)^n$ converges to a nonzero limit $E(a)$ as $n \to \infty$. The original positivity assumption by Szegő was relaxed over the years by many mathematicians, including G. Baxter, I. I. Hirshman, Jr., A. Devinatz, to the requirement that a satisfies some mild smoothness condition, has no zeros on $\mathbf{T}$, and has winding number zero about the origin. The constants $G(a)$ and $E(a)$ are then given by

$$G(a) = \exp(\log a)_0 \quad \text{and} \quad E(a) = \exp \sum_{k=1}^{\infty} k(\log a)_k (\log a)_{-k},$$

where $(\log a)_j$ denotes the jth Fourier coefficient of any continuous logarithm of a. Widom did two important things. First, he extended the theorem to block Toeplitz matrices, and secondly, he found a remarkably elegant operator theoretic proof with immense impact on subsequent research into the asymptotics of Toeplitz matrices.

In the block case, a is a function of $\mathbf{T}$ into $\mathbf{C}^{N\times N}$, the Fourier coefficients a_j are $N\times N$ matrices, $T_n(a)$ is accordingly a matrix of order nN, and it turns out that the function $\widetilde{a}$ associated with a and defined by $\widetilde{a}(t)=a(1/t)$ for $t\in\mathbf{T}$ plays an important role. In addition to the block Toeplitz operator $T(a)$, we need the block Hankel operator $H(a)$ defined by the infinite block Hankel matrix $(a_{j+k-1})_{j,k=1}^{\infty}$ on the $\mathbf{C}^N$-valued ℓ^2. Widom's smoothness assumption was that

$$\|a\| := \|a\|_\infty + \Big(\sum_{j=-\infty}^{\infty} |j|\,\|a_j\|^2\Big)^{1/2} < \infty.$$

In 1966, Mark Krein showed that such matrix functions form a Banach algebra. If a satisfies this smoothness condition, then $T(a)$ is bounded while $H(a)$ is a Hilbert–Schmidt operator. The theorem of [18] and [19] states that if $T(a)$ and $T(\widetilde{a})$ are Fredholm of index zero, then $T(a)T(a^{-1})-I$ is a trace class operator and $\det T_n(a)/G(a)^n \to E(a)$ where

$$G(a)=\exp(\log\det a)_0 \quad\text{and}\quad E(a)=\det T(a)T(a^{-1}).$$

This result is now in the textbooks as the Szegő–Widom limit theorem.

That $T(a)T(a^{-1})-I$ is a trace class operator follows from the nice identity

$$T(ab)=T(a)T(b)+H(a)H(\widetilde{b}),$$

which was established in [19]. This identity had been known and used for a long time, for example in the form $PabP = PaPbP + PaQbP$, but writing it in this form, with the Hankel operators, was one of Widom's strokes of genius. Why is $\det T(a)T(a^{-1})$ equal to Szegő's original constant in the scalar case? Widom observed that this follows from another remarkable identity, namely, the formula

$$\det(e^A e^B e^{-A} e^{-B}) = e^{\operatorname{tr}(AB-BA)},$$

which holds whenever A, B are bounded Hilbert space operators such that $AB-BA$ is of trace class. This formula was established independently by J. D. Pincus in 1972 and by J. W. Helton and R. E. Howe in 1973, and a simple proof was given by the third author [E2] in 2003. Widom expressed $T(a)T(a^{-1})$ as $e^A e^B e^{-A} e^{-B}$ so that the commutator $AB-BA$ is the product $H(c)H(\widetilde{c})$ of two Hankel operators generated by $c=\log a$, and since

$$\operatorname{tr} H(c)H(\widetilde{c}) = \operatorname{tr}\begin{pmatrix} c_1 & c_2 & c_3 & \dots \\ c_2 & c_3 & \dots & \\ c_3 & \dots & & \\ \dots & & & \end{pmatrix}\begin{pmatrix} c_{-1} & c_{-2} & c_{-3} & \dots \\ c_{-2} & c_{-3} & \dots & \\ c_{-3} & \dots & & \\ \dots & & & \end{pmatrix} = \sum_{k=1}^{\infty} k c_k c_{-k},$$

he arrived at Szegő's scalar case formula for the constant $E(a)$.

In [17], Widom derived an alternative expression for $E(a)$ in the case where a is a block Laurent polynomial and also used this in order to determine the limiting set of the eigenvalues of $T_n(a)$ in this situation, thus generalizing results by Schmidt, Spitzer, and Hirschman to the block case. In his 1989 paper [27], Widom showed in a direct way that $\det T(a)T(a^{-1})$ coincides with still another expression, which was obtained by I. Gohberg, M. A. Kaashoek, and F. van Schagen [GKS] in 1987.

As for Widom's proof in [19], we first of all want to remark that all previous proofs of the Szegő strong limit theorem were very complicated and rather indirect and did not convincingly reveal where the $E(a)$ actually comes from. This changed with Widom's operator theoretic proof. Instead of embarking on this proof here, we go some 25 years ahead. In 2000, Alexei Borodin and Andrei Okounkov [BO] established a formula which, in notation subsequently suggested by no-one but Widom, reads

$$\frac{\det T_n(a)}{G(a)^n} = \frac{\det(I - Q_n H(b)H(\widetilde{c})Q_n)}{\det(I - H(b)H(\widetilde{c}))}.$$

Here Q_n is projection onto the coordinates indexed by $n+1, n+2, \ldots$, a is assumed to have a Wiener–Hopf factorization $a = a_- a_+$, and b, c are defined by $b = a_- a_+^{-1}$, $c = a_-^{-1} a_+$. Since $Q_n \to 0$ strongly and $H(b)H(\widetilde{c})$ is of trace class, it follows that the right-hand side converges to $1/\det(I - H(b)H(\widetilde{c}))$, which can be shown to be just $\det T(a)T(a^{-1})$. The result is the Szegő–Widom limit theorem. Something like the Borodin–Okounkov formula was asked for by P. Deift and A. Its in 1999, and later it turned out that J. Geronimo and K. Case [GC] had a similar formula proved earlier in 1979. Borodin and Okounov's proof of their formula was very intricate. Simple operator theoretic proofs were subsequently given by Widom and Basor in [32], and by Widom and Böttcher in [35].

To mention at least one impact of Widom's proof in [19] on subsequent research, we note that [19] contains the beautiful identity

$$T_n(a)T_n(b) = T_n(ab) - P_n H(a)H(\widetilde{b})P_n - W_n H(\widetilde{a})H(b)W_n$$

for the product of two finite Toeplitz matrices. Here P_n is projection onto the first n coordinates, and W_n is P_n followed by reversal of the coordinates. We remark that Widom himself wrote Q_n instead of W_n. The W_n was introduced in [BS1] (which was written before [Sil] but appeared only after that paper), not only because Q_n is there used for $I - P_n$ but mainly to give merit to Widom. It was this eye-catching identity along with the observation that the products of the Hankel operators are compact if a or b is continuous which inspired Bernd Silbermann in 1980 to study the stability of the sequence $\{T_n(a)\}_{n=1}^{\infty}$ by embedding it into a Banach algebra of sequences in which sequences of the form

$$\{P_n K P_n + W_n L W_n + C_n\}_{n=1}^{\infty}$$

with compact K, L and $\|C_n\| \to 0$ form a closed two-sided ideal [Sil]. Since then, this idea has led to enormous progress in the foundation of plenty of approximation methods and numerical algorithms; see, e.g., [BS2, HRS1, HRS2, PS].

6. Toeplitz determinants with singular symbols

Symbols with discontinuities, zeros, poles, or nonzero winding number are referred to as singular symbols. If one of these four evils happens, Szegő's limit theorem breaks down. Widom's first paper on Toeplitz determinants in the strict sense is his 1971 paper [15], and there he considered a sheer monster of a singular symbol: the case where a is a positive function supported in $[\delta, 2\pi - \delta] \subset (0, 2\pi)$ and satisfying $a(\theta) = a(2\pi - \theta)$. He proved that then

$$\det T_n(a) \sim 2^{1/12} e^{3\zeta'(-1)} (\sin(\delta/2))^{-1/4} E_0(f)^2 G(f)^n n^{-1/4} (\cos(\delta/2))^{n^2},$$

where $G(f) = \exp(\log f)_0$ is as before, $E_0(f) = \exp(\frac{1}{4}\sum_{k=1}^{\infty} k(\log f)_k(\log f)_{-k})$, and f is given by

$$f(\theta) := a(2\arccos(\cos(\delta/2)\cos\theta)).$$

This paper was an important step toward the development connected with the sine kernel; see [CDI]. It should also be emphasized that the difficulties and asymptotics arising for symbols vanishing on entire arcs are far beyond the problems for the symbol class we will embark on now.

Namely, it was the 1968 paper [FH] by Michael Fisher and Robert Hartwig that set a big ball rolling. They introduced the class of singular symbols given by

$$a(e^{i\theta}) = b(e^{i\theta}) \prod_{r=1}^{R} |e^{i\theta} - e^{i\theta_r}|^{2\alpha_r}\, \varphi_{\beta_r,\theta_r}(e^{i\theta}),$$

where b is a nice function (smooth, nonvanishing on $\mathbf{T}$, and with winding number zero about the origin), $e^{i\theta_1}, \dots, e^{i\theta_R}$ are distinct points on $\mathbf{T}$, and the functions $\varphi_{\beta_r,\theta_r}$ are defined by

$$\varphi_{\beta_r,\theta_r}(e^{i\theta}) = \exp(i\beta_r \arg(-e^{i(\theta-\theta_r)}))$$

with the argument taken in $(-\pi, \pi]$. The function $\varphi_{\beta_r,\theta_r}$ satisfies

$$\varphi_{\beta_r,\theta_r}(e^{i(\theta_r+0)}) = e^{-\pi i\beta_r}, \quad \varphi_{\beta_r,\theta_r}(e^{i(\theta_r-0)}) = e^{\pi i\beta_r},$$

and it is continuous on $\mathbf{T} \setminus \{e^{i\theta_r}\}$. Such symbols a may have zeros ($\operatorname{Re}\alpha_r > 0$), poles ($\operatorname{Re}\alpha_r < 0$), oscillating discontinuities ($\operatorname{Re}\alpha_r = 0$), jumps ($\beta_r \notin \mathbf{Z}$), and nonzero winding numbers ($\beta_r \in \mathbf{Z}$).

Hartwig and Fisher raised the conjecture that

$$\det T_n(a)/G(a)^n \sim C(a)\, n^{\sum(\alpha_r^2-\beta_r^2)}$$

with some nonzero constant $C(a) = C(b, \theta_1, \dots, \theta_R, \alpha_1, \dots, \alpha_R, \beta_1, \dots, \beta_R)$. It is required that $\operatorname{Re}\alpha_r > -1/2$ for all r, which guarantees that a is in $L^1(0, 2\pi)$ and hence has well-defined Fourier coefficients. The assumption that $|\operatorname{Re}\beta_r| < 1/2$ for all r is a basic case of the conjecture. It avoids certain unpleasant ambiguities caused by larger exponents β_r, in particular by the situation where some of the numbers $\alpha_r \pm \beta_r$ are integers.

In special cases, the conjecture was confirmed by A. Lenard and by Fisher and Hartwig themselves. With his 1973 paper [16], Widom was the first to provide a rigorous proof of the conjecture in a sufficiently general case: he proved it under the assumption that $\beta_r = 0$ for all r. Hirschman writes in his review MR0331107 (48#9441):

> The present paper represents a jump of several quanta in depth and sophistication in an area which is not only of great interest to mathematicians, but to theoretical physicists as well.

In fact, Widom's proof is a gigantic piece of mathematical analysis that takes its starting point at a formula by Hirschman, which gives an exact expression for $\det T_n(a)$ in terms of the solutions p_n and q_n in $\mathbf{C}^n$ of the equations $T_n(a)p_n = e_1$ and $T_n(\overline{a})q_n = e_1$, where $e_1 = (1, 0, \dots, 0)^\top$. Widom also proved the conjecture for $R = 1$, $\alpha_1 > -1/2$, $-1/2 < \beta_1 < 1/2$, however, without determining the constant $C(a)$ in this case.

The Fisher–Hartwig conjecture was subsequently confirmed by the first author under the assumption that $\operatorname{Re}\beta_r = 0$ for all r (1978) or that $\alpha_r = 0$ and $|\operatorname{Re}\beta_r| < 1/2$ for all r (1979), by B. Silbermann and the second author in the case where $|\operatorname{Re}\alpha_r| < 1/2$ and $|\operatorname{Re}\beta_r| < 1/2$ for all r (1985), and by B. Silbermann and the third author for $R = 1$, $\operatorname{Re}\alpha_1 > -1/2$, $\beta_1 \in \mathbf{C}$ arbitrary (1996). See [BS2] for precise references. In each case, the constant $C(a)$ was completely identified. It was observed by several authors, for example by Silbermann and the second author already in 1981, that the conjecture is in general no longer true if $\alpha_r \pm \beta_r$ may assume values in $\mathbf{Z} \setminus \{0\}$. A new conjecture, which covers all possible cases, was formulated by Craig Tracy and the first author [BT] in 1991. This new conjecture was proved by the third author [E1] in 1997 in all cases in which it coincides with the original conjecture and by Percy Deift, Alexander Its, and Igor Krasovsky [DIK1] in 2009 in full generality. The entire development from Fisher and Hartwig's 1968 paper up to the present has both demanded and produced great progress in operator theory for Toeplitz and related matrices.

The Fisher–Hartwig conjecture has a continuous analogue for Wiener–Hopf determinants. In the 1983 paper [25] by Widom and the first author, this conjecture was proved for piecewise continuous symbols with a continuous argument, that is, for the case where $\alpha_r = 0$ and $\operatorname{Re}\beta_r = 0$ for all r. The idea of the proof is that Wiener–Hopf determinants when discretized become Toeplitz determinants. Unfortunately, one is led to determinants of the form $\det T_n(a^{(n)})$ in this way. Thus, not only the order of the determinant but also the symbol depend on n. However, sufficiently precise asymptotic results for Toeplitz matrices and determinants eliminate this obstacle. For general piecewise symbols, the continuous analogue of the Fisher–Hartwig conjecture was settled in 1994 in the papers [29] and [30] by Widom, Silbermann, and the second author. These papers are based on another idea. This time it is that Wiener–Hopf operators may be regarded as Toeplitz matrices with operator-valued entries.

Over the years it has become clear that the asymptotic behavior of Toeplitz and Wiener–Hopf determinants with several Fisher–Hartwig singularities can be determined by employing localization techniques, provided one knows the asymptotics for at least one symbol with a single Fisher–Hartwig singularity. In the Toeplitz case, such a symbol is $(1-t)^\gamma(1-1/t)^\delta$ $(t \in \mathbf{T})$ because we have the factorizations

$$|t-1|^{2\alpha} = (1-t)^\alpha(1-1/t)^\alpha,$$
$$\varphi_{\beta,0}(t) = \exp(i\beta\arg(-t)) = (1-t)^\beta(1-1/t)^{-\beta},$$

which gives $(1-t)^\gamma(1-1/t)^\delta$ with $\gamma = \alpha+\beta$ and $\delta = \alpha-\beta$. Both exact and asymptotic formulas for the corresponding Toeplitz determinants were found in 1985 by Silbermann and the second author, and two elementary derivations of these formulas are also in the 2005 paper [34].

In the Wiener–Hopf case, things are dramatically more complicated. Only in 2004, in [33], Widom and the first author were able to prove the predicted asymptotic behavior for the Wiener–Hopf determinants with the symbol

$$\left(\frac{\xi+0i}{\xi+i}\right)^\gamma \left(\frac{\xi-0i}{\xi-i}\right)^\delta \quad (\xi \in \mathbf{R}),$$

still requiring that $\gamma = \alpha+\beta$ and $\delta = \alpha-\beta$ with the real parts of α, β in $(-1/2, 1/2)$. The proof is highly sophisticated. Roughly speaking, it is based on introducing a parameter to regularize the symbol, on applying the Wiener–Hopf analogue of the Borodin–Okounkov formula, which was established in 2003 by Y. Chen and the first author, on considering the quotient of the Wiener–Hopf determinant over $(0, R)$ and an appropriate $n \times n$ Toeplitz determinant, on taking the limit $n, R \to \infty$ with $n/R \to 1$, and on finally returning to the original symbol by passing to the limit that makes the regularization parameter disappear.

In his journey from eigenvalue distribution problems to Szegő's theorem and generalizations for singular symbols, Widom sometimes made an excursion into other more general classes of operators. In a series of papers in the late 1970s, [20, 21, 22, 23], he proved a far-reaching extension of the classical Szegő theorem by developing a symbolic calculus for pseudodifferential operators. The context was general enough to include extensions with variable convolutions, higher dimensions, and general Riemannian manifolds. The applications ranged from the classical theorems in the Toeplitz case to heat expansions for Laplace–Beltrami operators. Many of these results entered his book [26].

Finally, we mention that in the 1995 paper [31] he found the asymptotic expansion of the Fredholm determinant of the sine kernel on a union of intervals. This Fredholm determinant is intimately connected to the eigenvalue distribution problem for random unitary matrices, and that paper is a beautiful bridge between his work in Toeplitz operators and his work in random matrix theory, which is the subject of the article by Corwin, Deift, and Its [CDI] in this issue.

Harold Widom has significantly enriched several fields of mathematics, and we hope we were able to convey an idea of the grand work done by him solely in Toeplitz and Wiener–Hopf operators as well as an idea of the pioneering spirit of the 1960s and of his achievements in those times. We remember with pleasure that his papers were among the first we read at the very beginnings of our careers, and since then up to the present, we count ourselves truly lucky for the many opportunities we had to learn from and to work with him and to benefit from his genius and his personality.

On a final personal note, the first author was privileged to be his graduate student. She was not only inspired by the strength and beauty of his mathematical creations, but also by his sense of integrity and fairness. Harold Widom was the perfect mentor, long before that word was popular. He knew when to give advice and be helpful, knew when to back off and let students mature, and was forever willing to lend an ear and inspire confidence.

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