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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
ELCHANAN MOSSEL

MR0039976 (12,624c) 90.0X

Arrow, Kenneth J.

Social Choice and Individual Values. (English)

Cowles Commission Monograph No. 12.

John Wiley & Sons, Inc., New York, N.Y.; Chapman & Hall, Ltd., London, 1951, xi+99 pp.

This book is concerned with the following problem. A collection of individuals and a set of social alternatives are given and it is assumed that each individual ranks the alternatives in accordance with his preference. Problem: to obtain a "social ordering" of the alternatives as a function of the individual orderings, which will represent the preferences of the community as a whole and which will satisfy certain requirements of compatibility with the preferences of the individuals. The problem is formalized as follows. A set S of alternatives is given. A weak ordering on S is defined to be a relation R which is transitive and such that any two alternatives are comparable. A set of n weak orderings $R_1, \cdots, R_n$ is given corresponding to n individuals. The problem is then to find a function which attaches to each such n-tuple of orderings an ordering R. Such a function is called a "social welfare function," and the ordering R is called the "social ordering." The author now gives a number of requirements which the social welfare function must satisfy, which we paraphrase roughly. (1) If two different sets of individual orderings $R_1, \cdots, R_n$ and $R_1', \cdots, R_n'$ are identical except that a particular alternative x is raised in preference by some of the individuals in the second set of orderings R_i', then this alternative will not be lowered in the corresponding social ordering R'. (2) The relative positions of two alternatives x and y in the social ordering R shall depend only on their relative positions in the individual orderings $R_1, \cdots, R_n$ and not on the positions of alternatives distinct from x and y. A social welfare function is called "imposed" if for some pair of alternatives x and y, xRy for every social ordering R. A social welfare function is called "dictatorial," if there exists an integer i, $1 \leq i \leq n$, such that for any $R_1, \cdots, R_n$, the social ordering R is the same as R_i.

The main result of the book can now be stated (General Possibility Theorem): If S contains more than two alternatives then any social welfare function satisfying the first and second conditions must be either imposed or dictatorial. A similar theorem is also proved for cases where restrictions are placed on the allowable individual orderings $R_1, \cdots, R_n$. A large portion of the book is taken up with giving economic justifications for the various axioms and conditions used in setting up the problem. However, the argument for (2) is not convincing. The following simple example may illustrate the difficulty. Two individuals are ranking 100 alternatives. Suppose x and y are two alternatives and suppose the first individual ranks x first and y last, the second ranks y first and x second. It then seems reasonable that the social ordering should rank x above y. On the other hand if the first individual ranks x first and y second, while the second ranks y first and x last the same reasoning would rank y above x in the social ordering. However, the author's second condition requires that x must also be ranked above y in this second case, which seems to contradict common sense. Thus if one accepts the author's remark that the result of the main theorem is "paradoxical" it would seem that paradoxes are already evident in his basic assumptions.

D. Gale
From MathSciNet, May 2022

MR1410265 (98d:90040) 90A28; 90-02, 90A08

Saari, Donald G.

Basic geometry of voting. (English)

Springer-Verlag, *Berlin*, 1995, xii+300 pp., \$39.00, ISBN 3-540-60064-7

This book was planned to be, according to the author, a student version of his previous book, *Geometry of voting* [Springer, Berlin, 1994; MR1297124]. However, it contains many previously unpublished recent results. I briefly recall Saari's main insight. Basically, voters rank candidates with no ties, i.e., they have a linear ordering over the candidates. We have then a list of linear orderings (one linear ordering for each voter), called a profile in the social choice literature. The information contained in such a profile, once a symmetry assumption is made on the voters' power—each voter has the same power—can be described in the following way. Since there are k candidates, there are $k!$ linear orderings. We can consider the proportion of voters whose ranking is any of these $k!$ rankings. Then, we have a $k!$-vector of rational numbers which can be approximated by a $k!$-vector in the $k!$-dimensional Euclidean space (in fact in its simplex). Consequently, we obtain a geometrical representation of the space of profiles. A voting rule associates to each profile a social outcome, which can be either a ranking of the candidates, a social preference (a binary relation which is a complete preorder or any other binary relation), or a social choice (a nonempty subset of the set of candidates). Saari's work has been mainly concerned with scoring rules where the social outcome is obtained on the basis of the summation of scores attributed to candidates according to their ranks in the voters' linear orderings (given a voter's ranking, Borda's rule, for instance, attributes scores such that the difference between a top-ranked candidate and a second-ranked candidate is the same as, say, between the sixth-ranked candidate and the seventh-ranked candidate, and this difference is the same for all voters).

In this book, compared with *Geometry of voting*, Saari has dropped the sections that were primarily of interest to experts, reorganized the presentation, and added exercises. He also introduces new material, mainly in Chapter 4, about Borda and Condorcet, and profile decomposition.

This reviewer also wants to draw the readers' attention to recent developments by Saari, only alluded to in this book (page 47). They concern difficulties due to assumptions such as independence of irrelevant alternatives in Arrow's theorem and minimal liberalism in Sen's analysis of the conflict between Paretianism and liberalism [see D. G. Saari, Math. Mag. **70** (1997), no. 2, 83–92 MR1448881; "Connecting and resolving Sen's and Arrow's theorem", Soc. Choice Welf., to appear]. My review of *Geometry of voting* was enthusiastic. A last comment: If you have a copy of it, get a copy of *Basic geometry of voting* also; if you haven't, get both.

Maurice Salles
From MathSciNet, May 2022

MR1942631 (2003h:91046) 91B14; 42C10, 91B08, 91B12

Kalai, Gil

A Fourier-theoretic perspective on the Condorcet paradox and Arrow's theorem.

Advances in Applied Mathematics **29** (2002), *no.* 3, 412–426.

Suppose that an agent has a strict preference relation over a set of alternatives. Given two alternatives a and b, we can encode the agent's preference using 1 to mean that a is preferred to b, and 0 otherwise. In social choice, the Arrowian axiom of independence of irrelevant alternatives requires that the social ranking of a versus b depends only on the individual rankings for a versus b of the n agents. Hence, social choice functions can be viewed as combinations of Boolean mappings from $\{0,1\}^n$ to $\{0,1\}$.

This paper suggests the use of Fourier expansions of Boolean mappings to study problems in social choice. The approach is illustrated by offering a simple proof for a known inequality regarding the probability of avoiding Condorcet paradoxes. Several related inequalities are obtained exploiting symmetry. A second application invokes symmetry assumptions stronger than usual to prove a special version of Arrow's theorem. Most interestingly, the paper also presents a probabilistic variant of this theorem stating that a neutral social choice function which is very likely to be a strict preference relation must be approximately a dictatorship.

Marco LiCalzi

From MathSciNet, May 2022

MR2044787 2004k:91072) 91B14; 55P20, 55Q05, 57N60

Weinberger, Shmuel

On the topological social choice model.

Journal of Economic Theory **115** (2004), *no.* 2, 377–384.

The author studies the appealing "topological approach to social choice", introduced by G. Chichilnisky and G. M. Heal [J. Econom. Theory **31** (1983), no. 1, 68–87; MR0720115]. Such a model was launched to avoid impossibility problems that arise in social choice, for instance in the Arrowian model introduced in the seminal work of K. J. Arrow [*Social Choice and Individual Values*, Wiley, New York, N. Y., 1951; MR0039976], who proved the famous "impossibility theorem", which states that under very mild and common sense conditions (unrestricted domain (U), weak Pareto principle (P), independence of irrelevant alternatives (I) and nondictatorship (D)), it happens that for a finite number of individuals and at least three distinct social alternatives, there is no social function satisfying the conditions U, P, I and D.

In these topological models, changing the axioms imposed to the aggregation rules in order to get "milder" situations, some "possibility" results appear.

Moreover, combinatorial techniques (as the ones used in the Arrowian approach) can also be "topologically reinterpreted", as shown in a crucial paper on this framework, namely [Yu. M. Baryshnikov, Adv. in Appl. Math. **14** (1993), no. 4, 404–415; MR1246414]. Baryshnikov developed a topological approach, mainly based on concepts coming from Algebraic Topology (e.g., "nerve of a covering", "simplicial complex", etc.) which allowed him to re-prove Arrow's impossibility result.

In the present paper, the author imagines n agents picking elements out of a choice space X. The problem is to give an aggregation of their choices. The "aggregation rule" is here understood as a continuous map $A\colon X^n \to X$ such that $A(x,\dots,x) = x$ $(x \in X)$ (this is the axiom of "respect of unanimity") and $A(x_1,\dots,x_n)$ is independent of the order (this is the axiom of "anonymity"). Chichilnisky and Heal proved that if X is a finite connected cellular CW-complex, it admits an aggregation rule for every $n \in \mathbb{N}$ if and only if X is a contractible space.

In the present paper, the author shows that if a finite connected CW complex admits a continuous, symmetric, and unanimous choice function for some number $n > 1$ of agents, then the choice space is contractible. It is also proved that if one removes the finiteness, some other spaces that allow "possibility results" appear; moreover, some of those new spaces are noncontractible. Thus, these results can be considered as an extension of earlier well-known results due to Chichilnisky and Heal.

However, I think that the kind of results achieved in that paper are not new. Some techniques based on "very old" problems arising in Algebraic Topology, such as the question of defining a "generalized mean" (see, e.g. [G. Aumann, Math. Ann. **109** (1933), 235–253; Zbl 0008.05601; Math. Ann. **111** (1935), 713–730; Zbl 0012.25205; Math. Ann. **119** (1944), 210–215; MR0012219; B. Eckmann, Comment. Math. Helv. **28** (1954), 329–340; MR0065920], could lead to similar results.

Therefore, the merit of this paper, a merit also shared by recent researchers on mathematical social choice, consists in realizing that those purely abstract results arising in topology can indeed be applied to prove deep results on aggregation of individual preferences to give a social one.

Esteban Induráin

From MathSciNet, May 2022

MR2115054 (2005k:91107) 91B14; 91A80, 91B02

Serrano, Roberto

The theory of implementation of social choice rules.

SIAM Review **46** (2004), *no.* 3, 377–414.

This paper surveys the static theory of implementation of social choice rules, concentrating on implementation in the following three solution concepts: dominant strategies, Nash equilibrium, and Bayesian Nash equilibrium. The following three results are reviewed, among others. The first is the non-implementability of non-dictatorial social choice rules in dominant strategies, the second is the impossibility of Nash implementation of non-monotonic social choice rules and the third is the virtual Nash implementability of every ordinal social choice rule. Proofs of all propositions stated are provided. The paper contains several illuminating examples. The final section discusses some other related topics.

Let N be a finite set of agents, $\mathcal{T}$ the (possibly infinite) set of states, and A the finite set of social alternatives. At each state t, each agent i is assumed to have a complete and transitive preference relation $\succeq_i^t$. For each state t, $\succeq^t = (\succeq_i^t)_{i\in N}$ is the preference profile at t. It is assumed that, for each agent i, there is a real-valued function u_i on $A \times \mathcal{T}$ such that, for each $t \in \mathcal{T}$ and each $a, b \in A$, $u_i(a,t) \geq u_i(b,t)$ if and only if $a \succeq_i^t b$. By Δ denote the set of probability distributions on A.

$\mathcal{T}$ satisfies no-total-indifference if for every $i \in N$ and every $t \in \mathcal{T}$ there is a pair a, a' of elements of A such that $a \succ_i^t a'$. $\mathcal{T}$ is independent if $\mathcal{T}$ has the structure of a Cartesian product, that is, $\mathcal{T} = \prod_{i \in N} \mathcal{T}_i$, where $\mathcal{T}_i$ is the set of types for agent i. Then for each agent i and each type t_i in $\mathcal{T}_i$ there is a preference relation $\succeq_i^{t_i}$ on A and $\succeq^t = (\succeq_i^{t_i})_{i \in N}$ for each $t = (t_i)_{i \in N} \in \mathcal{T}$. When $\mathcal{T}$ is independent, denote by $\mathcal{T}^S$ the set of states t such that $a \sim_i^{t_i} b$ if and only if $a = b$ for every $i \in N$ and every $a, b \in A$.

A social choice rule (resp. random social choice rule) is a mapping of $\mathcal{T}$ into $2^A \backslash \{\emptyset\}$ (resp. into $2^\Delta \backslash \{\emptyset\}$) and a social choice function (resp. random social choice function) is a single-valued social choice rule (resp. single-valued random social choice rule). A mechanism (resp. random mechanism) is an ordered pair $\Gamma = ((M_i)_{i \in N}, g)$ such that M_i is a message set for agent i and g is an outcome function of $\prod_{i \in N} M_i$ into A (resp. into Δ). The ordered pair (Γ, t) of a mechanism Γ and a state t can be regarded as a game. Given a game (Γ, t) and a game-theoretic solution concept $\mathcal{S}$, denote by $\mathcal{S}(\Gamma, t)$ the set of strategy profiles that are recommended by $\mathcal{S}$ in (Γ, t). A social choice rule F is said to be $\mathcal{S}$-implementable if there is a mechanism Γ such that $g(\mathcal{S}(\Gamma, t)) = F(t)$ for every $t \in \mathcal{T}$.

A social choice rule F is (a) ordinal if for every $t, t' \in \mathcal{T}$ such that $F(t) \neq F(t')$ there is some $i \in N$ and some $a, b \in A$ such that $a \succeq_i^t b$ and $b \succ_i^{t'} a$, (b) unanimous if $a \in F(t)$ for every $a \in A$ and every $t \in \mathcal{T}$ such that $a \succeq_i^t b$ for every $i \in N$ and every $b \in A$, (c) no-veto if $a \in F(t)$ for every $a \in A$ and every $t \in \mathcal{T}$ such that there is some $j \in N$ such that $a \succeq_i^t b$ for every $i \in N \backslash \{j\}$, (d) monotonic if $a \in F(t')$ for every $a \in A$ and every $t' \in \mathcal{T}$ such that there is some $t \in \mathcal{T}$ such that $a \in F(t)$ and $a \succeq_i^{t'} b$ for every $b \in A$ and every $i \in N$ such that $a \succeq_i^t b$, and (e) dictatorial if there is some $i \in N$ such that $a \succeq_i^t b$ for every $b \in A$ and every $t \in \mathcal{T}$ such that $a \in F(t)$.

Proposition 1. Every social choice rule which is implementable in some solution concept is ordinal.

Let (Γ, t) be any game. A message $\widehat{m}_i$ is a dominant strategy for i if $g(\widehat{m}_i, m_{-i}) \succeq_i^t g(m_i', m_{-i})$ for every $m_i' \in M_i$ and every $m_{-i} \in M_{-i} = \prod_{j \in N \backslash \{i\}} M_j$. Let $\mathcal{D}(\Gamma, t) = \prod_{i \in N} \mathcal{D}_i(\Gamma, t)$, where $\mathcal{D}_i(\Gamma, t)$ is the set of dominant strategies for i. A social choice rule R is implementable in dominant strategies if there is a mechanism Γ such that $g(\mathcal{D}(\Gamma, t)) = F(t)$ for every $t \in \mathcal{T}$. When $\mathcal{T}$ is independent, a social choice function f is (a) strategy-proof if $f(t_i, t_{-i}) \succeq_i^{t_i} f(t_i', t_{-i})$ for every $i \in N$, every $t_{-i} \in \mathcal{T}_{-i}$, and every $t_i, t_i' \in \mathcal{T}_i$ and (b) truthfully implementable in dominant strategies if, for every $t \in \mathcal{T}$ and every $i \in N$, t_i is a dominant strategy for i in the game $(((\mathcal{T}_j)_{j \in N}, f), t)$, and a social choice rule F is *truthfully implementable in dominant strategies* if there is a social choice function f that is a selection of F and is truthfully implementable in dominant strategies.

Proposition 2. For every social choice rule F such that $\mathcal{T}$ is independent and $\mathcal{T} = \mathcal{T}^S$, (a) if F is implementable in dominant strategies then F is single-valued and is truthfully implementable in dominant strategies and (b) if F is truthfully implementable in dominant strategies then every truthfully implementable selection of F is implementable in dominant strategies.

Proposition 5 (Gibbard-Satterthwaite). If $|A| \geq 3$ and $\mathcal{T}^S \subset \mathcal{T}$ then every strategy-proof social choice function f onto A is dictatorial.

Since strategy-proofness is necessary for truthful implementability in dominant strategies, Propositions 2 and 5 imply that there are no non-dictatorial social choice rules implementable in dominant strategies.

A strategy profile m^* in $\prod_{i\in N} M_i$ is a Nash equilibrium if, for every $i \in N$, $g(m^*) \succeq_i^t g(m_i, m_{-i}^*)$ for every $m_i \in M_i$. The set of Nash equilibria is denoted by $\mathcal{N}(\Gamma, t)$.

Propositions 6–7 (Maskin). For every social choice rule F, (a) if F is Nash implementable then F is monotonic and (b) if $|N| \geq 3$ and F is monotonic and satisfies no-veto then F is Nash implementable.

A random social choice rule F is (a) virtually Nash implementable if for every $\epsilon > 0$ there is a random mechanism Γ such that for every $t \in \mathcal{T}$ there is a bijection π_t of $F(t)$ onto $g(\mathcal{N}(\Gamma, t))$ such that $\sup_{p\in F(t)} \|p - \pi(p)\| < \epsilon$ and (b) ordinal if for every $t, t' \in \mathcal{T}$ there is some $i \in N$ and some $p, p' \in \Delta$ such that $u_i^*(p, t) \geq u_i^*(p', t)$ and $u_i^*(p', t') > u_i^*(p, t')$, where $u_i^*(p, t) = \sum_{a\in A} p(a)u_i(a, t)$ for each $i \in N$, each $p \in \Delta$, and each $t \in \mathcal{T}$.

Proposition 8 (Matsushima-Abreu-Sen). If $|N| \geq 3$ and $\mathcal{T}$ satisfies no-total-indifference then every ordinal random social choice rule is virtually Nash implementable.

To consider implementation in Bayesian Nash Equilibrium, suppose $\mathcal{T}$ is independent. It is assumed that for each $i \in N$ there is a probability distribution q_i on $\mathcal{T}$ such that for every $t_i \in \mathcal{T}_i$ there is some $t_{-i} \in \mathcal{T}_{-i}$ such that $q_i(t_i, t_{-i}) > 0$. For each $i \in N$ denote by T_i^* the set of $t \in \mathcal{T}$ such that $q_i(t) > 0$. It is further assumed that there is a subset $\mathcal{T}^*$ of $\mathcal{T}$ such that $\mathcal{T}^* = \mathcal{T}_i^*$ for every $i \in N$. For each random social choice function f, each $i \in N$, and each $t_i \in \mathcal{T}_i$, define $U_i(f, t_i'|t_i) = \sum_{t_{-i}\in\mathcal{T}_{-i}} q_i(t_{-i}, t_i)u_i^*(f(t_{-i}, t_i'), (t_{-i}, t_i))$ for each $t_i' \in \mathcal{T}_i$. A random social choice function f is equivalent to a random social choice function h if $f(t) = h(t)$ for every $t \in \mathcal{T}^*$. A Bayesian Nash equilibrium of the incomplete information game $(\Gamma, \mathcal{T})$ is a strategy profile $\sigma^* = (\sigma_i^*)_{i\in N}$, where σ_i^* is a function of $\mathcal{T}_i$ into M_i such that, for every $i \in N$ and every $t_i \in \mathcal{T}_i$, $U_i(g \circ \sigma^*, t_i|t_i) \geq U_i(g \circ (\sigma_{-i}^*, \sigma_i), t_i|t_i)$ for every function σ_i of $\mathcal{T}_i$ into M_i. For each random mechanism Γ, denote by $\mathcal{B}(\Gamma)$ the set of Bayesian Nash equilibria of the incomplete information game $(\Gamma, \mathcal{T})$. A random social choice function f is Bayesian implementable if there is a random mechanism Γ such that $g(\mathcal{B}(\Gamma))$ is equivalent to f. A random social choice function f is incentive compatible if, for every $i \in N$ and every $t_i \in \mathcal{T}_i$, $U_i(f, t_i|t_i) \geq U_i(f, t_i'|t_i)$ for every $t_i' \in \mathcal{T}_i$. A collection $\alpha = (\alpha_i)_{i\in N}$ is a deception if, for every $i \in N$, α_i is a mapping of $\mathcal{T}_i$ into itself and α_i is not the identity mapping for at least one $i \in N$. A random social choice function f is monotonic if for every deception α such that $f \circ \alpha$ is not equivalent to f there is some $i \in N$, some $t_i \in \mathcal{T}_i$, and some random social choice function y such that $U_i(y \circ \alpha, t_i|t_i) > U_i(f \circ \alpha, t_i|t_i)$ and $U_i(f, t_i'|t_i') \geq U_i(y_{\alpha_i(t_i)}, t_i'|t_i')$ for every $t_i' \in \mathcal{T}_i$, where $y_{\alpha_i(t_i)}$ is a random social choice function, defined by $y_{\alpha_i(t_i)}(t') = f(t_{-i}', \alpha_i(t_i))$ for each $t' \in \mathcal{T}$.

Proposition 9-10. For every Bayesian implementable random social choice function f, (a) there is an incentive compatible random social choice function $\widehat{f}$ which is equivalent to f and (b) there is a Bayesian monotonic random social choice function $\widehat{f}$ which is equivalent to f.

Considering economic environments, the author shows that a social choice rule is Nash implementable if and only if it is monotonic (Corollary 1) and presents a necessary and sufficient condition for virtual Bayesian implementability of a random social choice function (Proposition 12).

Hiroaki Osana
From MathSciNet, May 2022

MR3242898 91B14; 62A01, 91B06

Dietrich, Franz; Bradley, Richard; List, Christian

Aggregating causal judgments.

Philosophy of Science **81** (2014), *no.* 4, 491–515.

Causal judgments appear in many decision contexts. Jurors, for instance, investigate whether a given defendant did in fact cause damage to a plaintiff; aid agencies allocate resources to projects that cause poverty to be alleviated. Sometimes a decision maker is confronted with differing causal judgments from multiple sources; she then has to aggregate those causal judgments into a single one. It is such an aggregation of causal judgments that the paper under review studies.

The formalization of causal judgments in this paper follows J. Pearl [*Causality*, Cambridge Univ. Press, Cambridge, 2000; MR1744773] and P. Spirtes, C. Glymour and R. Scheines [*Causation, prediction, and search*, second edition, Adapt. Comput. Mach. Learn., MIT Press, Cambridge, MA, 2000; MR1815675]. Thus the set of causal judgments reported by each individual source is modelled as a Bayesian network. The authors distinguish between (what they call) a one-stage and a two-stage approach to the aggregation of causal judgments (modelled as Bayesian networks).

The one-stage approach is based on probabilistic opinion pooling: The individual Bayesian networks are each assumed to be derived from a probability measure; these individual probability measures then have to be merged, while satisfying certain responsiveness axioms such as *Zero Preservation* and *Event-Wise Independence*, known under slightly different names (viz. "Zero Probability Property" and "Weak Setwise Function Property") from the work of K. J. McConway [J. Amer. Statist. Assoc. **76** (1981), no. 374, 410–414; MR0624342], and a new property called *Independence Preservation*. The authors prove that in general Independence Preservation is inconsistent with the conjunction of Zero Preservation and Event-Wise Independence: all linear opinion pools violate Independence Preservation.

The two-stage approach is much more sophisticated and involves the aggregation of both qualitative causal judgments (causal relevance relations) and quantitative causal judgments (probability measures). The first stage itself consists of several steps: Initially, from each Bayesian network the causal relevance relation is deduced—that is, the assertions of negative or positive relevance or irrelevance of one variable for another, i.e. the directed edges of the Bayesian network. Then one is faced with the problem of aggregating these causal relevance relations. In the light of recent results from judgment aggregation theory, the aggregation rule for the causal relevance relations cannot simultaneously satisfy certain natural desiderata, viz. Universal Domain, Acyclicity, Non-Dictatorship and a new condition called *Unbiasedness*. Thus, one of these desiderata has to be dropped, and the authors provide a detailed discussion of various options to do so.

The second stage of the two-stage approach concerns the aggregation of the individual probability measures (quantitative causal judgments). In order to avoid the impossibility theorem for the first-stage approach, the problem is now addressed differently. From the causal relations within the Bayesian network, a causal ordering of the variables is derived, and the resulting sequence of random variables is assumed to be Markovian. Under this assumption, the authors provide a procedure for merging the probability assignments.

Frederik S. Herzberg
From MathSciNet, May 2022

MR3767233 91-02; 91A05, 91A22, 91B12, 91B14

Saari, Donald G.

Mathematics motivated by the social and behavioral sciences. (English)

CBMS-NSF Regional Conference Series in Applied Mathematics, 91.
Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2018, xvii+171 pp., ISBN 978-1-611975-17-8

Donald G. Saari's first book on mathematical applications to social sciences, in general, and to politics and economics, in particular, appeared more than a quarter century ago. It was a thought-provoking account of how geometrical ideas can enrich our understanding of voting procedures [*Basic geometry of voting*, Springer, Berlin, 1995; MR1410265]. Its background was the author's long-time interest in dynamical systems theory. It presented a new approach to the study of how institutions convert inputs into outputs, what is possible, what is impossible, which kinds of anomalies can be encountered and what are the causes of those anomalies. Over the decades since the publication of that book, Saari has systematically extended the coverage of the approach to include decisions, games and power, all fundamental elements of social, economic and political theory.

In addition to a multitude of articles, he has published four book-length treatises in the field: *Chaotic elections!* [Amer. Math. Soc., Providence, RI, 2001; MR1822218], *Decisions and elections* [Cambridge Univ. Press, Cambridge, 2001; MR2006479], *Disposing dictators, demystifying voting paradoxes* [Cambridge Univ. Press, Cambridge, 2008; MR2449532] and—most recently—the book under review. All these books share a common effort of their author to build bridges between the purely theoretical results and their intended fields of application. Although a mathematician by training, Saari strives to make abstract ideas accessible and useful to students (sometimes very young ones), teachers, interested citizens and practitioners. The book under review here continues along this path. In contradistinction to most of his previous books, Saari now primarily writes for his mathematician colleagues and students. The book is based on his lectures at the Conference Board of the Mathematical Sciences sponsored conference held in 2012. Most of the material is based on Saari's earlier works, which provide more detailed technical presentations of the theorems and proofs.

Most economists and political scientists know Saari for his geometrical approach to voting, but the new book starts from elementary games and develops this theme into a sketch of an evolutionary theory of games. In the first chapter the reader is guided from the familiar terrain of elementary 2×2 games and their local equilibrium analysis to more complex environments with multiple games and associated equilibria. The role of norms as guidelines of behavior is discussed in terms of equilibrium notions, and the crucial question of what brings about norm changes and the associated movements from one equilibrium to another is raised. The question is first approached using the standard techniques of dynamical systems theory, i.e., viewing elementary games as constituents of a dynamical system in which some laws of change are assumed to hold. Saari finds this approach suspicious in that it assumes what is to be found out. The functions (laws) characterizing the changes in macro systems are not in general known in social sciences. Hence he suggests a procedure whereby local data are utilized in interaction with tentative macro models. Complexity reduction is the key notion here. The approach is illustrated with examples from one-dimensional two-party contests where each party strives for total dominance. Another example used in the discussion of equilibrium types is the ultimatum game with several kinds of players (selfish, fair, selfless). The chapter provides an outline of a research strategy that combines the strengths of dynamical systems thinking and evolutionary game-theoretic reasoning.

The second chapter deals with voting, in general, and paradoxes related to voting, in particular. The chapter starts with a discussion of Oscar nominations and the criticism of the often used plurality and plurality runoff procedures. It then invokes geometry as a tool of describing, analyzing and explaining voting paradoxes. The reader of Saari's earlier works recognizes the main features of the reasoning. The chapter provides an overview of the much debated issue of the virtues of the Borda count especially in relation to the Condorcet winner (i.e., a candidate that—according to the profile of reported preferences of voters—would beat all other candidates in pairwise majority of voting). Saari has shown that among the positional methods the Borda count enjoys many desirable properties. Some of these are presented in the chapter. Towards the end of the chapter Saari takes up the all-important issue of how often we could expect to be faced with various kinds of voting paradoxes. This of course depends on what kinds of profiles will turn up in the future. As this information is typically not available, computer scientists, statisticians and mathematicians have studied the likelihood of various phenomena (the presence of Condorcet winners, identical choices resulting from several specific procedures, Borda paradoxes, etc.) under various types of profiles. In the second chapter Saari gives a brief overview of his contributions to this literature. Of particular interest is his result (jointly with M. Tataru) which states that, in closely contested three-candidate elections and under reasonable probability assumptions on profiles, the probability of the result depending solely on the chosen positional procedure in large electorates is about 70%. A distinctive feature in Saari's work on probabilistic social choice is the combination of geometry with probability calculations.

Chapter 3 extends the ideas developed in the voting context to the study of market equilibrium. Resorting to his results in voting theory, Saari questions the validity of the market equilibrium story maintaining that, much like in voting, with enough market actors almost anything can happen. The discussion then turns to other, non-positional voting rules and, in particular, to Copeland's, Dodgson's and Kemeny's rules. The first one is based on simply counting, for each candidate, the number of others it defeats in pairwise majority comparisons and then electing the candidate who defeats more competitors than any other candidate. Dodgson's method determines, given a profile of preferences, which candidate can be made the Condorcet winner with the smallest number of binary preference switches by the voters and picks that candidate as the winner. Kemeny's rule, finally, first generates all possible strict rankings of the candidates and then compares each constituent pair of the ranking with the same pair in individual preferences tallying the number of agreements. Summing the agreements over all voters gives the score of each ranking. The ranking with the largest score is then the Kemeny winner. All three rules are Condorcet extensions, i.e., result in the Condorcet winner when one exists. Using his earlier work Saari examines the similarity of these rules with the Borda count. This chapter also deals with the approval voting method, which allows each voter to give one or zero votes for each candidate, whereupon the candidate with the largest sum of votes wins. Saari shows that approval voting can lose any link at all between the observed preference profile and the voting outcome, i.e., profiles exist where an arbitrary ranking may result in any subset of candidates. An area that is only very briefly touched upon is multiple-criteria decision making where one would expect that Saari's voting theory has much to offer. Replacing voters with criteria of performance would provide a straightforward translation of voting theory into individual decision analysis. Regrettably, this chapter gives only a cursory view on this area of inquiry. The same is true of Simpson's paradox, an area in which Saari has published many works.

While the first three chapters mainly deal with exposing the strange behavior of voting rules and the conditions under which paradoxes can be expected to occur, Chapter 4 opens a new, explanatory angle to voting paradoxes. The main instruments of explaining are profile decompositions, i.e., representations of profiles as aggregates of sub-profiles, each with a significant role to play in determining the voting outcomes under various rules. The chapter repeats Saari's (in the reviewer's opinion) compelling argument for the Borda count in comparison with other positional procedures and, perhaps more importantly, with Condorcet extensions. The argument is based on the role of various profile components and shows that methods based on mere pairwise comparisons lose essential information in preference aggregation. The Borda count, in contradistinction, does not. Moreover, profile components that for all practical purposes should have no impact on the voting outcomes in fact do have such an impact when Condorcet extensions are resorted to, while the Borda count is immune to such influences.

Chapter 5 returns to what the book started from, viz., the analysis of games. It describes Saari's more recent work (with D. Jessie) on game decompositions and aims at an analysis of the matrix form games similar to the one applied to preference profiles in the preceding chapter. The decomposition applied consists of a Nash component or a portion which represents what each player can unilaterally achieve by the strategy choice, the behavioral component that indicates what payoff benefits each player can achieve through cooperation with others and the kernel which is a factor scaling all payoffs up or down. As such the last-mentioned component does not affect the strategic nature of the interaction underlying the game, but merely defines its significance for the players. Saari discusses the role of each component in games. It seems that more work on the behavioral component could significantly deepen our understanding of the reasons for why there often seems to be a discrepancy between what the standard game theory predicts and what people choose in game-like settings.

The final chapter takes a more general view of the topics discussed earlier and examines what is called the reductionistic approach to problem solving: given a problem, divide it into several smaller, more manageable parts, provide solutions to these component problems and unify them into a solution to the original one. Saari states a theorem—proved in his 1995 book—which turns out to be more general than Arrow's impossibility theorem and argues that no rule based on pairwise comparisons of candidates (or policy alternatives) is fair. In other words, as a general strategy the reductionist approach is bound to lead to aggregation paradoxes. Saari then moves on to explain why the reductionist strategy sometimes works. The chapter ends with a brief discussion on reductionism in the study of dark matter, thus connecting the theory of voting with astronomy. A grand finale, one could say.

The book is a perfect text for mathematically oriented scholars and students interested in how mathematics can be applied to political science, sociology and economics, but more importantly it also enlightens the reader about how the just-mentioned disciplines and their fundamental research foci can give rise to novel developments in mathematics. We know Saari as a pioneer of using geometric analysis in analyzing and explaining anomalies and paradoxes in social choice. In this book he goes a step further by demonstrating that empirical social sciences can also provide incentives to invent mathematical tools for solving perennial research problems that transcend disciplinary boundaries.

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A concept of sincerity for combinatorial voting.

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A problem in election mathematical models is the large number of Nash equilibria, since any profile where no player is pivotal (hence practically any profile in a large election) is a Nash equilibrium.

In a binary election, this issue is easily solved by sincere voting under majority rule. When there are more options, the problem is more tricky, and several Nash equilibrium refinements have been proposed in the literature [M. L. Balinski and R. Laraki, *Majority judgment*, MIT Press, Cambridge, MA, 2010; MR2767065; R. B. Myerson, J. Econom. Theory **103** (2002), no. 1, 219–251; MR1889750; R. B.Myerson and R. J. Weber, Amer. Polit. Sci. Rev. **87** (1993), no.1, 102–114, doi:10.2307/2938959]. If there is no general consensus on which concept is the most appropriate, there is some agreement that players should not use weakly dominated strategies.

This paper introduces an equilibrium refinement in elections having agents voting simultaneously on several binary issues. If voters have separable preferences over issues, voting honestly issue by issue is a weakly dominant strategy if majority rule is used issue by issue (as this paper assumes). When preferences are not separable, the authors propose issue sincerity.

More formally, suppose there are $N = \{1, \ldots, n\}$ voters and K issues (where $n \geq 3$ is an odd number). The strategy set $S_i = \{0, 1\}^K$ of voter $i \in N$ is a K-dimensional binary vector where $s_i^k \in \{0, 1\}$ is the vote of player i on issue k, where 0 means "vote for k" and 1 means "vote against k". An issue is adopted if and only if a strict majority of voters votes for it.

Each player i has a strict preference relation $\succ_i$ over 2^K, the set of all possible outcomes (e.g., the set of adopted issues).

Definition: $s_i \in S_i$ is issue sincere with respect to $K' \subset K$ if:
(i) for every $k \in K'$, $s_i^k = 1$ if and only if $K' \succ_i K'/\{k\}$, and
(ii) for every $k \notin K'$, $s_i^k = 1$ if and only if $K' \cup \{k\} \succ_i K'$.

$K' \subset K$ is outcome sincere if it is stable under issue sincerity, and it is an outcome sincere equilibrium if it is a Nash equilibrium under the (unique) strategy profile $s = (s_1, \ldots, s_n)$, under which it is issue sincere.

Thus, given $K' \subset K$, a voter votes for $k \in K'$ if and only if he prefers K to $K'/\{k\}$, and he votes for $k \notin K'$ if and only if he prefers $K' \cup \{k\}$ to K'.

This definition looks very natural. The main questions addressed in the paper are conditions of existence of a sincere outcome and the link with equilibrium refinement.

Unfortunately, a first example, with two issues $\{A, B\}$ and 3 players, shows that issue sincere outcomes may not exist:

$$\emptyset \succ_1 B \succ_1 AB \succ_1 A,$$
$$A \succ_2 \emptyset \succ_2 B \succ_2 AB,$$
$$AB \succ_3 A \succ_3 B \succ_3 \emptyset.$$

A first result provides a characterization showing that an outcome is sincere if and only if it is a *local Condorcet winner*, i.e., that for every $k \in K'$ a majority prefers K' to $K'/\{k\}$ and also, for $k \notin K'$, a majority prefers K to $K' \cup \{k\}$. If K' is a Condorcet winner, it is obviously local Condorcet. Surprisingly, existence of a Condorcet winner does not imply Nash equilibrium, as the following example shows:

$$\emptyset \succ_1 AB \succ_1 A \succ_1 B,$$
$$B \succ_2 AB \succ_2 A \succ_2 \emptyset,$$
$$A \succ_3 AB \succ_3 B \succ_3 \emptyset.$$

AB is a Condorcet winner, but the issue profile inducing it (player 1 votes for A and B, player 2 votes only for B and player 3 votes only for A) is not a Nash equilibrium because 1's unique best reply is to vote against A and B, which induces his best outcome: $\emptyset$. This is because A is pivotal in two issues.

A sufficient condition of existence of an issue sincere Nash equilibrium is existence of a local Condorcet winner (equivalent to sincere issue) and absence of a player who is pivotal in more than one issue in the corresponding issue sincere strategy profile.

The paper goes on and provides a link with a refinement of Nash equilibria introduced by Selten: perfect equilibrium. In this concept, players are supposed to make mistakes and the equilibrium is perfect if there is a sequence of mistakes with full support against which all players are best replying.

Translating this idea to this combinatorial voting game, the authors argue that mistakes on two or more issues should be much less likely than mistakes on one issue. This leads them to introduce b-perfectness (mistakes happen independently issue by issue). This is natural also and is shown to be equivalent to sincere issue Nash. The result is not surprising, but it clarifies the question of non-existence because Nash equilibria in behavioral strategies may not exist in extensive form games without perfect recall (Wichard, 2008).

Interestingly, issue sincere Nash equilibria are robust against an associated iterated reduction process. Finally, and not surprisingly, it is proved that being pivotal for more than one issue is very unlikely in large elections (using a replica model).

All in all, the paper proposes an interesting refinement of Nash equilibrium in multi-issue binary elections. Unfortunately, issue sincere outcomes (and so also equilibria) do not exist in all combinatorial voting games and, even in large elections, their existence is conditional on the existence of a local Condorcet winner. This is a drawback of this notion because refinements should always exist.

The authors may argue that pivotality in more than one issue is very unlikely in large elections, but in that case, even pivotality on one issue is unlikely, too.

An important question not addressed however is the existence of a local Condorcet winner. It would have been interesting to give domain conditions under which they exist, as is usually the case in social choice theory [C. D. Puppe and A. M. Slinko, Econom. Theory **67** (2019), no. 1, 285–318; MR3905426]. Another possibility is the calculation of the probability of existence of a local Condorcet winner; see, for example, [Y. Balasko and H. Crès, J. Econom. Theory **75** (1997), no. 2, 237–270; MR1470590] regarding super-majority rule.

Finally, this equivalence between local Condorcet winner and issue sincerity holds only because majority rule is used issue by issue. There are other voting methods to aggregate in multi-issue elections and it would have been interesting if the authors had discussed how their notion and results apply if one uses other voting methods such as the median rule in the judgment aggregation literature [C. List and C. Puppe, in *The handbook of rational and social choice*, 457–482, Oxford Univ. Press, Oxford, 2009; MR2599282; K. Nehring and M. Pivato, "The median rule in judgement aggregation", MPRA Paper No. 84258, 2018].

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