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SELECTED MATHEMATICAL REVIEWS

concerning publications of
YURI I. MANIN

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Manin, Yu. I.

A course in mathematical logic. (English)

Translated from the Russian by Neal Koblitz.

Graduate Texts in Mathematics, Vol. 53.

Springer-Verlag, New York-Berlin,, 1977, xiii+286 pp., ISBN 0-387-90243-0

As one might expect from a graduate text on logic by a very distinguished algebraic geometer, this book assumes no previous acquaintance with logic, but proceeds at a high level of mathematical sophistication. Chapters I and II form a short course. Chapter I is a very informal introduction to formal languages, e.g., those of first order Peano arithmetic and of ZFC set theory. Chapter II contains Tarski's definition of truth, Gödel's completeness theorem, and the Löwenheim-Skolem theorem. The emphasis is on semantics rather than syntax. Some rarely-covered side topics are included (unique readability for languages with parentheses, Mostowski's transitive collapse lemma, formalities of introducing definable constants and function symbols). Some standard topics are neglected. (The compactness theorem is not mentioned!) The latter part of Chapter II contains Smullyan's quick proof of Tarski's theorem on the undefinability of truth in formal arithmetic, and an account of the Kochen-Specker "no hidden variables" theorem in quantum logic. There are digressions on philosophical issues (formal logic vs. ordinary language, computer proofs). A wealth of material is introduced in these first 100 pages of the book.

The elements of set theory are expounded in an Appendix to Chapter II. Chapters III and IV are devoted to the continuum hypothesis. Chapter III contains the Scott-Solovay Boolean-valued models approach to Cohen's proof of the independence of CH, using the more readily intelligible proof of its independence from a system of axioms for the second order theory of the real numbers as preparation for the more difficult proof of independence from ZFC. The alternative forcing approach is covered in a supplementary section at the end of the chapter. Chapter IV contains Gödel's proof of the consistency of CH. In accord with the attitude of working set theorists, the emphasis is on semantics, constructing models, with syntax, finitist relative consistency proofs, relegated to a section at the end of the chapter. Some minor slips: p. 105 contains the absurd statement that CH is "the only known example" of a naturally-occurring undecidable proposition. It is in fact but the first-discovered of many, the latest being Whitehead's conjecture in group theory. CH is, however, the only example treated in this book, which does not contain Cohen's independence proof for AC, nor Gödel's consistency proof for the existence of non-measurable projective sets. Page 148 mentions the singular cardinals problem as an open question. Work of Silver, Magidor, et al. shortly before the publication of this book has settled it. In several places Frege's inconsistent notion of set—every property determines the set of all things with that property—is inaccurately called "Cantorian". On the whole Chapters III and IV have little connection with the rest of the book.

Chapter V introduces partial recursive functions and the so-called Church's thesis. One novelty is the presentation of some elementary facts in geometrical language in the last section of the chapter. Chapter VI is devoted to Hilbert's tenth problem and the lengthy proof that ($*$) every recursively enumerable set is Diophantine. Perhaps the hardest part of the proof of ($*$) is to produce one example of a function of exponential growth whose graph is Diophantine. In place of Matijasevič's original example using Fibonacci numbers, the author presents an original, simpler example based on Pell's equation. Once ($*$) is established, it is much exploited. For example, the proof of the enumeration theorem for partial recursive functions is based on it. Chapter VI closes with a discussion of Kolmogorov's measure of complexity of recursive functions (length of the shortest program).

Chapter VII is devoted to Gödel's incompleteness theorem. Gödel's proof had three components: (1) arithmetization of syntax, (2) definability of recursive functions in formal arithmetic, (3) self-referential arguments. Since (3) has to some extent been discussed on connection with Smullyan's work in Chapter II, and (2) is immediate from ($*$), Chapter VII is devoted to a detailed exposition of (1), including a careful explanation why in any two reasonable numberings of formulas, the same syntactic operations will correspond to recursive functions. Gödel's theorem is stated semantically: For first order Peano arithmetic or any related system, some true formula is not provable. The semantical fact that all provable formulas are true is used in the author's proof. The purely syntactic version of the theorem and its adaptation to finitely axiomatized fragments of Peano arithmetic are not present. So we do not get Gödel's second incompleteness theorem (on the impossibility of a consistency proof), Church's theorem (on the undecidability of elementary logic), nor the Trahtenbrot-Craig theorem (on the undecidability of the class of formulas valid in finite models). On the other hand, Chapter VII closes with an exposition, rare in logic texts, of Gödel's work on the length of proofs.

The last chapter, Chapter VIII, is devoted to the word problem for groups and Higman's characterization of recursively presented groups. Following Valiev, the author uses ($*$) to simplify the recursion-theoretic parts of the proof. So Chapter VIII consists mainly of detailed group-theoretic arguments.

Given the author's preference for semantics over syntax, it is no surprise that, for example, the Herbrand and Gentzen theorems do not appear in the book. It is more surprising that the standard topics from model theory (preservation theorems, elimination of quantifiers, ultraproducts, omitting of types) which are closely connected with algebra (Birkhoff's theorem, Artin-Schreier theory, the Ax-Kochen theorems, differentially closed fields) are also left out. For a graduate course this text would have to be supplemented from another source. On the other hand, many interesting unorthodox topics are included, and all the material is presented in a fresh way.

John P. Burgess
From MathSciNet, May 2023

MR0785261 (87b:58038) 58F07; 35Q20, 58A50
Manin, Yu. I.; Radul, A. O.
A supersymmetric extension of the Kadomtsev–Petviashvili hierarchy.
Communications in Mathematical Physics **98** (1985), *no.* 1, 65–77.

The authors introduce a radically new object into the theory of (continuous) integrable systems: Lax equations in an odd space-time. In some aspects, their construction can be thought of as both an odd extension and a nontrivial square root of the classical theory. Let K be a commutative superalgebra and let $\theta\colon K \to K$ be an odd derivation of K (e.g., $K = C^\infty(\mathbf{R}^1)[\xi]/(\xi^2)$, $\theta = \partial/\partial\xi + \xi\partial/\partial x$, so that $\theta^2 = \partial/\partial x$). Consider a ("pseudodifferential") operator L of the form (1) $L = \sum_{i=0 \text{ or } -\infty}^{n} u_i\theta^i$, and the associated Lax equations (2) $\partial L/\partial\tau = [P_+, L] = [-P_-, L]$, where P belongs to the centralizer $Z(L)$ of L in $K((\theta^{-1}))$, as in the classical theory, but τ, unlike the classical case, may be an "odd" time. The authors follow the method of fractional powers to study the equations (2), which means two things: (i) P runs over $\{(L^{1/n})^m : m \in \mathbf{N}\}$, and (ii) one deduces (2) from the same equation for $\sqrt[n]{L}$. But here they find an interesting problem: for $n \equiv 0 \bmod 2$, the nth root is not defined. For $n \equiv 1 \bmod 2$ everything is fine, and the authors sketch constructions of the whole gamut of things familiar in the usual (i.e., even) theory: ring of pseudodifferential operators, (super)residues, an infinity of conservation laws, variational derivatives of conservation laws and a sort of supercalculus of variations, a candidate for what may be called the first Hamiltonian structure, etc.

Reviewer's remarks: The paper is easy to understand and I restrict myself only to a few general remarks. The method of fractional powers, although the most direct from the point of view of deriving most of the results in the scalar (even) theory, is known to lack the power to deduce the commutativity of the flows in a purely algebraic fashion, i.e., bypassing the Hamiltonian formalism. This situation also persists in the odd case. Of course, it is not really that important as long as one considers only the scalar operators, as is the case here, but the full (future) theory should also handle the matrix operators, and here one would need to devise an odd extension of Wilson's deep method. (It is really not trivial: the central semisimplicity condition in Wilson's theory becomes hardly recognizable.) This remains, I think, the central problem in the theory; the odd Hamiltonian formalism and the odd variational calculus are not difficult to construct, by employing purely algebraic methods. Finally, when n in (1) is even, the theory should proceed by taking the $(n/2)$th root instead of the nth one (which does not exist). Then one obtains a truly odd extension of the classical theory (if $n = 2m$ then $\theta^n = (\frac{\partial}{\partial x})^m$, etc.), with plenty of new phenomena appearing (such as the disappearance of the second Hamiltonian structure, the odd character of the first Hamiltonian structure, etc.). The field has just started, and it is wide open.

B. A. Kupershmidt
From MathSciNet, May 2023

MR0787979 (86m:32001) 32-02; 14-02, 32L25, 58A50, 81-02

Manin, Yu. I

Kalibrovochnye polya i kompleksnaya geometriya. (Russian)

"Nauka", Moscow,, 1984, 336 pp., 2.70 r.

In the last two decades two new ideas were introduced into mathematical physics. The first of these is the notion of twistor geometry and the other is that of supersymmetry. In this book we find a beautiful blend of developments stemming from these two ideas written by a master expositor who uses the language of algebraic geometry to synthesize and unify the fundamental ideas involved. The fundamental idea of Penrose in his formulation of twistor geometry was to use the space of light-rays in Minkowski space as a new 5-dimensional background space on which to study physical problems. Information can be transformed from this new space (called the "twistor space" in this context) to Minkowski space by a variant of the classical Radon transform, referred to today as the Penrose transform. What appears is that solutions of the classical field equations (both linear and nonlinear varieties, including electromagnetic fields, Yang–Mills fields, gravitational fields, etc.) arise as the Penrose transform of algebraic data on twistor space. There are now a number of variants of this basic idea which have been developed, and these have been described in various places (recent references include books by the reviewer [*Complex geometry and mathematical physics*, Presses Univ. Montréal, Montreal, Que., 1982; MR0654864], R. Penroseand W. Rindler [*Spinors and space-time, Vol. 1*, Cambridge Univ. Press, Cambridge, 1984; MR0776784]; ibid., *Vol. 2*, 1986] and S. A. Huggettand K. P. Tod [*An introduction to twistor theory*, Cambridge Univ. Press, Cambridge, 1985]).

The theory of supersymmetry has evolved from both the point of view of graded Lie algebras and from the more geometric point of view of supermanifolds. Supermanifolds are manifolds which contain local coordinate systems in which some of the variables are commuting and some are noncommuting. This corresponds to the physicists' field theories which have some fields satisfying commutation relations, and others satisfying anticommutation relations. The mathematical theory of supermanifolds has been developed by Berezin, Kostant, Batchelor, Leĭtes, Rogers, the author (and his collaborators in Moscow), Rothstein and others. Recent references include a paper by D. A. Leĭtes [Uspekhi Mat. Nauk **35** (1980), no. 1(211), 3–57; MR0565567] and the collection *Mathematical aspects of superspace* [(Hamburg, 1983), Reidel, Dordrecht, 1984; MR0773076].

The book under review is in two parts. The first part develops twistor theory in algebro-geometric language, and is quite elegantly presented, but with more of an emphasis on the mathematical developments than on the physical field theories. The second part gives a thorough development of projective algebraic geometry in the context of supermanifolds. The supermanifolds developed here are defined sheaf-theoretically, where the structure sheaf is a Z_2-graded algebra modeled on an exterior algebra of a vector space, and where the grading is given by the even and odd elements of the algebra. The theory of super-Grassmannians and superflag-manifolds are developed, and the double fibrations of twistor geometry are carried over to this context. The general procedure of pulling back and taking direct images of vector bundles (the technical definition of the Penrose transform involves these ingredients) is generalized to this supermanifold setting. A fundamental observation of E. Witten [[]Phys. Lett. B **77** (1978), no. 4-5, 394–400] is incorporated into this more general setting, and representation of general Yang–Mills–Higgs fields on super–Minkowski space in terms of "super" vector bundles on the twistor spaces which naturally arise in this context is given. This is the general Penrose transform in this setting, and various examples of explicit geometric situations are worked out in the text.

The book is currently being translated into English, and will be published soon by Springer. It is a very beautiful book with a number of fine points which are not discussed in this review. On the other hand, it will not necessarily be easy to read for someone who is not familiar with the modern language of algebraic geometry or several complex variables, as the author makes full use of the power of this language, and this is not the place to learn about sheaves, cohomology, spectral sequences, etc. The book is highly recommended to those who can benefit from it, and it is an important contribution to the development of this area of research.

Raymond O. Wells Jr.

From MathSciNet, May 2023

MR0797416 (87j:14030) 14Gxx; 11Gxx, 14-02, 32Jxx, 58A50, 58C50, 58E99

Manin, Yu. I.

New dimensions in geometry.

Workshop Bonn 1984 (Bonn, 1984), 59–101, *Lecture Notes in Math., 1111, Springer, Berlin*, 1985.

The present article is a speculative article, drawing on the author's enormous breadth and depth of mathematical culture, and it is difficult, indeed impossible, for the reviewer to do full justice to all its many facets. The article is a "lecture" by the author to the 25th Arbeitstagung in Bonn (Atiyah summarized orally the prepared text reproduced here). The author seeks to pull together several major strands of current developments in geometry, arithmetic and physics, specifically: (i) arithmetic geometry in the sense of Arakelov–Faltings, or "A-geometry"; (ii) Kähler–Einstein metrics on algebraic varieties and their generalizations; (iii) supersymmetry, or graded structures on manifolds.

The paper is naturally divided into three parts: an introduction, three sections on A-geometry, followed by three sections on supermanifolds. The topic (ii) above is laced through the A-geometry section. I will just mention a selection of what the author chooses to review from these various fields, and will rather emphasize reproducing here some of his almost "aphoristic" remarks as a way of trying to entice the reader to peruse his suggestions personally.

The first premise of the paper is that Diophantine problems and physics have forced the enlargement of our concepts of geometry. Put succinctly, one replaces the coordinate ring $\mathbf{R}[x_1, \cdots, x_n]$ of Cartesian n-space by $\mathbf{Z}[x_1, \cdots, x_n; \xi_1, \cdots, \xi_m]$. Here $\mathbf{Z}$ represents the arithmetic aspect of geometry, the x_i are the usual (or "bosonic") geometric variables, and the ξ_j are anticommuting (or "fermionic") variables. The author's first aphorism: "All three types of geometric dimensions are on an equal footing". That the arithmetic and geometric (x_i) variables are of an equal stature goes back about 100 years or more; the equivalence of the x_i and ξ_j is the relatively recent import of supersymmetry in physics. The author proposes simply completing the triangle, and the point of the paper, is, in a way, to review current geometry with an eye towards evaluating whether we are evolving in that direction and to crystallize questions that he feels would help this trend.

A-geometry, the topic of Sections 1–3, seeks to compactify, in a natural way, a $\mathbf{Z}$-scheme by adding a variety at the infinite places. Of course, one knows what the variety is supposed to be, but one wants to add a metric structure at ∞, analogous to the norms at the ∞-places of a number field. In the case of an arithmetic surface, the key ingredient on the curve at infinity is a special metric and its Green's function. The author proposes using Kähler–Einstein metrics on varieties over infinite places, Hermite–Einstein metrics on stable bundles over these varieties, etc. He stops short of speculating on the role of odd variables in this A-geometry. Some of the questions he poses here are the following: (1) Do there exist "groups" mixing the arithmetic and geometric dimensions? (This means, obviously, more than group schemes/$\mathbf{Z}$ or adèle groups.) (2) If one considers Hermite–Einstein bundles as the "obvious" metrization of some coherent sheaves on X_∞, is there a categorical way to generate all coherent sheaves from these semistable ones, and does such a construction still have differential geometric content? (3) Do "canonical examples", such as moduli spaces of vector bundles on a curve, have canonical A-structures? For example, could one describe an A-geometric c_2 for higher rank bundles over the moduli space? (4) What is intersection theory and the Riemann–Roch theorem in this higher-dimensional context?

Some progress in local index theory for families, due to Bismut–Freed, Bost, Gillet–Soulé and others, should hopefully be relevant to this last question.

In the sections on superspace, the author recounts some of the basic definitions (Section 4), and then reports on the results of Vaĭntrob, Skornyakov, Voronov, Penkov and the author himself. These latter concern "super"-analogues of Kodaira–Spencer theory and of the Bruhat decomposition and Schubert cells for complete flag superspaces of classical type. Several interesting differences from the classical (ungraded) case arise, some already highlighted in Kac's representation theory for such algebras. I would again rather record here some of the more aphoristic suggestions and conjectures of the author. (1) (only hinted at) Supergeometry change-of-coordinate formulae involve derivatives of coordinate changes in the transformations of the odd coordinates—what is the role, seemingly forced, of distributions in continuous (as opposed to C^∞) supergeometry? (2) Is it possible to compactify superspaces along the odd directions? For example, Leĭtes asks what the purely odd projective space "$\operatorname{Proj}\mathbf{Z}[\xi_1,\cdots,\xi_n]$" should be. In much the same spirit, the author asks whether there is a cohomology on super-flag-manifolds with his super Bruhat cells (which are really "sub-super-schemes", and not just subvarieties of the flag supermanifolds as generators). In general, are there global geometric invariants of the odd dimensions? (Some steps in this direction might come from a paper by M. Rothstein [Trans. Amer. Math. Soc. **299** (1987), no. 1, 387–396].) (3) An even bolder question is, "Is the even geometry a collective effect in the infinite-dimensional odd geometry?" This has roots in both the equivalence of "wedge" and "spinor" pictures of representations of Kac–Moody algebras, and more philosophically from the perceived "primacy" of fermions in particle physics.

The final Section 6 treats the kinematics of supergravity from the point of view of creating a curved structure on superspace modelling the supergeometry of the Bruhat cells. Many interesting physical equations (especially Yang–Mills) have been treated "twistorially" in the last decade, and the equations of motion are usually related to such integrability conditions. (Cf. also a paper by the author [*Arithmetic and geometry, Vol. II*, 175–198, Birkhäuser, Boston, Mass., 1983; MR0717612].)

His current point of view is an extension of the twistor correspondence, the interpretation being given in Section 6.1 of this paper. A final calculation equates these integrability conditions with the pre-potential formalism of V. I. Ogievetskiĭ and E. S. Sokachev [Yadernaya Fiz. **31** (1980), no. 3, 821–839; MR0607671].

A final topic, which is not really touched upon, but simply pointed out: A true super-symmetric Kähler geometry will probably be quite sophisticated. Even in algebraic supergeometry, the value of classical projective techniques seems unfortunately limited, so the development of such a Kähler alternative might prove important.

In summary, then, the author has tried to "seize the moment", to discern a pattern crystallizing out of what seems a tantalizing chaos in the rapidly exploding frontiers of geometry. He has, in the opinion of this reviewer, done an exciting job of updating the vision of the late nineteenth century in the discovery of the fecund "parallels" between the theory of numbers and the theory of algebraic functions. The jewel of geometry is even more brilliant and fascinating with the dazzling interplay of flashes of light from any one of its new facets to another.

Daniel M. Burns Jr.

From MathSciNet, May 2023

MR0974910 (89m:11060) 11G35; 14G25; 14J20

Franke, Jens; Manin, Yuri I.; Tschinkel, Yuri

Rational points of bounded height on Fano varieties. (English)

Invent. Math., 2.

,, 1989, 421–435 pp.

Let K be a number field, V/K a variety, $\mathcal{L}$ a (metrized) line bundle on V/K, and $H_{\mathcal{L}}$ an absolute multiplicative height function on V relative to $\mathcal{L}$. If $\mathcal{L}$ is assumed to be ample, the number $N(V,\mathcal{L},H) = \#\{x \in V(K)\, H_{\mathcal{L}}(x) \leq H\}$ is finite, and the growth of $N(V,\mathcal{L},H)$ as a function of H is an important arithmetic invariant of V. The authors briefly discuss this problem for general V, and then they restrict to the case when V is a Fano variety, i.e. varieties for which the anti-canonical bundle $\mathcal{K}^{-1}$ is ample. Manin has conjectured that except for some degenerate cases, a Fano variety should satisfy the asymptotic relation $(*)$ $N(V,\mathcal{K}^{-1},H) \sim cH(\log H)^t$ for certain constants $c, t \geq 0$, and possibly even $t = \operatorname{rank}\operatorname{Pic}(V) - 1$. In this paper, the authors provide the following evidence for this conjecture. (1) Asymptotic $(*)$ with error term $O(H(\log H)^{t-1})$ is stable with respect to direct product of varieties. (2) For complete intersections in $\mathbf{P}^n$, $(*)$ is consistent with predictions from the Hardy-Littlewood method. (3) Let G be a semisimple algebraic group and P a parabolic subgroup. Then $(*)$ is true for the generalized flag manifold $P\backslash G$.

The proofs of (1) and (2) are clever, but relatively elementary, being respectively an application of Abel summation and the calculation of an adelic integral. The proof of (3) begins by identifying the Dirichlet series $Z(s) = \sum_{x\in(P\backslash G)(K)} H_{\mathcal{K}^{-1}}(x)^{-s}$ with a certain Langlands-Eisenstein series. Then general facts about meromorphic continuation and location of poles of such functions are used (in conjunction with a standard Tauberian argument) to deduce an asymptotic formula $(*)$. Thus the proof of (3) relies on the use of very deep machinery.

The paper concludes with a brief appendix describing numerical data for the rational points on the cubic surface $x^3 + 2y^3 + 3z^3 + 4w^3 = 0$.

Joseph H. Silverman

From MathSciNet, May 2023

MR1032922 (91g:11069) 11G35; 14G10, 14G40

Batyrev, V. V.; Manin, Yu. I.

Sur le nombre des points rationnels de hauteur borné des variétés algébriques. (French)

Math. Ann. **286** (1990), *no.* 1–3, 27–43.

In this very entertaining paper the authors raise the following question: How many rational points does an algebraic variety possess? More precisely, given a variety V defined over some number field k (or, more generally, some global field) and an ample sheaf L (or divisor class), we have the associated (exponential) height H_L [cf., e.g., S. Lang, *Fundamentals of Diophantine geometry*, Springer, New York, 1983; MR0715605] and the problem is to study the asymptotic behaviour of $N_{V,L,k}(H) = N_L(H) := \operatorname{card}\{x \in V(k)\colon H_L(x) \le H\}$. Actually the function H_L is defined only up to a function $\exp O(1)$ (or up to some choice of metrics) so we will speak only of the relation up to bounded functions $\gg\ll$ though the authors implicitly suggest one can give equivalent. We write $f \gg\ll g$ when there exist two positive constants such that $C_1 f \le g \le C_2 f$ and say that f and g are equivalent if asymptotically the inequality holds with any $C_1 < 1$ and $C_2 > 1$. Also, it is convenient to let H_L depend on the field k (so $H_{L,k}$ is not the absolute height $H_L(x) = H_{L,k}(x)^{1/[k\,\mathbf{Q}]}$). The authors relate this with a natural but, to our knowledge, new concept of zeta function: $Z_{V,L,k}(s) = Z_L(s) = \sum_{x \in V(k)} H_L(x)^{-s}$, and call $\beta_{V,L,k} = \beta_L$ the abscissa of convergence of this Dirichlet series. They then compare this with the following geometric object reminiscent of Nevanlinna theory [cf. P. Vojta, *Diophantine approximations and value distribution theory*, Lecture Notes in Math., 1239, Springer, Berlin, 1987]: call $\mathrm{NS}(V)$ the Néron-Severi group of V (tensored with $\mathbf{R}$), $N_{\mathrm{eff}}(V)$ the cone generated by effective divisors and K_V the canonical class, and then set $\alpha(L) = \inf\{r \in \mathbf{R}\; rL + K_V \in N_{\mathrm{eff}}(V)\}$. The function $\alpha(L)$ enjoys formally properties very similar to those of β_L, and the authors conjecture that for all V, L and positive ε there exists a dense Zariski open subset U such that $\beta_{U,L} \le \alpha(L) + \varepsilon$; they further conjecture that if the canonical class does not belong to N_{eff} then $\beta_{U,L} = \alpha(L)$ provided U is small enough and the ground field is large enough (such restrictions are trivially necessary). Results are of course scarce but these conjectures fit harmoniously with other Diophantine conjectures (cf. Vojta's book [op. cit.]) and suggest a link with the work of S. Mori [e.g., Ann. of Math. (2) **116** (1982), no. 1, 113–176; MR0662120]. The only conclusive evidence is for some Fano varieties (ample anticanonical class); the authors give in this case the following more precise conjecture: For U small enough and k large enough one should have: $N_{U,L,k} \gg\ll H^{\alpha(L)}(\log H)^{t(L)-1}$, where $t(L)$ is the codimension in $\mathrm{NS}(V)$ of the minimal face of $\partial N_{\mathrm{eff}}$ containing $\alpha(L)L + K_V$. The first known case is due to S. H. Schanuel [Bull. Soc. Math. France **107** (1979), no. 4, 433–449; MR0557080] who proved this for $V = \mathbf{P}^n$ (and hence product of such). Recently J. Franke, Manin and Yu. Tschinkel [Invent. Math. **95** (1989), no. 2, 421–435; MR0974910] have extended this to all "homogeneous" flag manifolds, that is, varieties obtained as the quotient of a semisimple linear algebraic group by a parabolic subgroup (in this case one can remain over the field of definition and keep the whole of V because of homogeneity). For all del Pezzo surfaces the authors prove that $\alpha(L) \leq \beta_{U,L} \leq (\beta_{-K_V})\alpha(L)$ for U small and k large and, even more, they prove that $N_{U,-K_V} \gg\ll H(\log H)^r$ for del Pezzo surfaces of degree $9-r$ and $0 \leq r \leq 3$. Hence the conjecture is true for these surfaces (to be precise, the proof of the last statement is given only over $\mathbf{Q}$).

As mentioned, the beautiful set of worked out examples and open problems displayed in this paper makes it extremely valuable and should prove fertile ground for further work.

Marc Hindry
From MathSciNet, May 2023

MR1095783 (92k:58024) 58B30; 14M30, 16W30, 17B37, 19D55, 32C11, 46L87, 58A50

Manin, Yuri I

Topics in noncommutative geometry. (English)

M. B. Porter Lecture.

Princeton University Press, *Princeton, NJ*, 1991, viii+164 pp., \$35.00,
ISBN 0-691-08588-9

This book brings together several distinct themes—Connes's noncommutative geometry, the theory of supervarieties, and the theory of quantum groups. All three of these subjects are instances of the principle, understood long ago but not realized until recently, that certain noncommutative rings, like their commutative counterparts, can successfully be regarded as geometric objects.

Chapter I, entitled An overview, provides several definitions from Connes's generalized theory of the de Rham complex, and from the theory of quantum groups. Though little is proved in this chapter, the reader who is unfamiliar with these subjects will find the framework that the author provides quite helpful.

After Chapter I, Connes's work recedes from the discussion, and supervarieties come to the fore. Chapter II is a beautiful summary of the theory of superalgebraic curves, in particular the so-called SUSY curves. These are $1|N$-dimensional supervarieties with a superconformal structure. The author introduces the notion of superconformal structure by considering first the superprojective spaces $\mathbf{P}^{1|N}$, $N = 1, 2$, regarding them as superhomogeneous spaces for the supergroups $\mathrm{PC}(2|N)$. These supergroups preserve, and are in turn defined by, certain distributions on $\mathbf{P}^{1|N}$. These distributions then provide a local model for the general definition of a superconformal structure on a $1|N$-dimensional supervariety. With the local model in hand, the author turns his attention to the moduli problem, which he approaches through Schottky superuniformization. Also in Chapter II is a summary and extension of the author's work on sheaves of Virasoro algebras, in which the central extension is realized invariantly in terms of an algebra of formal pseudodifferential operators canonically associated to the algebra of holomorphic vector fields on $\mathbf{C}$. The whole construction can be sheafified, and so takes place on any Riemann surface. A similar construction holds for SUSY curves. Perhaps the most fascinating aspect of this chapter is the speculation regarding the notion of Picard variety in supergeometry. It is well known that a straightforward imitation of the classical definitions does not lead to a satisfactory theory. The author outlines ideas of Levin and Skornyakov which suggest that the moduli space of rank $1|1$ sheaves endowed with an odd automorphism should play the central role.

Next the discussion turns to flag superspaces and Schubert supercells. The discussion here is rather brief. The reader is introduced to the flag Weyl groups, the singularities of the Schubert supervarieties, and their Bott-Samelson desingularization. Root systems are also discussed briefly.

The final topic of the book is the author's approach to quantum groups. The theme here is that among all Hopf algebras, those which deserve to be called quantum groups are the ones which are in some sense the "symmetry groups" of quantum spaces. A quantum space is defined by its algebra of functions, A, together with a subset $A_1 \subset A$ which generates A. One may think of A as the polynomials on the quantum space and A_1 as the linear functions. In the author's approach, A should in fact be a superalgebra—quantum groups are automatically quantum supergroups. The author's emphasis on the quantum spaces lends clarity and motivation to the subject. Appropriately, this chapter is not a textbook on quantum groups. Indeed, the literature on quantum groups is growing so rapidly that no text on the subject can remain current for very long. Rather, what the author provides here is a source of ideas and perspectives that will be of lasting value.

Mitchell Rothstein

From MathSciNet, May 2023

MR1291244 (95i:14049) 14N10; 53C15, 58D10, 58F05

Kontsevich, M.; Manin, Yu.

Gromov-Witten classes, quantum cohomology, and enumerative geometry. (English)

Comm. Math. Phys. **164** (1994), *no.* 3, 525–562.

This interesting and innovative paper gives an axiomatic treatment of the Gromov-Witten invariants of symplectic manifolds, including a geometric interpretation of the quantum cohomology ring in terms of a flat connection (the Dubrovin connection). The authors also explain the relation between the associativity law and a certain system of quadratic third-order partial differential equations (the WDVV equations). As a result they obtain explicit recursion formulae for certain enumerative problems in algebraic geometry.

Briefly, the Gromov-Witten invariants are defined by counting pseudo-holomorphic curves in a $2n$-dimensional compact symplectic manifold (X, ω) with a compatible almost complex structure J. Given a homology class $A \in H^2(X, \mathbf{Z})$, integers $p, g \geq 0$, and cohomology classes $\alpha_1, \cdots, \alpha_p \in H^*(X)$ (integral cohomology modulo torsion), the GW-invariant

$$\Phi_{A,p,g}(\alpha_1, \cdots, \alpha_p) = \int_{M_{A,p,g}} {e_1}^*\alpha_1 \wedge \cdots \wedge {e_p}^*\alpha_p$$

can roughly be understood as the number of pseudo-holomorphic curves of genus g representing the homology class A and intersecting p given cycles Poincaré dual to the cohomology classes α_ν. The complex structure on the Riemann surface is allowed to vary, as are the points of intersection. The α_ν must satisfy the dimension condition

$$\sum_{\nu=1}^{p} \deg(\alpha_\nu) = \dim M_{A,p,g} = (2-2g)(n-3) + 2p + 2c_1 \cdot A$$

with $c_1 = c_1(TX, J) \in H^2(X, \mathbf{Z})$. Here $M_{A,p,g}$ is the moduli space of unparametrized pseudo-holomorphic curves in X of genus g with p marked points representing the homology class A and $e_i \colon M_{A,p,g} \to X$ denotes the evaluation map at the ith marked point.

The ideas for defining such invariants go back to the work of M. L. Gromov [Invent. Math. **82** (1985), no. 2, 307–347; MR0809718] and E. Witten [in *Surveys in differential geometry (Cambridge, MA, 1990)*, 243–310, Lehigh Univ., Bethlehem, PA, 1991; MR1144529]. Care must be taken with the definition of the invariants, firstly, because of the presence of multiply covered curves and, secondly, because the moduli spaces $M_{A,p,g}$ are, in general, not compact and one needs to work with Gromov's compactification. The rigorous treatment requires some analysis, the details of which were carried out by Y. Ruan and G. Tian [Math. Res. Lett. **1** (1994), no. 2, 269–278; MR1266766; J. Differential Geom., to appear] (for general Riemann surfaces) and D. McDuff and the reviewer [*J-holomorphic curves and quantum cohomology*, Amer. Math. Soc., Providence, RI, 1994; MR1286255] (for genus zero). The construction also requires a suitable perturbation of the Cauchy-Riemann equations; however, the resulting invariants are independent of the almost complex structure J and the perturbation used to define them and they depend only on the isotopy class of the symplectic form ω.

In the terminology of the paper under review, the invariants $\Phi_{A,p,g}$ are the codimension-zero classes. For the case of genus zero and codimension zero, the axioms in this paper take the following form. (Zero) The invariants $\Phi_{A,p,0}(\alpha_1, \cdots, \alpha_p)$ vanish unless $\int_A \omega \geq 0$. (Symmetry) The number $\Phi_{A,p,0}(\alpha_1, \cdots, \alpha_p)$ is equivariant under the action of the permutation group S_p, where the action on $\mathbf{Z}$ is given by the sign of the permutation. (Fundamental class) If $\alpha_p = 1_X \in H^0(X)$ is the Poincaré dual of the fundamental class $[X]$, then the only nonvanishing invariants are $\Phi_{0,3,0}(\alpha_1, \alpha_2, 1_X) = \int_X \alpha_1 \wedge \alpha_2$. (Mapping to point) For $A = 0$ the only nonzero invariants are

$$\Phi_{0,3,0}(\alpha_1, \alpha_2, \alpha_3) = \int_X \alpha_1 \wedge \alpha_2 \wedge \alpha_3.$$

(Divisor) If $p \geq 4$ and $\deg(\alpha_p) = 2$, then

$$\Phi_{A,p,0}(\alpha_1, \cdots, \alpha_p) = \Phi_{A,p-1,0}(\alpha_1, \cdots, \alpha_{p-1}) \cdot \int_A \alpha_p.$$

(Splitting) A special case of this axiom asserts that the integer invariant

$$\Psi_{A,4,0}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \sum_B \sum_i \Phi_{A-B,3,0}(\alpha_1, \alpha_2, \phi_i) \Phi_{B,3,0}(\psi_i, \alpha_3, \alpha_4)$$

is equivariant under the action of the permutation group S_4. Here the ϕ_i form a basis of $H^*(X)$ and the ψ_i are the dual basis with respect to Poincaré duality (i.e. $\deg(\phi_i) + \deg(\psi_i) = 2n$ and $\phi_i \cdot \psi_j = \delta_{ij}$). The classes α_ν satisfy $\sum_\nu \deg(\alpha_\nu) = 2n + 2c_1 \cdot A$ and the sum runs over all pairs (B, i) with $\deg(\alpha_3) + \deg(\alpha_4) + \deg(\psi_i) = 2n + 2c_1 \cdot B$.

To date, the definition of the invariants requires the assumption that there be no pseudo-holomorphic spheres with negative Chern number. This holds, for example, when X has dimension at most 6, or when $c_1 \cdot A = 0$ for every $A \in \pi_2(M)$, or when X is monotone (the symplectic version of Fano variety: there exists a constant $\lambda > 0$ such that $[\omega] \cdot A = \lambda c_1 \cdot A$ for every $A \in \pi_2(M)$). The invariants should be well defined without these assumptions but so far this has not been established. Some ideas in this direction are contained in a paper by Kontsevich ["Enumeration of rational curves via torus actions", Preprint, http://xxx.lanl.gov/abs/hep-th/9405035].

In this paper Kontsevich and Manin show how to use these axioms and their appropriate generalizations to higher genus and classes of higher codimension in order to determine all the GW-invariants for genus zero from the basic invariants $\Phi_{A,3,0}(\alpha_1, \alpha_2, \alpha_3)$ with $c_1 \cdot A \leq 2n+1$, $\sum_\nu \deg(\alpha_\nu) = 2n + 2c_1 \cdot A$ and $\deg(\alpha_3) = 2$. This holds whenever the cohomology ring is generated by $H^2(X)$.

Witten [op. cit.; MR1144529] organized these data into a generating function $\Phi = \Phi_\omega^X : H^*(X, \mathbf{C}) \to \mathbf{C}$ which depends on the symplectic form ω and is defined by

$$\Phi_\omega^X(\alpha) = \sum_p \sum_A \sum_{i_1, \cdots, i_p} \frac{\exp\left(-\int_A \omega\right)}{p!} \Phi_{A,p,0}(\alpha_{i_1}, \cdots, \alpha_{i_p}).$$

Here $\alpha = \alpha_0 \oplus \cdots \oplus \alpha_{2n}$ with $\alpha_i \in H^i(X, \mathbf{C})$ and the sum runs over all $A \in H_2(X)$ and (unordered) multi-indices $i_1, \cdots, i_p \in \{0, \cdots, 2n\}$ with $\sum_{\nu=1}^p \deg(\alpha_{i_\nu}) = 2n - 6 + 2p + 2c_1 \cdot A$. There is a nontrivial convergence problem. In the monotone case the sum is finite for every p and the convergence follows from the axioms. Moreover, in this case the factor $\exp(-\int_A \omega)$ can be dropped. In the Calabi-Yau case with $c_1 = 0$ the sum is infinite for every p, the factor $\exp(-\int_A \omega)$ is essential, and convergence is a conjecture. The function $\Phi = \Phi_\omega^X$ is called the Gromov-Witten potential. In terms of this potential the axioms can be restated as a system of partial differential equations, namely $\partial_0 \partial_i \partial_j \Phi = g_{ij}$ and the WDVV equations

$$\sum_{\mu,\nu} \partial_i \partial_j \partial_\mu \Phi \, g^{\mu\nu} \, \partial_\nu \partial_k \partial_l \Phi = (-1)^{d_i(d_j + d_k)} \sum_{\mu,\nu} \partial_j \partial_k \partial_\mu \Phi \, g^{\mu\nu} \, \partial_\nu \partial_i \partial_l \Phi.$$

Here the function Φ is expressed in terms of complex coordinates $x^0, \cdots, x^m$ with $\alpha = \sum_i x^i \phi_i$. The ϕ_i form a homogeneous basis of $H^*(X)$ with $\deg(\phi_i) = d_i$ and $\phi_i \cdot \phi_j = g_{ij}$. The g^{ij} represent the inverse matrix of g_{ij}. The derivatives are to be understood in the sense of supermanifolds with $\partial_i \partial_j \Phi = (-1)^{d_i d_j} \partial_j \partial_i \Phi$. The proof that, in the monotone case, the GW-potential Φ_ω satisfies the WDVV equations was given by Ruan and Tian [op. cit.; MR1266766]. It also follows from the gluing theorem for J-holomorphic spheres in papers by G. Liu ["Associativity of quantum multiplication", Preprint, SUNY, Stony Brook, NY, 1994; per revr.] and the book by McDuff and the reviewer [op. cit.; MR1286255 (Appendix A)].

Kontsevich and Manin explain how the WDVV equations give rise to a potential Dubrovin structure on $H^*(X, \mathbf{C})$, understood as a supermanifold. Thus each tangent space is equipped with a metric given by Poincaré duality, and a multiplicative structure

$$x * y = \sum_{i,j,l} A_{ij}^l x^i y^j \phi_l, \quad A_{ij}^l = \sum_k \partial_i \partial_j \partial_k \Phi \, g^{kl},$$

where $x = \sum_i x^i \phi_i$, $y = \sum_j y^j \phi_j$. This is the quantum deformation of the cup-product and the WDVV equations are equivalent to associativity. The ordinary cup-product appears as the limit of $\Phi_{t\omega}$ with $t \to \infty$.

Example (Kontsevich): In the case $X = \mathbf{C}\mathrm{P}^2$ the GW-potential (without the term $\exp(-\int_A \omega)$) is in standard coordinates x, y, z on $H^*(\mathbf{C}\mathrm{P}^2, \mathbf{C}) = \mathbf{C}^3$ given by

$$\Phi_0^{\mathbf{C}\mathrm{P}^2}(x, y, z) = \frac{xy^2 + x^2 z}{2} + \sum_{d=1}^{\infty} N(d) \frac{z^{3d-1}}{(3d-1)!} e^{dy},$$

where $N(d) = \Phi_{dL,3d-1,0}(\mathrm{pt}, \cdots, \mathrm{pt})$ denotes the number of holomorphic spheres of degree d passing through $3d - 1$ generic points. The WDVV equations are now equivalent to an explicit recursion formula for the numbers $N(d)$. The first few values are $N(1) = 1$, $N(2) = 1$, $N(3) = 12$, $N(4) = 620$, $N(5) = 87\,304$, $N(6) = 26\,312\,976$, $N(7) = 14\,616\,808\,192$. Kontsevich and Manin also show how to explicitly compute numerical invariants of this type for any projective space and for Del Pezzo surfaces.

Remark 1: S. Piunikhin, the reviewer and M. Schwarz ["Symplectic Floer-Donaldson theory and quantum cohomology", Preprint, Univ. Warwick, Coventry, 1995; per revr.] have shown that the quantum multiplication structure at the origin $\alpha = 0$ is equivalent to the pair-of-pants product in Floer homology.

Remark 2: Explicit computations of the quantum cohomology ring structure can be found in papers by A. Givental and B. Kim [Comm. Math. Phys. **168** (1995), no. 3, 609–641] for flag manifolds, C. Vafa [in *Essays on mirror manifolds*, 96–119, Internat. Press, Hong Kong, 1992; MR1191421], Witten [in *Geometry, topology, & physics*, 357–422, Internat. Press, Cambridge, MA, 1995], and B. Siebert and Tian ["Quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator", Preprint, 1994; per revr.] for complex Grassmannians, A. Astashkevich and V. Sadov ["Quantum cohomology of partial flag manifolds", Preprint, http://xxx.lanl.gov/abs/hep-th/9401103] for partial flag manifolds, and S. K. Donaldson [in *Vector bundles in algebraic geometry (Durham, 1993)*, 119–138, Cambridge Univ. Press, Cambridge, 1995] for the moduli space of flat SO(3)-connections over a Riemann surface of genus 2. M. Callahan [Ph.D. Thesis, Univ. Oxford, Oxford, in preparation] uses a generalization of the quantum cohomology structure to distinguish isotopy classes of symplectomorphisms on such moduli spaces.

Remark 3: In a recent paper, C. H. Taubes [Math. Res. Lett. **2** (1995), no. 2, 221–238] has related the Gromov-Witten invariants of symplectic 4-manifolds to the new invariants of smooth 4-manifolds by Seiberg and Witten. He considers moduli spaces $M_{A,p,g}$ which consist of embedded curves. In view of the adjunction formula this means that $c_1 \cdot A = A \cdot A + 2 - 2g$ and hence $\dim M_{A,0,g} = 2p$, $p = A \cdot A + 1 - g$. Taubes proves that the GW-invariant $\Phi_{A,p,g}(\mathrm{pt}, \cdots, \mathrm{pt})$ agrees with the Seiberg-Witten invariant $\mathrm{SW}(X, c_1 + 2\mathrm{PD}(A))$.

Dietmar A. Salamon

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MR1702284 (2001g:53156) 53D45; 14H10, 14N35, 18D50, 32G34

Manin, Yuri I

Frobenius manifolds, quantum cohomology, and moduli spaces. (English)

American Mathematical Society Colloquium Publications, 47.
American Mathematical Society, *Providence, RI*, 1999, xiv+303 pp., \$55.00,
ISBN 0-8218-1917-8

This book is a good introduction to the theory of Frobenius manifolds and quantum cohomology. Many topics discussed in the book were developed by the author in his earlier work. The theory of Frobenius manifolds was developed by B. A. Dubrovin [in *Integrable systems and quantum groups (Montecatini Terme, 1993)*, 120–348, Lecture Notes in Math., 1620, Springer, Berlin, 1996; MR1397274], but the formal treatment of the author might be more accessible to an algebraist or algebraic geometer.

The theory of Frobenius manifolds relates very different areas of mathematics. They can be constructed from the quantum cohomology of a smooth projective manifold, from a deformation of a singularity of a holomorphic function, and from differential Gerstenhaber-Batalin-Vilkovisky (dGBV) algebras. There is a connection between Frobenius manifolds and integrable systems. One can formulate the mirror conjecture as an isomorphism between two Frobenius manifolds associated to two different manifolds. Many insights in this subject come from theoretical physics.

Roughly speaking, the book consists of two parts. In the first part (Chapters 1–4) the author discusses Frobenius manifolds in general, and gives the most important examples. In the second part (Chapters 5–6) the quantum cohomology and the moduli spaces of stable maps are studied in detail. This requires sophisticated algebraic-geometric techniques.

A Frobenius algebra is a commutative associative algebra with a unity and a symmetric nondegenerate invariant bilinear form. A Frobenius manifold M has a structure of a Frobenius algebra on each tangent space T_tM such that this structure varies smoothly with t, and the corresponding metric on M is flat. It is also required that the structure constants of the Frobenius algebras are the partial derivatives of a function F, the potential of M. In addition, one usually requires that there exists a flat identity, that is, a flat vector field which is the identity for the multiplication, and a grading Euler vector field. (The flat identity and the Euler vector field are incorporated in Dubrovin's definition of a Frobenius manifold.) The fact that M is a Frobenius manifold with the Euler field E can be expressed as flatness of a structure connection on $M \times \mathbf{C}^*$.

Many examples of Frobenius manifolds coming from geometry are semisimple. A Frobenius manifold M is semisimple if for a general point $t \in M$ the Frobenius algebra structure on T_tM is semisimple, that is, it has no nilpotent elements. Since all such algebras of the same dimension are isomorphic, it gives rise to the canonical coordinates on M, which are different from the flat coordinates. The existence of two sets of coordinates plays an important role in the study of semisimple Frobenius manifolds.

In Chapter 2 the author identifies the semisimple Frobenius manifolds with flat identity and Euler vector field as special solutions to Schlessinger's differential equations. It follows that semisimple Frobenius manifolds can be identified with the special initial conditions to Schlesinger's equation. The author explicitly describes the set of special initial conditions. In order to obtain the above identification he studies meromorphic connections on $M \times \mathbf{P}^1$ using results of Malgrange. The original structure connection has a pole of order 2 along $M \times \{\infty\}$. (It has been studied by Dubrovin and Y. Zhang [Compositio Math. **111** (1998), no. 2, 167–219; MR1606165; Comm. Math. Phys. **198** (1998), no. 2, 311–361; MR1672512; Selecta Math. (N.S.) **5** (1999), no. 4, 423–466 MR1740678].) However, the author notices that there is another structure connection on $M \times \mathbf{P}^1$ having only simple poles, which is related to the original one by a formal Laplace transform.

In Chapters 3 (the first part) and 4 the author conducts the study of formal Frobenius manifolds. A key observation is that a structure of a formal Frobenius manifold on a vector space V (that is, on the underlying affine manifold) is equivalent to a structure of a cohomological field theory (CohFT) of genus 0 on V. A (complete) CohFT of rank r is a triple $(V, \eta, I_{g,n})$, where V is a vector space of dimension r, η is a (super)symmetric nondegenerate bilinear form on V, and the linear maps

$$I_{g,n}\colon H^\bullet(\overline{M}_{g,n}, \mathbf{Q}) \to V^{\otimes n}$$

satisfy certain conditions when restricted to the boundary strata. Here, $\overline{M}_{g,n}$ is the moduli space of stable curves of genus g with n punctures. To define a structure of a genus 0 CohFT one needs to provide $I_{g,n}$ only for $g = 0$. There is yet another point of view explained in Chapter 4. The collection $\{H_\bullet(\overline{M}_{0,n}, \mathbf{Q})\}$, $n \geq 3$, possesses a natural structure of a (cyclic) operad. A CohFT in genus 0 is then an algebra over this operad. This allows one to apply the machinery of operads to the study of formal Frobenius manifolds.

There is a natural way to form the tensor product of CohFTs. Consequently, one can form the tensor product of two formal Frobenius manifolds. The author, following R. M. Kaufmann [Internat. J. Math. **10** (1999), no. 2, 159–206; MR1687157], explains how one forms the tensor product of two analytic Frobenius manifolds.

In Chapter 3 the author also explains three constructions of Frobenius manifolds [cf. Y. I. Manin, Asian J. Math. **3** (1999), no. 1, 179–220; MR1701927]. Since the Gromov-Witten invariants of a smooth projective variety produce an example of a cohomological field theory, this also gives an example of a formal Frobenius manifold. (At this point the author presents only the axioms of Gromov-Witten invariants, postponing more detailed discussion until later chapters.) Another way to construct a Frobenius manifold comes from deformations of singularities. This structure was first studied by K. Saito.

The third way uses dGBV algebras, and was motivated by work of S. A. Barannikov and M. Kontsevich [Internat. Math. Res. Notices **1998**, no. 4, 201–215; MR1609624]. The author shows that each dGBV algebra satisfying certain conditions gives rise to a formal Frobenius manifold. Each (weak) Calabi-Yau manifold determines a dGBV algebra satisfying these conditions, and, consequently, produces a formal Frobenius manifold.

Chapter 5 is devoted to the construction of the moduli spaces of stable maps, and their properties. In order to do this, the author discusses the formalism of stacks since the (fine) moduli spaces of stable maps are stacks rather than schemes [cf. K. A. Behrend and Yu. I. Manin, Duke Math. J. **85** (1996), no. 1, 1–60; MR1412436].

Gromov-Witten invariants are defined as intersection numbers on the moduli spaces of stable maps, and a generating function having these numbers as its coefficients is the potential of the corresponding Frobenius manifold. However, in general these spaces can be singular, and not of the expected dimension. Therefore one needs to construct virtual fundamental classes. The virtual fundamental classes were first constructed by J. Li and G. Tian [J. Amer. Math. Soc. **11** (1998), no. 1, 119–174; MR1467172; in *Topics in symplectic* 4*-manifolds (Irvine, CA, 1996)*, 47–83, Internat. Press, Cambridge, MA, 1998; MR1635695] in the symplectic category, and by K. A. Behrend and B. Fantechi [Invent. Math. **128** (1997), no. 1, 45–88; MR1437495] in the analytic category. The construction is fairly complicated, and the author presents the axioms of the virtual fundamental classes along with a brief sketch of the Behrend-Fantechi construction.

The rest of the chapter is concerned with gravitational descendants and the Virasoro conjecture [T. Eguchi, K. Hori and C. S. Xiong, Phys. Lett. B **402** (1997), no. 1-2, 71–80; MR1454328]. Correlators with gravitational descendants are the intersection numbers on the moduli spaces of stable maps involving the first Chern classes of the tautological line bundles associated to the punctures. The corresponding generating function G is called the large phase space potential. The Gromov-Witten invariants are called primary correlators. Eguchi, Hori, and Xiong conjectured that certain explicitly constructed differential operators $L_i,\ i \geq -1$, satisfying the Virasoro relations annihilate G. The author discusses this conjecture, and first proves two equations: $L_{-1}G = 0,\ L_0 G = 0$. The reader can find a review of the developments on the Virasoro conjecture in [E. Getzler, in *Algebraic geometry: Hirzebruch 70 (Warsaw, 1998)*, 147–176, Contemp. Math., 241, Amer. Math. Soc., Providence, RI, 1999; MR1718143].

This book is of interest to a broad mathematical audience.

Alexandre I. Kabanov
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Manin, Yuri I.; Marcolli, Matilde

Continued fractions, modular symbols, and noncommutative geometry. (English)

Selecta Math. (N.S.) **8** (2002), *no.* 3, 475–521.

The rotation algebra is a well-studied C^*-algebra, providing a ready example on which to test notions related to foliations and crossed products. It is defined as the C^*-algebra A_θ, where $\theta \in \mathbb{R}$, generated by two unitaries U and V satisfying

$$VU = \exp(2\pi i\theta)UV.$$

By an important result of M. A. Rieffel [Pacific J. Math. **93** (1981), no. 2, 415–429; MR0623572; J. Pure Appl. Algebra **5** (1974), 51–96; MR0367670], the algebras A_θ and $A_{\theta'}$, $\theta, \theta' \in \mathbb{R}$, are strongly Morita equivalent if and only if θ and θ' are in the same orbit of the fractional linear action of $\mathrm{PSL}(2,\mathbb{Z})$ on the projective real line $\mathrm{P}_1(\mathbb{R})$. Strongly Morita equivalent C^*-algebras have the same space of classes of irreducible representations, canonically isomorphic K-theory groups, and share many other properties [see A. Connes, *Noncommutative geometry*, Academic Press, San Diego, CA, 1994; MR1303779 (Chapter II, Appendix A)]. On putting $U = \exp(2\pi i x)$, $V = \exp(2\pi i y)$, $x, y \in \mathbb{R}$ and $\theta = 0$, we see that A_0 is the algebra of continuous functions on the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$. In Connes's theory of noncommutative geometry, a dense subalgebra $\mathcal{A}_\theta$ of "smooth elements" of the rotation algebra A_θ is known as the "noncommutative 2-torus $\mathbb{T}_\theta$" and is an important case study in this theory. A generic element of $a \in \mathcal{A}_\theta$ is a formal sum

$$a = \sum_{(m,n)\in\mathbb{Z}^2} a(m,n)U^m V^n,$$

where the sequence $(a(m,n))_{(m,n)\in\mathbb{Z}^2}$ is of rapid decay. The algebra A_θ has a canonical trace function $\tau(\cdot)$ determined by

$$\tau\,(a) = a(0,0).$$

In [Inst. Hautes Études Sci. Publ. Math. No. 62 (1985), 257–360; MR0823176] Connes computed the Hochschild and periodic cyclic cohomology of $\mathcal{A}_\theta$. He showed that the dimension of the Hochschild cohomology spaces depends on the Diophantine properties of θ, whereas the periodic cyclic cohomology is complex 2-dimensional in both odd and even degrees. The bases of the periodic cyclic cohomology can be described in terms of the trace τ and the natural derivations on $\mathcal{A}_\theta$ determined by

$$\delta_1(U^mV^n) = 2\pi imU^mV^n, \quad \delta_2(U^mV^n) = 2\pi inU^mV^n.$$

The noncommutative torus $\mathbb{T}_\theta$ was used by J. Bellissard [in *Statistical mechanics and field theory: mathematical aspects (Groningen, 1985)*, 99–156, Lecture Notes in Phys., 257, Springer, Berlin, 1986; MR0862832; in *Localization in disordered systems (Bad Schandau, 1986)*, 61–74, Teubner, Leipzig, 1988; MR0965981] in an application of noncommutative geometry to the Quantum Hall Effect (QHE) in physics. Bellissard gave a complete explanation of certain stable numerical quantities described by the QHE in terms of noncommutative topological invariants of $\mathbb{T}_\theta$, thereby completing earlier work of Novikov and Thouless.

The 2-torus $\mathbb{R}^2/\mathbb{Z}^2$ can be given a complex structure. The different possibilities for this structure are parametrized by the Poincaré upper half plane $\mathcal{H}$ of complex numbers with positive imaginary part. To $z \in \mathcal{H}$, we may associate the complex 1-dimensional torus

$$T_z = \mathbb{C}^2/(\mathbb{Z} + z\mathbb{Z}),$$

given by the complex numbers modulo translation by the lattice $\mathbb{Z} + z\mathbb{Z}$. The complex isomorphism classes of these tori, also known as elliptic curves from their geometric description, are in bijective correspondence with the orbits of the fractional linear action of $\mathrm{PSL}(2,\mathbb{Z})$ on $\mathcal{H}$. The quotient space $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathcal{H}$ can be compactified by adding a point at infinity corresponding to the 1-point quotient or "cusp" $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathrm{P}_1(\mathbb{Q})$. Adding extra structure to the isomorphism classes of elliptic curves leads to replacing $\mathrm{PSL}(2,\mathbb{Z})$ by certain of its finite index subgroups G_0. The corresponding modular curves $G_0\backslash\mathcal{H}$ can be compactified by adding the finite set of cusps $G_0\backslash\mathrm{P}_1(\mathbb{Q})$.

The central point of the paper under review is that this traditional picture bypasses the Morita classes of noncommutative tori that would appear if the boundary of the modular curve $G_0\backslash\mathcal{H}$ were considered instead to be the "noncommutative modular curve" $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathrm{P}_1(\mathbb{R})$. This is in the spirit of J. Bost and Connes [Selecta Math. (N.S.) **1** (1995), no. 3, 411–457; MR1366621], Connes [Selecta Math. (N.S.) **5** (1999), no. 1, 29–106; MR1694895], Y. Soibelman [Lett. Math. Phys. **56** (2001), no. 2, 99–125 MR1854130] and others, as discussed in the paper under review and its references. The paper contributes completely new concrete examples of how mathematics usually applied to the commutative case relates to that traditionally applied to the noncommutative case. It opens up a new field in "noncommutative number theory", aimed at combining the mathematics of classical spaces of automorphic functions with that of noncommutative algebras. For further work in this direction by the authors see [Yu. I. Manin, "Real multiplication and noncommutative geometry", preprint, arXiv.org/abs/math/0202109; "Von Zahlen und Figuren", preprint, arXiv.org/abs/math/0201005; M. Marcolli, J. Number Theory **98** (2003), no. 2, 348–376 MR1955422].

We now outline the main results of the paper. A matrix $A \in \mathrm{PSL}(2,\mathbb{R})$ is hyperbolic if its trace has absolute value greater than 2. In this case, it has two hyperbolic fixed points θ, with $A'(\theta) < 1$, and θ', with $A'(\theta') > 1$. The oriented geodesic in $\mathcal{H}$ from θ' to θ is invariant under the action of A and is called the axis of A. If $A \in G_0$, then the axis of A becomes a closed geodesic in $G_0\backslash\mathcal{H}$. Moreover θ and θ' are then irrational conjugates in a real quadratic field. Conversely, every closed geodesic in $G_0\backslash\mathcal{H}$ represents the conjugacy class of a primitive hyperbolic transformation in G_0. Furthermore, closed geodesics for the modular group are known to be coded by "minus" continued fractions [D. B. Zagier, *Zetafunktionen und quadratische Körper*, Springer, Berlin, 1981; MR0631688].

For $\theta \in \mathbb{R}$, we may try to understand in what sense $\mathbb{T}_\theta$ is a limit of T_z as z tends to θ along a geodesic in $\mathcal{H}$. In this vein, the authors extend the classical definition of modular symbols to "limiting modular symbols", with limits along geodesics in the upper half plane ending at points on the "noncommutative boundary". They show that quadratic irrationalities give rise to limiting cycles whereas generic irrational points give rise to cycles vanishing in a suitable averaged sense.

Let $X_{G_0} = X_{G_0}(\mathbb{C})$ denote the smooth compactification of $G_0\backslash\mathcal{H}$ by the finite number of cusps in bijection with $G_0\backslash \mathrm{P}_1(\mathbb{Q})$ and let φ be the corresponding covering map. As in [Yu. I. Manin, Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 19–66; MR0314846; also in *Selected papers of Yu. I. Manin*, World Sci. Publishing, River Edge, NJ, 1996; MR1408904 (pp. 202–247); L. Merel, Manuscripta Math. **80** (1993), no. 3, 283–289; MR1240651], for any two points α, β in $\mathcal{H} \cup \mathrm{P}_1(\mathbb{Q})$, we can define a real homology class or "modular symbol" $\{\alpha, \beta\} \in H^1(X_{G_0}, \mathbb{R})$ by integrating lifts $\varphi^*(\omega)$ of differentials ω of the first kind on X_{G_0} along the geodesic path connecting α to β:

$$\int_{\alpha,\beta} \omega := \int_\alpha^\beta \varphi(\omega).$$

When α, β are cusps, the modular symbol represents a rational homology class. To extend the definition to "limiting modular symbols", when either endpoint is real irrational, the authors define

$$\{\{*, \beta\}\}_{G_0} := \lim \frac{1}{T(x,y)} \{x, y\}_{G_0} \in H^1(X_{G_0}, \mathbb{R}),$$

where $x, y \in \mathcal{H}$ are two points on the geodesic joining α to β, x is arbitrary but fixed, $T(x, y)$ is the geodesic distance between them, and the limit is taken as y tends to β. If the limit exists, the authors show that it depends neither on x nor on α, which justifies the notation. These integrals can be related to finite (when α, β are cusps), stably periodic (when α, β are two fixed points of a hyperbolic element of G_0 as described above), or general infinite continued fractions. The different cases are treated using results from [Yu. I. Manin, op. cit., 1972] and [J. B. Lewis and D. B. Zagier, in *The mathematical beauty of physics (Saclay, 1996)*, 83–97, World Sci. Publishing, River Edge, NJ, 1997; MR1490850]. Continued fractions that eventually agree up to a shift of index can be identified by $\mathrm{PGL}(2, \mathbb{Z})\backslash \mathrm{P}_1(\mathbb{R})$. In this paper all of this noncommutative boundary is considered.

A striking result of the paper (Theorem 0.2.2), which is derived from certain averaging techniques over successive convergents in infinite continued fractions, uses modular symbols to relate Mellin transforms of weight-two cusp forms for $G_0 = \Gamma_0(N)$ to quantities defined entirely on the noncommutative boundary of the corresponding modular curve. This gives a concrete example of how the two types of tori $\mathbb{T}_\theta$, $\theta \in \mathbb{R}$, and T_z, $z \in \mathcal{H}$, give information about each other.

These results on the limiting modular symbols rely on certain properties, involving spectral analysis, of the Ruelle transfer operator or Gauss-Kuzmin operator for the shift of the continued fraction expansion, generalized to subgroups G of finite index in $\mathrm{GL}(2,\mathbb{Z})$. In particular the authors generalize the Gauss-Kuzmin-Lévy formula (Theorem 0.1.2). This result gives a formula for the limit of the pullback of the Lebesgue measure on $(0,1)\times \mathrm{GL}(2,\mathbb{Z})/G$ with respect to $g_n(\alpha)$ acting on α and t simultaneously, where

$$g_n(\alpha) = \begin{pmatrix} p_{n-1}(\alpha) & p_n(\alpha) \\ q_{n-1}(\alpha) & q_n(\alpha) \end{pmatrix},$$

and $p_n(\alpha)/q_n(\alpha)$ are the successive convergents to α.

A different direction, with a related philosophy, is the study of the K-theory of the noncommutative modular curves in the spirit of Connes noncommutative geometry [A. Connes, op. cit., 1985]. In this theory, the quotient space $G\backslash \mathrm{P}_1(\mathbb{R})$, where G is of finite index in $\mathrm{PSL}(2,\mathbb{Z})$, can be studied "topologically" via its associated crossed product algebra $C(\mathrm{P}_1(\mathbb{R}))\rtimes G$ or the strongly Morita equivalent $C(\widehat{X})\rtimes \mathrm{PSL}(2,\mathbb{Z})$, where $\widehat{X} = \mathrm{P}_1(\mathbb{R})\times \mathrm{PSL}(2,\mathbb{Z})/G$. M. V. Pimsner [Invent. Math. **86** (1986), no. 3, 603–634; MR0860685] (see also [M. Laca and J. S. Spielberg, J. Reine Angew. Math. **480** (1996), 125–139; MR1420560]) has studied the K-theory of $C(\widehat{X})$ and its crossed product with $\Gamma = \mathrm{PSL}(2,\mathbb{Z}) = \mathbb{Z}/2 * \mathbb{Z}/3$, $\Gamma_0 = \mathbb{Z}/2$ and $\Gamma_1 = \mathbb{Z}/3$. The K-theory in degrees 0 and 1 is related by a six-term exact sequence. On the other hand, in [Yu. I. Manin, op. cit., 1972] and [L. Merel, op. cit.] the homology groups $H_1(X_G;\mathbb{Z})$ and relative homology groups $H^{\mathrm{cusps}} := H_1(X_G, \mathrm{cusps};\mathbb{Z})$ were studied via the "modular complex" and "relative modular complex" (with respect to the elliptic and parabolic (cuspidal) fixed points of $\mathrm{PSL}(2,\mathbb{Z})$). This homology is based on the n-cells, $n = 0,1,2$, of the $\mathrm{PSL}(2,\mathbb{Z})/G$-orbit of the fundamental region of $\mathrm{PSL}(2,\mathbb{Z})$ built from geodesics joining those fixed points. The authors show (Theorem 4.4.1) that there is a natural isomorphism between a four-term exact sequence derived from Pimsner's exact sequence and an exact sequence derived from the modular complexes. Essentially, this relates H^{cusps} to the noncommutative topology of $G\backslash \mathrm{P}_1(\mathbb{R})$. The authors also relate the modular complex to homological constructions of noncommutative geometry via the periodic cyclic cohomology of the "smooth" crossed product algebras associated to $G\backslash \mathrm{P}_1(\mathbb{R})$.

These innovative and ground-breaking results reveal the mutual influence of the mathematics of a commutative geometric object and that of its natural noncommutative boundary.

Paula Tretkoff

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Manin, Yu. I.

Real multiplication and noncommutative geometry (ein Alterstraum). (English)

The legacy of Niels Henrik Abel, 685–727, *Springer, Berlin*, 2004.

This is the type of visionary paper that makes waves through different mathematical communities and is bound to have a long-lasting effect and influence. The paper is a pleasure to read, full of thought-provoking ideas, centered around the unexpected interplay between noncommutative geometry and abelian class field theory. The first subject, in the version developed by Alain Connes, is functional analytic in nature and originates from the theory of operator algebras, while the latter is a beautiful chapter of classical number theory.

A goal of the paper is to provide evidence for a new approach via noncommutative geometry to the explicit class field theory problem (Hilbert's 12th problem) and the related Stark conjectures for real quadratic fields.

In the case of imaginary quadratic fields, it is well known that the explicit class field theory problem (generators of the maximal abelian extension and the explicit Galois action) has a geometric counterpart which is based on the theory of elliptic curves with complex multiplication. This paper proposes the idea that there should be a parallel for real quadratic fields of the theory of elliptic curves with complex multiplication, based on the (noncommutative) geometry of quantum tori with real multiplication.

The geometry of quantum tori (or noncommutative tori, also known as irrational rotation algebras) was developed in [A. Connes, C. R. Acad. Sci. Paris Sér. A-B **290** (1980), no. 13, A599–A604; MR0572645] as a very important test case of the main ideas and tools of noncommutative geometry such as cyclic cohomology [cf. A. Connes, in *Géométrie différentielle (Paris, 1986)*, 33–50, Hermann, Paris, 1988; MR0955850].

In noncommutative geometry the good notion of isomorphism of spaces is given by Morita equivalences. It was shown in [A. Connes, op. cit., 1980] and [M. A. Rieffel, Pacific J. Math. **93** (1981), no. 2, 415–429; MR0623572] that the orbits of the action of $\mathrm{SL}_2(\mathbf{Z})$ on the parameter θ of the irrational rotation give the equivalence classes of quantum tori. In particular, if θ is a quadratic irrationality, the algebra has nontrivial Morita self-equivalences.

Manin identifies this property as the natural analog for a noncommutative torus of complex multiplication for elliptic curves. Such noncommutative tori are said to have real multiplication. The value of the modulus θ of a noncommutative torus with real multiplication is an element of a real quadratic field, just as in the case of elliptic curves with complex multiplication the modulus τ is an element of an imaginary quadratic field.

There is an equivalence of categories between rank two lattices in a complex line and elliptic curves realized by the period functor. Moreover, elliptic curves (with level structure) have moduli spaces given by the modular curves. There are three classical approaches to the construction of abelian extensions of imaginary quadratic fields: via the elliptic curves with complex multiplication, in terms of the values at torsion points of the Weierstrass $\wp$ function; via the modular curves, in terms of the values at a CM point τ of modular functions; via the Stark numbers, that is, by considering zeta functions of lattices and obtaining algebraic units in abelian extensions from the exponential of the derivative of such zeta functions.

This paper presents analogs of the three approaches mentioned above, in the case of noncommutative tori. The author introduces a category of pseudolattices and a Morita category of finitely generated right modules over an algebra with morphisms given by isomorphism classes of projective bimodules corresponding to projections. There is an equivalence of categories between pseudolattices and the Morita category of noncommutative tori realized by the K_0 functor. This result relies on the theory of noncommutative tori developed in [A. Connes, op. cit., 1980; in *Algèbres d'opérateurs (Sém., Les Plans-sur-Bex, 1978)*, 19–143, Lecture Notes in

Math., 725, Springer, Berlin, 1979; MR0548112; M. A. Rieffel, op. cit.; F.-P. Boca, Comm. Math. Phys. **202** (1999), no. 2, 325–357; MR1690050]: the basic aspects of the theory are recalled in the paper.

This first part of the paper contains in addition interesting discussions on how to regard noncommutative tori as "limits" (or degenerations) of elliptic curves, or on viewing elliptic curves themselves as noncommutative spaces, which explain and motivate the ideas and results of the paper. The topic of noncommutative tori as limits of elliptic curves is illustrated by a reinterpretation of "Hecke's substitution" (lifting of closed geodesics on the modular curve to the space of lattices) as a way of passing from pseudolattices to lattices, or from a noncommutative torus to a family of elliptic curves.

Another main part of the paper is dedicated to a discussion of Stark's numbers for real quadratic fields. These can be formulated in terms of zeta functions $\zeta(L, l_0, s)$ associated to a pair (L, l_0) of an integral ideal L in a real quadratic field K and an element l_0 in the ring of integers of K, with suitable conditions. The Stark numbers are then of the form $S_0(L, l_0) = \exp(\zeta'(L, l_0, 0))$. Stark conjectured [see H. M. Stark, Advances in Math. **22** (1976), no. 1, 64–84; MR0437501] that these numbers are algebraic units and generate abelian extensions of K.

The author discusses Hecke's approach to the computation of sums of the type that occurs in the definition of $\zeta(L, l_0, s)$ in terms of a suitable class of theta functions for pseudolattices. These can be obtained by averaging ordinary theta constants along geodesics (via the Hecke substitution). Poisson's formula gives a functional equation for the theta functions of pseudolattices.

The last part of the paper develops another related theme, a theory of quantized theta functions. These are different from the theta functions of pseudolattices introduced earlier in the paper and play a role in defining morphisms of noncommutative tori. After recalling the relation between theta functions and Heisenberg groups, the author recalls a result of Boca [op. cit.] that constructs projections in irrational rotation algebras in terms of theta functions. The main result of this section generalizes Boca's calculation and describes a construction for quantized theta function (in the sense previously developed by the present author in [Progr. Theoret. Phys. Suppl. No. 102 (1990), 219–228 (1991); MR1182167; in *Moduli of abelian varieties (Texel Island, 1999)*, 231–254, Progr. Math., 195, Birkhäuser, Basel, 2001; MR1827023; Lett. Math. Phys. **56** (2001), no. 3, 295–320; MR1855265]). This result clarifies the relation between Manin's quantized theta functions and the representations of quantum tori.

This paper is complemented by another paper of the present author ["Von Zahlen und Figuren", preprint, `arxiv.org/abs/math/0201005`] where the theme of class field theory for real quadratic field and noncommutative tori with real multiplication is further developed.

Matilde Marcolli
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Manin, Yuri I.

Iterated integrals of modular forms and noncommutative modular symbols.

Algebraic geometry and number theory, 565–597, *Progr. Math., 253*, *Birkhäuser Boston, Boston, MA*, 2006.

This interesting and inspiring paper sets up the foundations of the theory of "iterated noncommutative modular symbols" by generalizing simultaneously the classical theory of modular symbols [Yu. I. Manin, Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 19–66; MR0314846; *Selected papers of Yu. I. Manin*, World Sci. Publ., River Edge, NJ, 1996; MR1408904 (pp. 202–247); Mat. Sb. (N.S.) **92(134)** (1973), 378–401, 503; MR0345909; *Selected papers of Yu. I. Manin*, World Sci. Publ., River Edge, NJ, 1996; MR1408904 (pp. 268–290); V. V. Šokurov, Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), no. 2, 443–464, 480; MR0571104; Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), no. 3, 670–718, 720; MR0582162; L. Merel, "Quelques aspects arithmétiques et géometriques de la théorie des symboles modulaires", thèse de doctorat, Univ. Paris VI, Paris, 1993] and that of multiple zeta values [cf., e.g., D. B. Zagier, in *First European Congress of Mathematics, Vol. II (Paris, 1992)*, 497–512, Progr. Math., 120, Birkhäuser, Basel, 1994; MR1341859]. The main goal of the paper is that of extending the theory of periods of elliptic modular forms by replacing (single) integration along geodesics in the complex upper-half plane with a process of iterated integration of modular forms (i.e. cusp forms and Eisenstein series), possibly multiplied by a (complex) power of the standard complex coordinate variable, along geodesics connecting two cusps.

The theory of multiple zeta values (MZV) and in particular the definition of MZV via m-multiple iterated integrals ($m = m_1 + \cdots + m_k$),

$$(1) \qquad \zeta(m_1, \ldots, m_k) = \int_0^1 \frac{dz_1}{z_1} \int_0^{z_1} \frac{dz_2}{z_2} \int_0^{z_2} \cdots \int_0^{z_{m_{k-1}}} \frac{dz_{m_k}}{1 - z_{m_k}} \cdots \in \mathbb{C}, \%tag1$$

where the sequence of differential forms in the iterated integral consists of consecutive subsequences of the form $\frac{dz}{z}, \ldots, \frac{dz}{z}, \frac{dz}{1-z}$ of lengths $m_k, m_{k-1}, \ldots, m_1$, is the author's inspiration for the definition of a total iterated integral:

$$(2) \quad J_\gamma(\Omega) = 1 + \sum_{n=1}^{\infty} \int_0^1 \gamma^*(\Omega)(t_1) \int_0^{t_1} \gamma^*(\Omega)(t_2) \cdots \int_0^{t_{n-1}} \gamma^*(\Omega)(t_n) \in \mathbb{C}\langle\langle A_V \rangle\rangle.$$

Here, a finite sum $\Omega = \sum_{v \in V} A_v \omega_v$ of sections $A_v \omega_v$ of the sheaf of bi-modules $\Omega^1_X(U)\langle\langle A_V \rangle\rangle$ over the rings $\mathcal{O}_X(U)\langle\langle A_V \rangle\rangle$ of holomorphic functions, as U varies among the open sets of a connected (not necessarily compact) Riemann surface X, with coefficients noncommuting free formal variables A_v ($A_V = \{A_v | v \in V\}$, $V =$ finite set, $\mathbb{C}\langle\langle A_V \rangle\rangle =$ ring of associative formal series), is integrated over the simplex $0 < t_n < \cdots < t_1 < 1$, along a piecewise smooth path $\gamma\colon [0,1] \to U$. This construction describes the first step toward a generalization of the geometry associated to MZV, namely

$$(\mathbf{P}^1(\mathbb{C}), \{0, 1, \infty\}) \simeq \Gamma_0(4) \backslash (\mathfrak{H} \cup \mathbf{P}^1(\mathbb{Q}), \{\text{cusps}\}).$$

The congruence subgroup $\Gamma_0(4)$ of $\mathrm{SL}_2(\mathbb{Z})$ is replaced here by an arbitrary (congruence) subgroup Γ, by use of which the Riemann surface X gets uniformized. The

Eisenstein series of weight 2 for $\Gamma_0(4)$ (i.e. the lifting to the Poincaré upper-half plane $\mathfrak{H}$ of the meromorphic differential forms $\frac{dz}{z}$, $\frac{dz}{1-z}$ with logarithmic singularities at $\{0,1,\infty\}$) are replaced by cuspidal elliptic modular forms with respect to Γ. In relation to this point, the integral formula (1) is not quite covered by the formalism of (2), because the integrands in (1) have logarithmic poles at the boundary. In the last part of the paper the author develops the case when the integration limits of the iterated integral are logarithmic singularities of the form Ω (differentials of the third kind). By using the Manin-Drinfel′d theorem on cusps, he suggests a generalization of Drinfel′d's associator [V. G. Drinfeld, Algebra i Analiz **2** (1990), no. 4, 149–181; MR1080203] and extends to this case a part of the identities satisfied by the latter.

In the case of differentials of the first kind, there is an equivalent version of (2) which is a generating series of iterated integrals and is defined by the following function of the variable z:

$$J_a^z(\omega_V) = 1 + \sum_{n=1}^{\infty} \sum_{(v_1,\dots,v_n)\in V^n} A_{v_1}\cdots A_{v_n} \times \int_a^z \omega_{v_1}(z_1) \int_a^{z_1} \omega_{v_2}(z_2) \cdots \int_a^{z_{n-1}} \omega_{v_n}(z_n),$$

where $z_i = \gamma(t_i) \in X$, $a = \gamma(0)$, $z = \gamma(1)$. This description allows one to recover, in a multiplicative/noncommutative version, the usual properties of the integrals such as the additivity of simple integrals with respect to the union of integration paths (cyclicity) and the variable change formula (functoriality). These basic relations between total iterated integrals are complemented by a group-like property described in terms of a co-multiplication on the ring of formal series

$$\Delta\colon \mathbb{C}\langle\langle A_V\rangle\rangle \to \mathbb{C}\langle\langle A_V\rangle\rangle \widehat{\otimes}_{\mathbb{C}} \mathbb{C}\langle\langle A_V\rangle\rangle, \quad \Delta(A_v) = A_v\otimes 1 + 1\otimes A_v,$$

which extends to the series with coefficients in $\mathcal{O}_X$ and Ω_X^1 producing the formula

$$\Delta(J_a^z(\omega_V)) = J_a^z(\omega_V)\widehat{\otimes}_{\mathcal{O}_X} J_a^z(\omega_V). \tag{3}$$

The identity in (3) is a multiplicative version of the additivity of a simple integral as a functional of the integration form. It also encodes all shuffle relations between the iterated integrals of the forms ω_v and is equivalent to the fact that $\log(J_a^z(\omega_V))$ can be expressed as a series in commutators (of arbitrary length) of the variables A_v.

The central part of the paper generalizes the theory of ordinary (non-iterated) integrals by showing, in particular, how the classical theory is recast within the linear (in A_v) terms of this new iterated construction. For 1-forms f of cusp modular type of weight $k=2r$ for a congruence subgroup $\Gamma=\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$, the theory of ordinary integrals is suitably extended, by covering and generalizing the following points.

(a) It is well known that the classical Mellin transform $\Lambda(f;s) = \int_{i\infty}^0 f(z)z^{s-1}dz$ of a function in $(dz)^{-r}((\Omega_{\mathfrak{H}}^1)^{\otimes r})^\Gamma$, which is Γ-normalized by the involution $g_N = \left(\begin{smallmatrix} 0 & -1 \\ N & 0\end{smallmatrix}\right)$ (i.e. $g_N^*(f(z)(dz)^r) = \epsilon_f f(z)(dz)^r$, $\epsilon_f = \pm 1$), satisfiess the functional equation

$$\Lambda(f;s) = \epsilon_f N^{r-s}\Lambda(f;k-s). \tag{4}$$

This result is generalized to iterated integrals by introducing an iterated version of the Mellin transform associated to a finite sequence $f_1, \dots, f_k$ of cusp forms with respect to a congruence subgroup

$$(5)\quad M(f_1, \dots, f_k; s_1, \dots, s_k) := \\ I^0_{i\infty}(\omega_1, \dots, \omega_k) = \int_{i\infty}^0 \omega_1(z_1) \int_{i\infty}^{z_1} \omega_2(z_2) \cdots \int_{i\infty}^{z_{n-1}} \omega_n(z_n).$$

Here we write $\omega_j(z) = f_j(z) z^{s_j - 1} dz$. Then, the total Mellin transform associated to the finite family $f_V = \{f_v | v \in V\}$ of cusp forms ($s_V = \{s_v | v \in V\}$)

$$TM(f_V; s_V) = J^0_{i\infty}(\omega_V) = \\ \sum_{n=0}^{\infty} \sum_{(v_1, \dots, v_n) \in V^n} A_{v_1} \cdots A_{v_n} M(f_{v_1}, \dots, f_{v_n}; s_{v_1}, \dots, s_{v_n})$$

satisfies, under the assumption of stability of the family f_V with respect to g_N and for an appropriate linear transformation $(g_N)_*$ of the formal variables A_v, the following functional equation ($k_V = (k_v)$, $k_v =$ weight of $f_v(z)$):

$$(6)\qquad TM(f_V; s_V) = g_*(TM(f_V; k_V - s_V))^{-1}.$$

The reviewer thinks that it may be useful to some readers to report explicitly the following comment of the author related to the above functional equation. "The individual Mellin transforms (5) do not fulfill any generalized form of functional equation, simply because applying g_N to the integration limits in them one gets an expression which is not a Mellin transform in the sense developed in the paper. It is only when one puts them all together that one defines the necessary environment for replacing the overall minus sign on the right hand side of (4) by the overall exponent -1 on the right hand side of (6)."

(b) The classical theory of modular symbols deals with the space $MS_k(\Gamma)$ of $\mathbb{R}$-linear functionals on the space $S_k(\Gamma)$ of cusp forms. This space is spanned by the Shimura integrals

$$f(z) \mapsto \int_\alpha^\beta f(z) z^{m-1} dz; \quad 1 \le m \le k-1; \quad \alpha, \beta \in \mathbf{P}^1(\mathbb{Q}).$$

Three descriptions of $MS_k(\Gamma)$ are known: (i) formal, i.e. given in terms of generators and relations; (ii) geometric, i.e. as (part of) the middle homology of a Kuga-Sato variety; (iii) cohomological, i.e. as the dual space to the group cohomology $H^1(\Gamma, \mathbb{V}_{k-2})$ with coefficients in the $(k-2)$-th symmetric power of the basic representation of SL. The author develops in the paper an iterated extension of this third description, by introducing the notion of noncommutative modular symbol as a cohomology class ζ_a of the noncommutative group cohomology $H^1(\Gamma, \Pi)$. Here, Π denotes the multiplicative group of power series in (A_v) with constant term 1. More precisely, a family (ω_v) of Shimura differentials $f_v(z) z^{m_v - 1} dz$, where f_v form a basis of the $\mathbb{C}$-vector space $\bigoplus_i S(k_i, \Gamma)$, and for a fixed weight, m_v ranges over all critical integers for this weight, spans a Γ-invariant space. Let Π be the group of group-like and $(-\mathrm{id})_*$-invariant elements of $(1 + \sum_{v \in V} A_v \mathbb{C}\langle\langle A_V \rangle\rangle)^*$. The left action of Γ on Π is implemented by the functorial action $J \mapsto g_* J$. Then, for any $a \in \mathfrak{H} \cup \mathbf{P}^1(\mathbb{Q})$, the map

$$\Gamma \to \Pi; \quad \gamma \mapsto J^a_{\gamma a}(\Omega)$$

defines a noncommutative 1-cocycle $\zeta_a \in Z^1(\Gamma,\Pi)$ whose class in $H^1(\Gamma,\Pi)$ does not depend on the choice of a. The author eventually shows that the cohomology class ζ_a belongs to the cuspidal subset $H^1(\Gamma,\Pi)_{\text{cusp}}$ consisting of those cohomology classes whose restriction on all stabilizers of Γ-cusps is trivial. Because the group Γ generally does not act transitively on cusps, the components of cocycles ζ_a do not contain iterated (Shimura) integrals along all geodesics connecting two cusps. In the paper the author uses his previously developed techniques of continued fractions in order to express all such integrals through a finite number of them.

(c) In the classical setting, one knows that the Mellin transform $\Lambda(f;s)$ of a cusp form $f(z)$ of weight $2r$ and with Fourier expansion $f(z)=\sum_{n=1}^\infty a_n e^{2\pi i n z}$ can be expressed in terms of the associated Dirichlet series as

$$\Lambda(f;s) = -\frac{\Gamma(s)}{(-2\pi i)^s}\sum_{n=1}^\infty \frac{a_n}{n^s}.$$

This result is generalized in the paper to show that the iterated Mellin transforms (5) at integer values of their Mellin arguments (i.e. integral points of the critical, multidimensional strip) can be expressed as a linear combination of multiple Dirichlet series of a special form. The precise formula reads as follows. Starting with the family of 1-forms on $\mathfrak{H}$,

$$\omega_v(z) = \sum_{n=1}^\infty c_{v,n} e^{2\pi i n z} z^{m_v-1}dz, \quad c_{v,n}\in\mathbb{C},\ m_v\in\mathbb{Z},\ m_v\geq 1,\ c_{n,v}=O(n^C),$$

put

$$L(z;\omega_{v_k},\dots,\omega_{v_1};j_k,\dots,j_1) =$$

$$(7)\quad (2\pi i z)^{j_k}\times \sum_{n_1,\dots,n_k\geq 1}\frac{c_{v_1,n_1}\cdots c_{v_k,n_k}e^{2\pi i(n_1+\cdots+n_k)z}}{n_1^{m_{v_1}+j_0-j_1}(n_1+n_2)^{m_{v_2}+j_1-j_2}\cdots(n_1+\cdots+n_k)^{m_{v_k}+j_{k-1}-j_k}}.$$

The presence of the exponential terms ensures absolute convergence of this series, for any $z\in\mathbb{C}$ with $\mathrm{Im}(z)>0$, so that it defines a holomorphic function on $\mathfrak{H}$ (a formal substitution $z=0$ may lead to divergence). Then, for any $k\geq 1$, $(v_1,\dots,v_k)\in V^k$ and $\mathrm{Im}(z)>0$, we have

$$\begin{aligned}(2\pi i)^{m_{v_1}+\cdots+m_{v_k}} I_{i\infty}^z(\omega_{v_k},\dots,\omega_{v_1}) = \\ (-1)^{\sum_{i=1}^k(m_{v_i}-1)}\sum_{j_1=0}^{m_{v_1}-1}\sum_{j_2=0}^{m_{v_2}-1+j_1}\cdots\sum_{j_k=0}^{m_{v_k}-1+j_{k-1}}(-1)^{j_k} \\ \times\frac{(m_{v_1}-1)!(m_{v_2}-1+j_1)!\cdots(m_{v_k}-1+j_{k-1})!}{j_1!j_2!\cdots j_k!} \\ \times L(z;\omega_{v_k},\dots,\omega_{v_1};j_k,\dots,j_1).\end{aligned}$$

The limit $z\to 0$ and a description of $I_{i\infty}^0(\omega_{v_k},\dots,\omega_{v_1})$ is also discussed in the paper. Under the assumption that $z^{1-m_v}\omega_v(z)$ are a basis of a space of cusp modular forms for the subgroup $\Gamma_0(N)$, or more in general for any modular subgroup which is normalized by the involution $z \mapsto g_N z$, one obtains

$$J^0_{i\infty}(\omega_V) = ((g_N)_*(J^{\frac{i}{\sqrt{N}}}_{i\infty}(\omega_V)))^{-1} J^{\frac{i}{\sqrt{N}}}_{i\infty}(\omega_V). \tag{8}$$

By replacing the coefficients of the formal series at the right hand side of (8) with their convergent representations via multiple Dirichlet series with exponents, one gets the sought for description of $I^0_{i\infty}(\omega_{v_k}, \ldots, \omega_{v_1})$, by avoiding this way, possible divergences at $z = 0$.

In addition to the integral representation (1), multiple zeta values have an equivalent description given in terms of k-multiple Dirichlet series

$$\zeta(m_1, \ldots, m_k) = \sum_{0<n_1<\cdots<n_k} \frac{1}{n_1^{m_1} \cdots n_k^{m_k}} \tag{9}$$

which converge for all integers $m_i \geq 1$ and $m_k > 1$. Easy combinatorial considerations allow one to express in two different ways products $\zeta(l_1, \ldots, l_k) \cdot \zeta(m_1, \ldots, m_k)$ as $\mathbb{Z}$-linear combinations of multiple zeta values, described by a so-called shuffle product. Depending upon a chosen description of the MZV, one talks of harmonic product (i.e. the multiplication rule corresponding to a formal multiplication and rearrangement of the terms in the sum (9)) or of shuffle product (i.e. the product corresponding to the product of two integrals in (1)). The harmonic product, for example, determines so-called harmonic shuffle relations, that is relations involving sums over shuffles, with repetitions, which enumerate simplices occurring in the natural simplicial decomposition of the product. In the paper, the author considers (formal) multiple Dirichlet series of a special form generalizing the formal series considered in (7) and deduces bilinear relations between them generalizing the classical harmonic shuffle relations involving shuffle with repetitions. To obtain this result one has to extend considerably the initial system of series coming from 1-forms of modular type. In relation to this point, the author states an interesting open problem related to the description of some nontrivial spaces of Dirichlet series, containing periods of cusp forms, closed with respect to the series shuffle relations and consisting entirely of periods in the sense of M. Kontsevich and Zagier [in *Mathematics unlimited—2001 and beyond*, 771–808, Springer, Berlin, 2001; MR1852188].

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