

CONTEMPORARY MATHEMATICS

720

Centre de Recherches Mathématiques Proceedings

Spectral Theory and Applications

CRM Summer School
Spectral Theory and Applications
July 4–14, 2016
Université Laval, Québec, Canada

Alexandre Girouard
Editor

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Preface

The *2016 CRM Summer School in Québec City* took place at Université Laval from July 4–14. The topic of the meeting was *Spectral Theory and Applications*. The event was sponsored by the Centre de Recherches Mathématiques (CRM), Institut des Sciences Mathématiques (ISM), the National Science Foundation (NSF), Groupe Interdisciplinaire de Recherche en Éléments Finis (GIREF), and Université Laval.

The summer school brought together students and internationally renowned experts from several subfields of spectral theory. The program consisted of six minicourses introducing fundamentals of spectral theory, applications to physics and partial differential equations, as well as spectral geometry and numerical methods. The minicourses were complemented by exercise sessions and computer labs. There were also some shorter presentations touching upon various related research topics, including classical integral operators, network analysis, and random matrices. The school featured about 50 participants from 12 different countries. These lectures should be useful to graduate as well as advanced undergraduate students.

The lecture notes by Thomas Ransford (Université Laval) cover the fundamentals of spectral theory in Hilbert spaces, up to the spectral theorem for compact self-adjoint operators. Applications to Sturm–Liouville differential equations are also presented. The presentation is streamlined and will appeal to any student who wants to quickly learn the most useful and concrete aspects of spectral theory on Hilbert spaces.

The lecture notes by Richard Laugesen (University of Illinois at Urbana-Champaign) focus on spectral theory of partial differential operators. The emphasis is on basic examples, such as the Laplace operator on bounded Euclidean domains. It starts from separation of variables for the Dirichlet and Neumann Laplacians and builds up to a proof of the spectral theorem for these operators, using weak solutions and quadratic forms on Hilbert spaces. The variational characterization of eigenvalues is then presented, with several applications, such as the monotonicity of Dirichlet eigenvalues under inclusion. Applications to the stability of reaction-diffusion equations are also discussed.

The lectures by Richard Froese (University of British Columbia) provide a mathematical background on classical and quantum mechanics. A novel aspect of the lectures is the continual blend and comparison of ideas from classical and quantum mechanics. An introduction to Lagrangian submanifolds and Legendre transforms is included, and the Hamilton–Jacobi equation is used to construct solutions to the Schrödinger equation. Hidden variables are discussed as a means to illustrate the essential strangeness of the quantum description of nature. This is an area that is neither easy to access nor widely known.

The lectures by Felix Kwok (Hong Kong Baptist University) introduce numerical methods for approximating the eigenvalues and eigenfunctions of partial differential operators. Finite difference methods are first presented, followed by finite element methods. Applications to vibrating plates and their nodal patterns are presented.

The lectures by Yaiza Canzani (University of North Carolina at Chapel Hill) give an overview of spectral geometry on Riemannian manifolds and start with an informal discussion on the ubiquity and importance of the Laplacian. This serves as a road map to the lectures, which cover isospectrality, spectral asymptotics, and heat equation methods. A review of basic Riemannian geometry is included.

The paper by Ram Band (Technion) and Sven Gnutzmann (University of Nottingham) teaches basic quantum graph theory through a well-chosen set of exercises. It covers various vertex conditions, the secular function, and scattering, as well as a trace formula related to periodic orbits.

The paper by Dmitry Khavinson (University of South Florida) presents several open problems in the study of spectral and geometric properties of classical integral operators, such as the Cauchy operator, and single and double layer potentials.

The proceedings of the *2016 CRM Summer School in Québec City on Spectral Theory and its Applications* cover a large variety of topics and methods, combining geometric, analytic, and numerical ideas. We hope that this volume will serve as a reference for young mathematicians who are eager to learn the basics of this fascinating area of mathematics.

The editor would like to express his gratitude to all the contributing authors, as well as to all the speakers at the summer school.

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This book is a collection of lecture notes and survey papers based on the minicourses given by leading experts at the 2016 CRM Summer School on Spectral Theory and Applications, held from July 4–14, 2016, at Université Laval, Québec City, Québec, Canada.

The papers contained in the volume cover a broad variety of topics in spectral theory, starting from the fundamentals and highlighting its connections to PDEs, geometry, physics, and numerical analysis.



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