

# CONTEMPORARY MATHEMATICS

721

## Nonassociative Mathematics and its Applications

Fourth Mile High Conference on  
Nonassociative Mathematics  
July 29 - August 5, 2017  
University of Denver, Denver, Colorado

Petr Vojtěchovský  
Murray R. Bremner  
J. Scott Carter  
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## Introduction

This volume consists of the proceedings of the Fourth Mile High Conference on Nonassociative Mathematics that took place at the University of Denver, Denver, Colorado, July 29–August 5, 2017. The Mile High Conferences cover all aspects of nonassociative mathematics, including quasigroups, loops, latin squares, Lie algebras, Jordan algebras, octonions, quandles, and their applications.

Nonassociative mathematics is concerned with operations that violate the associative law  $x(yz) = (xy)z$ . Given that two out of the four basic arithmetical operations are not associative, nonassociative mathematics is of ancient origin, but as a research discipline it is relatively recent. Latin squares can be traced back to Euler, octonions were constructed by Graves in 1843 just two years after Hamilton discovered quaternions, and Lie algebras appeared during the 1870s. Self-distributivity is present already in Artin’s 1925 work on braid groups and in Burstin and Mayer’s 1929 paper on distributive quasigroups. Jordan algebras were introduced in 1933, and Moufang proved her eponymous theorem in 1935 as an analog of Artin’s theorem on two-generated subalgebras of alternative algebras. The theory of quasigroups and loops then developed rapidly, first from a geometric point of view in the late 1930s in the works of Bol and then algebraically around 1943 at Albert’s Chicago school, with subsequent major contributions by Belousov and Bruck. Toward mathematical physics, the Freudenthal-Tits magic square emerged in 1950s and 1960s as an organizing principle for exceptional Lie algebras and Lie groups, while the Yang-Baxter equation was formulated in 1968. Leibniz algebras, a generalization of Lie algebras, were introduced by Bloh in 1965 and systematically investigated by Loday in 1993. Finally, Joyce and Matveev independently developed the theory of quandles in connection with knot invariants in the early 1980s.

While it is impossible to give justice to the modern developments in nonassociative mathematics in the space of this introduction—and we will therefore not try—one can point out three approaches that are responsible for many recent results and that can also be discerned in this volume.

The first approach is based on careful analysis of standard proofs of classical results and techniques in the associative setting. In rare situations, it can be observed that the classical argument does not require the full force of associativity and that the corresponding result can therefore be extended to areas not previously considered. Much more common, however, is the scenario when associativity is substantially present not only in the proofs but also in the encountered concepts. For a given concept, it is then advisable to develop and study several generalizations that coincide under the assumption of associativity. The resulting theories can be seen



as refinements of the original ideas, and they tend to be well-behaved and more profound than other somewhat arbitrary generalizations.

The second approach is to translate a problem about nonassociative structures into an equivalent, albeit quite technical, problem based on associative algebras or on objects familiar from the associative world, such as permutations or matrices. The resulting problem is rarely straightforward, but at least many tools and deep results become available to the investigator. (To illustrate, the only known proof of Lagrange’s theorem for Moufang loops is based on the study of groups with triality that requires the classification of finite simple groups.)

The third approach is to employ extensive computations, both numerical and symbolic. Linear algebra and Gröbner bases are often used to classify “small” nonassociative structures that hint at larger theories. Specialized computational packages exist as add-ons to standard algebra systems to aid in calculations with quandles, quasigroups, alternative algebras, and so on, where hand calculations are extremely impractical. Finally, automated deduction and finite model builders are used prominently, perhaps more than in any other branch of mathematics, to gain insight into previously inaccessible theories and to speed up the cycle “example  $\rightarrow$  conjecture  $\rightarrow$  theorem”.

We present these proceedings as a small but representative selection of active areas of investigation in nonassociative mathematics as well as a sampling of applications to set theory, low-dimensional topology, and supergravity. Several papers are of a survey character and are therefore suitable as introductions to their respective subjects.

We thank the National Science Foundation and the Associate Provost for Research and the Dean of the Division of Natural Sciences and Mathematics at the University of Denver as well as the Gudder Trust for their ongoing financial support of the Mile High Conferences. We also thank local organizers Ryan DeMUSE, Wesley Fussner, Hwajin Park, and Seung Yeop Yang for their help with the conference and Christine Thivierge at the American Mathematical Society for her assistance with the preparation of this volume.

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Nonassociative mathematics is a broad research area that studies mathematical structures violating the associative law  $x(yz) = (xy)z$ . The topics covered by nonassociative mathematics include quasigroups, loops, Latin squares, Lie algebras, Jordan algebras, octonions, racks, quandles, and their applications.

This volume contains the proceedings of the Fourth Mile High Conference on Nonassociative Mathematics, held from July 29–August 5, 2017, at the University of Denver, Denver, Colorado.

Included are research papers covering active areas of investigation, survey papers covering Leibniz algebras, self-distributive structures, and rack homology, and a sampling of applications ranging from Yang-Mills theory to the Yang-Baxter equation and Laver tables.

An important aspect of nonassociative mathematics is the wide range of methods employed, from purely algebraic to geometric, topological, and computational, including automated deduction, all of which play an important role in this book.



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